Uncertain Portfolio with Fuzzy Investment Proportion Based on Possibilistic Theory

Xue Deng, Yingxian Lin and Huidan Zhuang

***Abstract*—Due to the uncertainty of the financial market and insufficient knowledge of investors, it is very difficult for investors to determine the investment proportion. Therefore, we do some research on uncertain portfolio when the investment proportion is regarded as a fuzzy number. First, we discuss the two cases where the return and the investment proportion are either triangular fuzzy numbers or trapezoidal fuzzy numbers. Then, based on possibilistic theory and fuzzy theory, we derive the mathematical expressions of the possibilistic returns and variances of portfolio with fuzzy investment proportions. Later, a tri-objective model is constructed with liquidity and fuzzy constraints. It is transformed into a single objective model with parameters by the fuzzy linear programming. Finally, the feasibility of the model is illustrated. We explore the effects of variable parameter values by a numerical example. In fact, our proposed model in this paper can guide investors to make proper investment decisions when the economy is in recession and the investment information is not comprehensive.**

***Index Terms*—Portfolio selection, Uncertain investment proportion, Triangular fuzzy number, Trapezoidal fuzzy number, Turnover rate**

I. INTRODUCTION

The mean-variance model constructed by Markowitz [1] quantitatively analyzed the portfolio problem, which opened up a new way for scholars to study the portfolio in a random environment. Interested readers can refer to Konno and Yamazaki [2], Speranza [3], Yu et al. [4], Huang [5], Wei and Ye [6], Sun et al. [7], Villena and Reus [8], Yang [9], Kim et al. [10], etc.

Since the financial market is full of uncertainty and complexity, investors are faced with imperfect information. In 1965, Zadeh [11] presented fuzzy set theory, which provided a significant way to study fuzzy problems and phenomenon with quantitative description and analysis. After that, scholars explored portfolio problems in a fuzzy environment more widely. Some scholars presented portfolio selection models with lower and upper possibility distributions, for example, Watada [12], Ramaswamy [13] and Tanaka [14]. Carlsson [15] and Zhang [16] studied fuzzy numbers and introduced possibilistic mean, variance and covariance. Interested readers can learn more about further research by referring to Bhattacharyya et al. [17], Zhang et al. [18], Vernic [19], Ruiz et al. [20], Mehlawat [21], Deng et al. [22], Nazir [23], Peachavanish [24], Deng and Pan [25], Zheng and Yao [26], Wang [27], etc.

Although many scholars study fuzzy portfolio selection models, few scholars attempt to explore and analyze the fuzzy investment proportion of portfolios. In the investment process, if the information and knowledge about the investments is incomplete or the economic conditions are poor, it is difficult for investors to make a certain investment proportion. At this point, the fuzzy investment proportion will help investors to make portfolio decisions better. Tsaur [28] only considered the investment proportion as symmetric triangular fuzzy numbers. Therefore, we consider more complicated situations in this paper, that is, the investment proportion as triangular fuzzy numbers or trapezoidal fuzzy numbers. Furthermore, we construct a tri-objective portfolio model with turnover rate based on return and risk, and discuss its feasibility with a numerical example.

This paper is organized as follows. In Section 2, some concepts of possibilistic theory are reviewed. In Section 3, we give conclusions under two kinds of investment proportions, namely triangular fuzzy numbers and trapezoidal fuzzy numbers. In Section 4, a tri-objective portfolio model with fuzzy investment proportions is constructed and transformed. In Section 5, a detailed numerical example is provided and the constructed model is analyzed in depth. In Section 6, we summarize the contributions of this research.

II. PRELIMINARIES

We review some concepts of the upper and lower possibilistic mean values and variances which will be used in the remainder of this paper.

Definition 1 $A, B \in F$ are two fuzzy numbers with $[A]^{\gamma} = [\alpha_1(\gamma), \alpha_2(\gamma)]$ and $[B]^{\gamma} = [\beta_1(\gamma), \beta_2(\gamma)]$ for $\forall \gamma \in [0,1]$. Then

$$[AB]^{\gamma} = [\alpha_1(\gamma)\beta_1(\gamma), \alpha_2(\gamma)\beta_2(\gamma)] = [A]^{\gamma} \times [B]^{\gamma}. \quad (1)$$

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Definition 2 $A \in F$ is a fuzzy number with $[A]^{\gamma} = [\alpha_1(\gamma), \alpha_2(\gamma)]$ for $\forall \gamma \in [0,1]$, the corresponding lower and upper possibilistic mean values are

$$M_*(A) = \frac{\int_0^1 Pos[A \le \alpha_1(\gamma)]\alpha_1(\gamma)d\gamma}{\int_0^1 Pos[A \le \alpha_1(\gamma)]d\gamma} = 2\int_0^1 \gamma\alpha_1(\gamma)d\gamma, \quad (2)$$

$$M^*(A) = \frac{\int_0^1 Pos[A \ge \alpha_2(\gamma)]\alpha_2(\gamma)d\gamma}{\int_0^1 Pos[A \ge \alpha_2(\gamma)]d\gamma} = 2\int_0^1 \gamma\alpha_2(\gamma)d\gamma, \quad (3)$$

where

$$Pos[A \le \alpha_1(\gamma)] = \prod[-\infty, \alpha_1(\gamma)] = \sup_{\mu \le \alpha_1(\gamma)} A(\mu) = \gamma, \quad (4)$$

$$Pos[A \ge \alpha_2(\gamma)] = \prod[\alpha_2(\gamma), \infty] = \sup_{\mu \ge \alpha_2(\gamma)} A(\mu) = \gamma. \quad (5)$$

the corresponding possibilistic mean value of A is

$$M(A) = \frac{1}{2}\left[M_*(A) + M^*(A)\right]. \quad (6)$$

Lemma 1 $A, B \in F$ are two fuzzy numbers and $\mu \in R$, then we have

$$M(\mu A) = \mu M(A), \quad (7)$$

$$M(A+B) = M(A) + M(B). \quad (8)$$

Definition 3 $A \in F$ is a fuzzy number with $[A]^{\gamma} = [\alpha_1(\gamma), \alpha_2(\gamma)]$ for $\forall \gamma \in [0,1]$, then the lower and upper possibilistic variances are

$$Var_*(A) = 2\int_0^1 \gamma[M_*(A) - \alpha_1(\gamma)]^2 d\gamma, \quad (9)$$

$$Var^*(A) = 2\int_0^1 \gamma[M^*(A) - \alpha_2(\gamma)]^2 d\gamma. \quad (10)$$

the corresponding possibilistic variance of A is

$$Var(A) = \frac{1}{2}\left[Var_*(A) + Var^*(A)\right]. \quad (11)$$

III. Some New Results with the Fuzzy Investment Proportion

Due to the incomplete information and knowledge of portfolio or poor economic conditions, it is rather difficult for investors to make a certain investment proportion. Therefore, we consider the investment proportion as a fuzzy number in the decision-making process. In this section, we will discuss the cases where the investment proportion is a triangle fuzzy number or a trapezoidal fuzzy number.

A. Discussions on Taking the Investment Proportion as a Triangular Fuzzy Number

In subsection A, the return rate and proportion of the asset i are triangular fuzzy numbers $\tilde{r}_i = (f_i; \varpi_i, \vartheta_i)$ and $\tilde{x}_i = (e_i; \varsigma_i, \kappa_i)$, $i = 1, 2, \cdots, n$, where values in parentheses from left to right are the central value, left and right spread values. Thus, the γ-levels of $\tilde{r}_i$ and $\tilde{x}_i$ are

$$[\tilde{r}_i]^{\gamma} = [f_i - \varpi_i(1-\gamma), f_i + \vartheta_i(1-\gamma)], \quad (12)$$

$$[\tilde{x}_i]^{\gamma} = [e_i - \varsigma_i(1-\gamma), s_i + \kappa_i(1-\gamma)]. \quad (13)$$

Thus, it can be deduced that the lower possibilistic return of the asset i is (14) and the upper possibilistic return is (15).

For two assets that have the fuzzy number construction method described above, the lower possibilistic return of the portfolio is (16), the upper possibilistic return is (17). Accordingly, the lower possibilistic variance of the portfolio is (18) and the upper possibilistic variance is (19).

$$\begin{aligned}(\tilde{r}_i\tilde{x}_i)_*^{\gamma} &= [f_i - \varpi_i(1-\gamma)] \times [e_i - \varsigma_i(1-\gamma)] \\ &= f_ie_i - f_i\varsigma_i(1-\gamma) - e_i\varpi_i(1-\gamma) + \varpi_i\varsigma_i(1-\gamma)^2 \\ &= \varpi_i\varsigma_i\gamma^2 + (f_i\varsigma_i + e_i\varpi_i - 2\varpi_i\varsigma_i)\gamma + (f_ie_i - f_i\varsigma_i - e_i\varpi_i + \varpi_i\varsigma_i).\end{aligned} \quad (14)$$

$$\begin{aligned}(\tilde{r}_i\tilde{x}_i)^{*\gamma} &= [f_i + \vartheta_i(1-\gamma)] \times [e_i + \kappa_i(1-\gamma)] \\ &= f_ie_i + f_i\kappa_i(1-\gamma) + e_i\vartheta_i(1-\gamma) + \vartheta_i\kappa_i(1-\gamma)^2 \\ &= \vartheta_i\kappa_i\gamma^2 - (f_i\kappa_i + e_i\vartheta_i + 2\vartheta_i\kappa_i)\gamma + (f_ie_i + f_i\kappa_i + e_i\vartheta_i + \vartheta_i\kappa_i).\end{aligned} \quad (15)$$

$$\begin{aligned}M_*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) &= M_*(\tilde{r}_1\tilde{x}_1) + M_*(\tilde{r}_2\tilde{x}_2) \\ &= 2\int_0^1 \gamma[\varpi_1\varsigma_1\gamma^2 + (f_1\varsigma_1 + e_1\varpi_1 - 2\varpi_1\varsigma_1)\gamma + (f_1e_1 - f_1\varsigma_1 - e_1\varpi_1 + \varpi_1\varsigma_1)]d\gamma \\ &\quad + 2\int_0^1 \gamma[\varpi_2\varsigma_2\gamma^2 + (f_2\varsigma_2 + e_2\varpi_2 - 2\varpi_2\varsigma_2)\gamma + (f_2e_2 - f_2\varsigma_2 - e_2\varpi_2 + \varpi_2\varsigma_2)]d\gamma \\ &= \frac{1}{6}(\varpi_1\varsigma_1 + \varpi_2\varsigma_2) - \frac{1}{3}(f_1\varsigma_1 + f_2\varsigma_2 + e_1\varpi_1 + e_2\varpi_2) + (f_1e_1 + f_2e_2).\end{aligned} \quad (16)$$

$$\begin{aligned}M^*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) &= M^*(\tilde{r}_1\tilde{x}_1) + M^*(\tilde{r}_2\tilde{x}_2) \\ &= 2\int_0^1 \gamma[\vartheta_1\kappa_1\gamma^2 - (f_1\kappa_1 + e_1\vartheta_1 + 2\vartheta_1\kappa_1)\gamma + (f_1e_1 + f_1\kappa_1 + e_1\vartheta_1 + \vartheta_1\kappa_1)]d\gamma \\ &\quad + 2\int_0^1 \gamma[\vartheta_2\kappa_2\gamma^2 - (f_2\kappa_2 + e_2\vartheta_2 + 2\vartheta_2\kappa_2)\gamma + (f_2e_2 + f_2\kappa_2 + e_2\vartheta_2 + \vartheta_2\kappa_2)]d\gamma \\ &= \frac{1}{6}(\vartheta_1\kappa_1 + \vartheta_2\kappa_2) + \frac{1}{3}(f_1\kappa_1 + f_2\kappa_2 + e_1\vartheta_1 + e_2\vartheta_2) + (f_1e_1 + f_2e_2).\end{aligned} \quad (17)$$

$$\begin{aligned}
Var_*(\tilde{r}_1\tilde{x}_1+\tilde{r}_2\tilde{x}_2) &= 2\int_0^1 \gamma\left\{M_*(\tilde{r}_1\tilde{x}_1+\tilde{r}_2\tilde{x}_2)-[(\tilde{r}_1\tilde{x}_1)_*^{\gamma}+(\tilde{r}_2\tilde{x}_2)_*^{\gamma}]\right\}^2 d\gamma \\
&= 2\int_0^1 \gamma[\frac{1}{6}(\varpi_1\varsigma_1+\varpi_2\varsigma_2)-\frac{1}{3}(f_1\varsigma_1+f_2\varsigma_2+e_1\varpi_1+e_2\varpi_2)+(f_1e_1+f_2e_2) \\
&\quad -(\varpi_1\varsigma_1+\varpi_2\varsigma_2)\gamma^2-(f_1\varsigma_1+f_2\varsigma_2+e_1\varpi_1+e_2\varpi_2-2\varpi_1\varsigma_1-2\varpi_2\varsigma_2)\gamma \\
&\quad -(f_1e_1+f_2e_2-f_1\varsigma_1-f_2\varsigma_2-e_1\varpi_1-e_2\varpi_2+\varpi_1\varsigma_1+\varpi_2\varsigma_2)]^2 d\gamma \\
&= \frac{1}{18}[(f_1\varsigma_1+f_2\varsigma_2+e_1\varpi_1+e_2\varpi_2)-\frac{4}{5}(\varpi_1\varsigma_1+\varpi_2\varsigma_2)]^2+\frac{1}{300}(\varpi_1\varsigma_1+\varpi_2\varsigma_2)^2.
\end{aligned} \tag{18}$$

$$\begin{aligned}
Var^*(\tilde{r}_1\tilde{x}_1+\tilde{r}_2\tilde{x}_2) &= 2\int_0^1 \gamma\left\{M^*(\tilde{r}_1\tilde{x}_1+\tilde{r}_2\tilde{x}_2)-[(\tilde{r}_1\tilde{x}_1)^{*\gamma}+(\tilde{r}_2\tilde{x}_2)^{*\gamma}]\right\}^2 d\gamma \\
&= 2\int_0^1 \gamma[\frac{1}{6}(\vartheta_1\kappa_1+\vartheta_2\kappa_2)+\frac{1}{3}(f_1\kappa_1+f_2\kappa_2+e_1\vartheta_1+e_2\vartheta_2)+(f_1e_1+f_2e_2) \\
&\quad -(\vartheta_1\kappa_1+\vartheta_2\kappa_2)\gamma^2+(f_1\kappa_1+f_2\kappa_2+e_1\vartheta_1+e_2\vartheta_2+2\vartheta_1\kappa_1+2\vartheta_2\kappa_2)\gamma \\
&\quad -(f_1e_1+f_2e_2+f_1\kappa_1+f_2\kappa_2+e_1\vartheta_1+e_2\vartheta_2+\vartheta_1\kappa_1+\vartheta_2\kappa_2)]^2 d\gamma \\
&= \frac{1}{18}[(f_1\kappa_1+f_2\kappa_2+e_1\vartheta_1+e_2\vartheta_2)+\frac{4}{5}(\vartheta_1\kappa_1+\vartheta_2\kappa_2)]^2+\frac{1}{300}(\vartheta_1\kappa_1+\vartheta_2\kappa_2)^2.
\end{aligned} \tag{19}$$

More generally, for n assets, we can infer the lower and upper possibilistic returns and variances of the portfolio according to the following theorems:

Theorem 1: Let the returns and proportions of n assets be triangular fuzzy numbers $\tilde{r}_i=(f_i;\varpi_i,\vartheta_i)$ and $\tilde{x}_i=(e_i;\varsigma_i,\kappa_i)$, $i=1,2,\cdots,n$. Thus, the lower possibilistic return and the upper possibilistic return of the portfolio are (20) and (21).

Proof: Firstly, the lower possibilistic return of n assets is (22). Similarly, the upper possibilistic return of n assets is (23).

□

Therefore, the possibilistic return of n assets can be expressed as (24).

Theorem 2: Let the returns and proportions of n assets be triangular fuzzy numbers $\tilde{r}_i=(f_i;\varpi_i,\vartheta_i)$ and $\tilde{x}_i=(e_i;\varsigma_i,\kappa_i)$, $i=1,2,\cdots,n$. Thus, the lower and upper possibilistic variances of the portfolio are (25) and (26).

Proof: Firstly, the lower possibilistic variance of n assets is (27). Similarly, the upper possibilistic variance of n assets is (28).

□

$$M_*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)=\frac{1}{6}\sum_{i=1}^{n}\varpi_i\varsigma_i-\frac{1}{3}\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)+\sum_{i=1}^{n}f_ie_i. \tag{20}$$

$$M^*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)=\frac{1}{6}\sum_{i=1}^{n}\vartheta_i\kappa_i+\frac{1}{3}\sum_{i=1}^{n}(f_i\kappa_i+e_i\vartheta_i)+\sum_{i=1}^{n}f_ie_i. \tag{21}$$

$$\begin{aligned}
M_*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i) &= M_*(\tilde{r}_1\tilde{x}_1+\tilde{r}_2\tilde{x}_2+\cdots+\tilde{r}_n\tilde{x}_n) \\
&= M_*(\tilde{r}_1\tilde{x}_1)+M_*(\tilde{r}_2\tilde{x}_2)+\cdots+M_*(\tilde{r}_n\tilde{x}_n) \\
&= \frac{1}{6}\varpi_1\varsigma_1-\frac{1}{3}(f_1\varsigma_1+e_1\varpi_1)+f_1e_1+\frac{1}{6}\varpi_2\varsigma_2-\frac{1}{3}(f_2\varsigma_2+e_2\varpi_2)+f_2e_2+\cdots+\frac{1}{6}\varpi_n\varsigma_n-\frac{1}{3}(f_n\varsigma_n+e_n\varpi_n)+f_ne_n \\
&= \frac{1}{6}\sum_{i=1}^{n}\varpi_i\varsigma_i-\frac{1}{3}\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)+\sum_{i=1}^{n}f_ie_i.
\end{aligned} \tag{22}$$

$$\begin{aligned}
M^*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i) &= M^*(\tilde{r}_1\tilde{x}_1+\tilde{r}_2\tilde{x}_2+\cdots+\tilde{r}_n\tilde{x}_n) \\
&= M^*(\tilde{r}_1\tilde{x}_1)+M^*(\tilde{r}_2\tilde{x}_2)+\cdots+M^*(\tilde{r}_n\tilde{x}_n) \\
&= \frac{1}{6}\vartheta_1\kappa_1+\frac{1}{3}(f_1\kappa_1+e_1\vartheta_1)+f_1e_1+\frac{1}{6}\vartheta_2\kappa_2+\frac{1}{3}(f_2\kappa_2+e_2\vartheta_2)+f_2e_2+\cdots+\frac{1}{6}\vartheta_n\kappa_n+\frac{1}{3}(f_n\kappa_n+e_n\vartheta_n)+f_ne_n \\
&= \frac{1}{6}\sum_{i=1}^{n}\vartheta_i\kappa_i+\frac{1}{3}\sum_{i=1}^{n}(f_i\kappa_i+e_i\vartheta_i)+\sum_{i=1}^{n}f_ie_i.
\end{aligned} \tag{23}$$

$$M(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)=\frac{M^*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)+M_*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)}{2}=\frac{\frac{1}{6}\sum_{i=1}^{n}(\vartheta_i\kappa_i+\varpi_i\varsigma_i)+\frac{1}{3}\sum_{i=1}^{n}(f_i\kappa_i+e_i\vartheta_i-f_i\varsigma_i-e_i\varpi_i)+2\sum_{i=1}^{n}f_ie_i}{2}. \tag{24}$$

$$Var_*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)=\frac{1}{18}[\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)-\frac{4}{5}\sum_{i=1}^{n}(\varpi_i\varsigma_i)]^2+\frac{1}{300}(\sum_{i=1}^{n}\varpi_i\varsigma_i)^2. \tag{25}$$

$$Var^*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)=\frac{1}{18}[\sum_{i=1}^{n}(f_i\kappa_i+e_i\vartheta_i)+\frac{4}{5}(\sum_{i=1}^{n}\vartheta_i\kappa_i)]^2+\frac{1}{300}(\sum_{i=1}^{n}\vartheta_i\kappa_i)^2. \tag{26}$$

$$\begin{aligned}Var_*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)&=2\int_0^1\gamma[M_*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)-(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)_*^{\gamma}]^2d\gamma\\&=2\int_0^1\gamma[\frac{1}{6}\sum_{i=1}^{n}\varpi_i\varsigma_i-\frac{1}{3}\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)+\sum_{i=1}^{n}f_ie_i-\gamma^2\sum_{i=1}^{n}\varpi_i\varsigma_i-\gamma\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i-2\varpi_i\varsigma_i)\\&\quad-\sum_{i=1}^{n}(f_ie_i-f_i\varsigma_i-e_i\varpi_i+\varpi_i\varsigma_i)]^2d\gamma\\&=2\int_0^1\gamma[(-\gamma^2+2\gamma-\frac{5}{6})(\sum_{i=1}^{n}\varpi_i\varsigma_i)+(\frac{2}{3}-\gamma)\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)]^2d\gamma\\&=\frac{1}{18}[\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)-\frac{4}{5}(\sum_{i=1}^{n}\varpi_i\varsigma_i)]^2+\frac{1}{300}(\sum_{i=1}^{n}\varpi_i\varsigma_i)^2.\end{aligned} \tag{27}$$

$$\begin{aligned}Var^*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)&=2\int_0^1\gamma[M^*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)-(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)^{*\gamma}]^2d\gamma\\&=2\int_0^1\gamma[\frac{1}{6}\sum_{i=1}^{n}\vartheta_i\kappa_i+\frac{1}{3}\sum_{i=1}^{n}(f_i\kappa_i+e_i\vartheta_i)+\sum_{i=1}^{n}f_ie_i-\gamma^2\sum_{i=1}^{n}\vartheta_i\kappa_i+\gamma\sum_{i=1}^{n}(f_i\kappa_i+e_i\vartheta_i+2\vartheta_i\kappa_i)\\&\quad-\sum_{i=1}^{n}(f_ie_i+f_i\kappa_i+e_i\vartheta_i+\vartheta_i\kappa_i)]^2d\gamma\\&=2\int_0^1\gamma[(-\gamma^2+2\gamma-\frac{5}{6})(\sum_{i=1}^{n}\vartheta_i\kappa_i)+(\gamma-\frac{2}{3})\sum_{i=1}^{n}(f_i\kappa_i+e_i\vartheta_i)]^2d\gamma\\&=\frac{1}{18}[\sum_{i=1}^{n}(f_i\kappa_i+e_i\vartheta_i)+\frac{4}{5}(\sum_{i=1}^{n}\vartheta_i\kappa_i)]^2+\frac{1}{300}(\sum_{i=1}^{n}\vartheta_i\kappa_i)^2.\end{aligned} \tag{28}$$

$$\sigma_{1*}{}^2=\frac{1}{18}[\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)-\frac{4}{5}(\sum_{i=1}^{n}\varpi_i\varsigma_i)]^2. \tag{29}$$

$$\sigma_{2*}{}^2=\frac{1}{300}(\sum_{i=1}^{n}\varpi_i\varsigma_i)^2. \tag{30}$$

$$\sigma_1^{*2}=\frac{1}{18}[\sum_{i=1}^{n}(f_i\kappa_i+e_i\vartheta_i)+\frac{4}{5}(\sum_{i=1}^{n}\vartheta_i\kappa_i)]^2. \tag{31}$$

$$\sigma_2^{*2}=\frac{1}{300}(\sum_{i=1}^{n}\vartheta_i\kappa_i)^2. \tag{32}$$

$$\sigma_1=\frac{\sqrt{\sigma_1^{*2}}+\sqrt{\sigma_{1*}{}^2}}{2}=\frac{\frac{1}{3\sqrt{2}}[\sum_{i=1}^{n}(f_i\kappa_i+e_i\vartheta_i)+\frac{4}{5}(\sum_{i=1}^{n}\vartheta_i\kappa_i)]+\frac{1}{3\sqrt{2}}[\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)-\frac{4}{5}(\sum_{i=1}^{n}\varpi_i\varsigma_i)]}{2}. \tag{33}$$

$$\sigma_2=\frac{\sqrt{\sigma_2^{*2}}+\sqrt{\sigma_{2*}{}^2}}{2}=\frac{\frac{1}{10\sqrt{3}}(\sum_{i=1}^{n}\vartheta_i\kappa_i)+\frac{1}{10\sqrt{3}}(\sum_{i=1}^{n}\varpi_i\varsigma_i)}{2}. \tag{34}$$

Here, we use the method of [28], for the sake of simplicity, (25) is divided into two parts, namely, the variance of the portfolio and the fuzzy variance of the incomplete information, respectively. Then the lower possibilistic variance of *n* assets are expressed as (29) and (30).

Similarly, the upper possibilistic variance of *n* assets are expressed as (31) and (32).

Therefore, the standard deviations of the portfolio can be expressed as (33), the standard deviations of the incomplete information can be expressed as (34).

From above, we can see that when the triangular fuzzy returns and proportions degenerate to symmetric triangular fuzzy numbers, that is $\varpi_i=\vartheta_i$ and $\varsigma_i=\kappa_i$, the resulting conclusions are the corresponding ones in [28]. Therefore, [28] is a special case of subsection A.

B. Discussion on Taking the Investment Proportion as a Trapezoidal Fuzzy Number

In subsection B, the return rate and proportion of the asset i are trapezoidal fuzzy numbers $\tilde{r}_i=(f_i,h_i;\varpi_i,\vartheta_i)$ and $\tilde{x}_i=(e_i,t_i;\varsigma_i,\kappa_i)$, $i=1,2,\cdots,n$, where $[f_i,h_i]$, ϖ_i and ϑ_i are the central interval, left and right spread values of $\tilde{r}_i$, and $\tilde{x}_i$ has a composition similar to $\tilde{r}_i$. Then the γ-levels of $\tilde{r}_i$ and $\tilde{x}_i$ can be derived as

$$[\tilde{r}_i]^\gamma = [f_i - \varpi_i(1-\gamma), h_i + \vartheta_i(1-\gamma)], \tag{35}$$

$$[\tilde{x}_i]^\gamma = [e_i - \varsigma_i(1-\gamma), t_i + \kappa_i(1-\gamma)]. \tag{36}$$

Thus, it can be deduced that the lower possibilistic return of the asset i is (37) and the upper possibilistic return is (38).

For two assets that have the fuzzy number construction method described above, the lower possibilistic return of the portfolio is (39) and the upper possibilistic return is (40). Accordingly, the lower possibilistic variance of the portfolio is (41) and the upper possibilistic variance is (42).

More generally, for n assets, we can derive the lower and upper possibilistic returns and variances of the portfolio according to the following theorems:

Theorem 3: Let the returns and proportions of n assets be trapezoidal fuzzy numbers $\tilde{r}_i = (f_i, h_i; \varpi_i, \vartheta_i)$ and $\tilde{x}_i = (e_i, t_i; \varsigma_i, \kappa_i)$, $i = 1, 2, \cdots, n$. Thus, the lower possibilistic return and the upper possibilistic return of the portfolio are (43) and (44).

$$\begin{aligned}(\tilde{r}_i\tilde{x}_i)_*^\gamma &= [f_i - \varpi_i(1-\gamma)] \times [e_i - \varsigma_i(1-\gamma)] \\ &= f_i e_i - f_i\varsigma_i(1-\gamma) - e_i\varpi_i(1-\gamma) + \varpi_i\varsigma_i(1-\gamma)^2 \\ &= \varpi_i\varsigma_i\gamma^2 + (f_i\varsigma_i + e_i\varpi_i - 2\varpi_i\varsigma_i)\gamma + (f_i e_i - f_i\varsigma_i - e_i\varpi_i + \varpi_i\varsigma_i).\end{aligned} \tag{37}$$

$$\begin{aligned}(\tilde{r}_i\tilde{x}_i)^{*\gamma} &= [h_i + \vartheta_i(1-\gamma)] \times [t_i + \kappa_i(1-\gamma)] \\ &= h_i t_i + h_i\kappa_i(1-\gamma) + t_i\vartheta_i(1-\gamma) + \vartheta_i\kappa_i(1-\gamma)^2 \\ &= \vartheta_i\kappa_i\gamma^2 - (h_i\kappa_i + t_i\vartheta_i + 2\vartheta_i\kappa_i)\gamma + (h_i t_i + h_i\kappa_i + t_i\vartheta_i + \vartheta_i\kappa_i).\end{aligned} \tag{38}$$

$$\begin{aligned}M_*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) &= M_*(\tilde{r}_1\tilde{x}_1) + M_*(\tilde{r}_2\tilde{x}_2) \\ &= 2\int_0^1 \gamma[\varpi_1\varsigma_1\gamma^2 + (f_1\varsigma_1 + e_1\varpi_1 - 2\varpi_1\varsigma_1)\gamma + (f_1e_1 - f_1\varsigma_1 - e_1\varpi_1 + \varpi_1\varsigma_1)]d\gamma \\ &\quad + 2\int_0^1 \gamma[\varpi_2\varsigma_2\gamma^2 + (f_2\varsigma_2 + e_2\varpi_2 - 2\varpi_2\varsigma_2)\gamma + (f_2e_2 - f_2\varsigma_2 - e_2\varpi_2 + \varpi_2\varsigma_2)]d\gamma \\ &= \frac{1}{6}(\varpi_1\varsigma_1 + \varpi_2\varsigma_2) - \frac{1}{3}(f_1\varsigma_1 + f_2\varsigma_2 + e_1\varpi_1 + e_2\varpi_2) + (f_1e_1 + f_2e_2).\end{aligned} \tag{39}$$

$$\begin{aligned}M^*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) &= M^*(\tilde{r}_1\tilde{x}_1) + M^*(\tilde{r}_2\tilde{x}_2) \\ &= 2\int_0^1 \gamma[\vartheta_1\kappa_1\gamma^2 - (h_1\kappa_1 + t_1\vartheta_1 + 2\vartheta_1\kappa_1)\gamma + (h_1t_1 + h_1\kappa_1 + t_1\vartheta_1 + \vartheta_1\kappa_1)]d\gamma \\ &\quad + 2\int_0^1 \gamma[\vartheta_2\kappa_2\gamma^2 - (h_2\kappa_2 + t_2\vartheta_2 + 2\vartheta_2\kappa_2)\gamma + (h_2t_2 + h_2\kappa_2 + t_2\vartheta_2 + \vartheta_2\kappa_2)]d\gamma \\ &= \frac{1}{6}(\vartheta_1\kappa_1 + \vartheta_2\kappa_2) + \frac{1}{3}(h_1\kappa_1 + h_2\kappa_2 + t_1\vartheta_1 + t_2\vartheta_2) + (h_1t_1 + h_2t_2).\end{aligned} \tag{40}$$

$$\begin{aligned}Var_*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) &= 2\int_0^1 \gamma\left\{M_*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) - [(\tilde{r}_1\tilde{x}_1)_*^\gamma + (\tilde{r}_2\tilde{x}_2)_*^\gamma]\right\}^2 d\gamma \\ &= 2\int_0^1 \gamma[\frac{1}{6}(\varpi_1\varsigma_1 + \varpi_2\varsigma_2) - \frac{1}{3}(f_1\varsigma_1 + f_2\varsigma_2 + e_1\varpi_1 + e_2\varpi_2) + (f_1e_1 + f_2e_2) \\ &\quad - (\varpi_1\varsigma_1 + \varpi_2\varsigma_2)\gamma^2 - (f_1\varsigma_1 + f_2\varsigma_2 + e_1\varpi_1 + e_2\varpi_2 - 2\varpi_1\varsigma_1 - 2\varpi_2\varsigma_2)\gamma \\ &\quad - (f_1e_1 + f_2e_2 - f_1\varsigma_1 - f_2\varsigma_2 - e_1\varpi_1 - e_2\varpi_2 + \varpi_1\varsigma_1 + \varpi_2\varsigma_2)]^2 d\gamma \\ &= \frac{1}{18}[(f_1\varsigma_1 + f_2\varsigma_2 + e_1\varpi_1 + e_2\varpi_2) - \frac{4}{5}(\varpi_1\varsigma_1 + \varpi_2\varsigma_2)]^2 + \frac{1}{300}(\varpi_1\varsigma_1 + \varpi_2\varsigma_2)^2.\end{aligned} \tag{41}$$

$$\begin{aligned}Var^*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) &= 2\int_0^1 \gamma\left\{M^*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2) - [(\tilde{r}_1\tilde{x}_1)^{*\gamma} + (\tilde{r}_2\tilde{x}_2)^{*\gamma}]\right\}^2 d\gamma \\ &= 2\int_0^1 \gamma[\frac{1}{6}(\vartheta_1\kappa_1 + \vartheta_2\kappa_2) + \frac{1}{3}(h_1\kappa_1 + h_2\kappa_2 + t_1\vartheta_1 + t_2\vartheta_2) + (h_1t_1 + h_2t_2) \\ &\quad - (\vartheta_1\kappa_1 + \vartheta_2\kappa_2)\gamma^2 + (h_1\kappa_1 + h_2\kappa_2 + t_1\vartheta_1 + t_2\vartheta_2 + 2\vartheta_1\kappa_1 + 2\vartheta_2\kappa_2)\gamma \\ &\quad - (h_1t_1 + h_2t_2 + h_1\kappa_1 + h_2\kappa_2 + t_1\vartheta_1 + t_2\vartheta_2 + \vartheta_1\kappa_1 + \vartheta_2\kappa_2)]^2 d\gamma \\ &= \frac{1}{18}[(h_1\kappa_1 + h_2\kappa_2 + t_1\vartheta_1 + t_2\vartheta_2) + \frac{4}{5}(\vartheta_1\kappa_1 + \vartheta_2\kappa_2)]^2 + \frac{1}{300}(\vartheta_1\kappa_1 + \vartheta_2\kappa_2)^2.\end{aligned} \tag{42}$$

$$M_*(\sum_{i=1}^n \tilde{r}_i\tilde{x}_i) = \frac{1}{6}\sum_{i=1}^n \varpi_i\varsigma_i - \frac{1}{3}\sum_{i=1}^n (f_i\varsigma_i + e_i\varpi_i) + \sum_{i=1}^n f_i e_i. \tag{43}$$

$$M^*(\sum_{i=1}^n \tilde{r}_i\tilde{x}_i) = \frac{1}{6}\sum_{i=1}^n \vartheta_i\kappa_i + \frac{1}{3}\sum_{i=1}^n (h_i\kappa_i + t_i\vartheta_i) + \sum_{i=1}^n h_i t_i. \tag{44}$$

Proof: Firstly, the lower possibilistic return of n assets is (45). Similarly, the upper possibilistic return of n assets is (46).

□

Therefore, the possibilistic return of n assets can be expressed as (47).

Theorem 4: Let the returns and proportions of n assets be trapezoidal fuzzy numbers $\tilde{r}_i = (f_i, h_i; \varpi_i, \vartheta_i)$ and $\tilde{x}_i = (e_i, t_i; \varsigma_i, \kappa_i)$, $i = 1, 2, \cdots, n$. Thus, the lower possibilistic variance and the upper possibilistic variance of the portfolio are (48) and (49).

Proof: Firstly, the lower possibilistic variance of n assets is (50). Similarly, the upper possibilistic variance of n assets is (51).

□

Similarly, the lower possibilistic variance of n assets is divided into (52) and (53), the upper possibilistic variance of n assets can be divided into (54) and (55).

Therefore, the standard deviations of the portfolio can be expressed as (56), and (57) is the standard deviations of the incomplete information.

From above, we can see that when $f_i = h_i$ and $e_i = t_i$, that is, the trapezoidal fuzzy returns and proportions degenerate to triangular fuzzy numbers, the resulting conclusions are the corresponding ones in subsection A. Therefore, subsection A is a special case of subsection B.

$$\begin{aligned} M_*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i) &= M_*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2 + \cdots + \tilde{r}_n\tilde{x}_n) \\ &= M_*(\tilde{r}_1\tilde{x}_1) + M_*(\tilde{r}_2\tilde{x}_2) + \cdots + M_*(\tilde{r}_n\tilde{x}_n) \\ &= \frac{1}{6}\varpi_1\varsigma_1 - \frac{1}{3}(f_1\varsigma_1 + e_1\varpi_1) + f_1e_1 + \frac{1}{6}\varpi_2\varsigma_2 - \frac{1}{3}(f_2\varsigma_2 + e_2\varpi_2) + f_2e_2 + \cdots + \frac{1}{6}\varpi_n\varsigma_n - \frac{1}{3}(f_n\varsigma_n + e_n\varpi_n) + f_ne_n \\ &= \frac{1}{6}\sum_{i=1}^{n}\varpi_i\varsigma_i - \frac{1}{3}\sum_{i=1}^{n}(f_i\varsigma_i + e_i\varpi_i) + \sum_{i=1}^{n} f_ie_i. \end{aligned} \tag{45}$$

$$\begin{aligned} M^*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i) &= M^*(\tilde{r}_1\tilde{x}_1 + \tilde{r}_2\tilde{x}_2 + \cdots + \tilde{r}_n\tilde{x}_n) \\ &= M^*(\tilde{r}_1\tilde{x}_1) + M^*(\tilde{r}_2\tilde{x}_2) + \cdots + M^*(\tilde{r}_n\tilde{x}_n) \\ &= \frac{1}{6}\vartheta_1\kappa_1 + \frac{1}{3}(h_1\kappa_1 + t_1\vartheta_1) + h_1t_1 + \frac{1}{6}\vartheta_2\kappa_2 + \frac{1}{3}(h_2\kappa_2 + t_2\vartheta_2) + h_2t_2 + \cdots + \frac{1}{6}\vartheta_n\kappa_n + \frac{1}{3}(h_n\kappa_n + t_n\vartheta_n) + h_nt_n \\ &= \frac{1}{6}\sum_{i=1}^{n}\vartheta_i\kappa_i + \frac{1}{3}\sum_{i=1}^{n}(h_i\kappa_i + t_i\vartheta_i) + \sum_{i=1}^{n}h_it_i. \end{aligned} \tag{46}$$

$$M(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i) = \frac{M^*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i) + M_*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)}{2} = \frac{\frac{1}{6}\sum_{i=1}^{n}(\vartheta_i\kappa_i + \varpi_i\varsigma_i) + \frac{1}{3}\sum_{i=1}^{n}(h_i\kappa_i + t_i\vartheta_i - f_i\varsigma_i - e_i\varpi_i) + \sum_{i=1}^{n} f_ie_i + \sum_{i=1}^{n}h_it_i}{2}. \tag{47}$$

$$Var_*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i) = \frac{1}{18}[\sum_{i=1}^{n}(f_i\varsigma_i + e_i\varpi_i) - \frac{4}{5}\sum_{i=1}^{n}(\varpi_i\varsigma_i)]^2 + \frac{1}{300}(\sum_{i=1}^{n}\varpi_i\varsigma_i)^2. \tag{48}$$

$$Var^*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i) = \frac{1}{18}[\sum_{i=1}^{n}(h_i\kappa_i + t_i\vartheta_i) + \frac{4}{5}(\sum_{i=1}^{n}\vartheta_i\kappa_i)]^2 + \frac{1}{300}(\sum_{i=1}^{n}\vartheta_i\kappa_i)^2. \tag{49}$$

$$\begin{aligned} Var_*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i) &= 2\int_0^1\gamma[M_*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i) - (\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)_*^{\gamma}]^2 d\gamma \\ &= 2\int_0^1\gamma[\frac{1}{6}\sum_{i=1}^{n}\varpi_i\varsigma_i - \frac{1}{3}\sum_{i=1}^{n}(f_i\varsigma_i + e_i\varpi_i) + \sum_{i=1}^{n} f_ie_i - \gamma^2\sum_{i=1}^{n}\varpi_i\varsigma_i - \gamma\sum_{i=1}^{n}(f_i\varsigma_i + e_i\varpi_i - 2\varpi_i\varsigma_i) \\ &\quad - \sum_{i=1}^{n}(f_ie_i - f_i\varsigma_i - e_i\varpi_i + \varpi_i\varsigma_i)]^2 d\gamma \\ &= 2\int_0^1\gamma[(-\gamma^2 + 2\gamma - \frac{5}{6})(\sum_{i=1}^{n}\varpi_i\varsigma_i) + (\frac{2}{3} - \gamma)\sum_{i=1}^{n}(f_i\varsigma_i + e_i\varpi_i)]^2 d\gamma \\ &= \frac{1}{18}[\sum_{i=1}^{n}(f_i\varsigma_i + e_i\varpi_i) - \frac{4}{5}(\sum_{i=1}^{n}\varpi_i\varsigma_i)]^2 + \frac{1}{300}(\sum_{i=1}^{n}\varpi_i\varsigma_i)^2. \end{aligned} \tag{50}$$

$$Var^*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)=2\int_0^1\gamma[M^*(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)-(\sum_{i=1}^{n}\tilde{r}_i\tilde{x}_i)^{*\gamma}]^2d\gamma$$

$$=2\int_0^1\gamma[\frac{1}{6}\sum_{i=1}^{n}\vartheta_i\kappa_i+\frac{1}{3}\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i)+\sum_{i=1}^{n}h_it_i-\gamma^2\sum_{i=1}^{n}\vartheta_i\kappa_i+\gamma\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i+2\vartheta_i\kappa_i)$$

$$-\sum_{i=1}^{n}(h_it_i+h_i\kappa_i+t_i\vartheta_i+\vartheta_i\kappa_i)]^2d\gamma \tag{51}$$

$$=2\int_0^1\gamma[(-\gamma^2+2\gamma-\frac{5}{6})(\sum_{i=1}^{n}\vartheta_i\kappa_i)+(\gamma-\frac{2}{3})\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i)]^2d\gamma$$

$$=\frac{1}{18}[\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i)+\frac{4}{5}(\sum_{i=1}^{n}\vartheta_i\kappa_i)]^2+\frac{1}{300}(\sum_{i=1}^{n}\vartheta_i\kappa_i)^2.$$

$$\sigma_{1*}{}^2=\frac{1}{18}[\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)-\frac{4}{5}(\sum_{i=1}^{n}\varpi_i\varsigma_i)]^2. \tag{52}$$

$$\sigma_{2*}{}^2=\frac{1}{300}(\sum_{i=1}^{n}\varpi_i\varsigma_i)^2. \tag{53}$$

$$\sigma_1^{*2}=\frac{1}{18}[\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i)+\frac{4}{5}(\sum_{i=1}^{n}\vartheta_i\kappa_i)]^2. \tag{54}$$

$$\sigma_2^{*2}=\frac{1}{300}(\sum_{i=1}^{n}\vartheta_i\kappa_i)^2. \tag{55}$$

$$\sigma_1=\frac{\sqrt{\sigma_1^{*2}}+\sqrt{\sigma_{1*}{}^2}}{2}=\frac{\frac{1}{3\sqrt{2}}[\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i)+\frac{4}{5}(\sum_{i=1}^{n}\vartheta_i\kappa_i)]+\frac{1}{3\sqrt{2}}[\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)-\frac{4}{5}(\sum_{i=1}^{n}\varpi_i\varsigma_i)]}{2}. \tag{56}$$

$$\sigma_2=\frac{\sqrt{\sigma_2^{*2}}+\sqrt{\sigma_{2*}{}^2}}{2}=\frac{\frac{1}{10\sqrt{3}}(\sum_{i=1}^{n}\vartheta_i\kappa_i)+\frac{1}{10\sqrt{3}}(\sum_{i=1}^{n}\varpi_i\varsigma_i)}{2}. \tag{57}$$

IV. Fuzzy Portfolio Model with Fuzzy Investment Proportion

In the process of the investment, we not only hope for greater returns and lower risks of the assets, but also need to consider their liquidity. Generally, a low turnover rate indicates poor liquidity, and vice versa. In addition, the liquidity of assets is uncertain and influenced by the subjective will of investors. Therefore, let the liquidity of the asset i be a trapezoidal fuzzy number $\tilde{l}_i=(lf_i,lh_i;l\varpi_i,l\vartheta_i)$, where $[lf_i,lh_i]$, $l\varpi_i$ and $l\vartheta_i$ are the central value, left and right spread values of $\tilde{l}_i$, $i=1,2,\cdots,n$. So, the γ-level of $\tilde{l}_i$ is

$$[\tilde{l}_i]^\gamma=[lf_i-l\varpi_i(1-\gamma),lh_i+l\vartheta_i(1-\gamma)]. \tag{58}$$

According to (47), (59) is the possibilistic mean value of the fuzzy turnover rate. Here, we consider the return, investment proportion and turnover rate as fuzzy trapezoidal numbers. Therefore, we construct the following fuzzy multi-objective model (60).

The first objective function represents the maximum expected return; the latter two objective functions represent the minimum standard variance of the portfolio and the incomplete information, respectively. The first constraint assures that the turnover rate is no less than the given value $M(l_o)$; the second constraint requires that the sum of the fuzzy investment proportion is essentially less than or equal to 1; the last constraint implies that u_i denotes the upper level of the left and right spread values of the investment proportion, that is, ς_i and κ_i are not lower than 0 and not higher than the value of u_i.

For the proposed fuzzy multi-objective model (60), we transform the two minimized objective functions into constraints, thereby obtaining the simplified single objective model (61). m and n denote the maximum levels of risks of the portfolio and the incomplete information that investors can accept, respectively.

$$M(l)=M(\sum_{i=1}^{n}\tilde{l}_i\tilde{x}_i)=\frac{\frac{1}{6}\sum_{i=1}^{n}(l\vartheta_i\kappa_i+l\varpi_i\varsigma_i)+\frac{1}{3}\sum_{i=1}^{n}(lh_i\kappa_i+t_il\vartheta_i-lf_i\varsigma_i-e_il\varpi_i)+\sum_{i=1}^{n}lf_ie_i+\sum_{i=1}^{n}lh_it_i}{2}. \tag{59}$$

$$\begin{cases} \max & \dfrac{\frac{1}{6}\sum_{i=1}^{n}(\vartheta_i\kappa_i+\varpi_i\varsigma_i)+\frac{1}{3}\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i-f_i\varsigma_i-e_i\varpi_i)+\sum_{i=1}^{n}f_ie_i+\sum_{i=1}^{n}h_it_i}{2} \\ \min & \dfrac{\frac{1}{3\sqrt{2}}[\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i)+\frac{4}{5}(\sum_{i=1}^{n}\vartheta_i\kappa_i)]+\frac{1}{3\sqrt{2}}[\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)-\frac{4}{5}(\sum_{i=1}^{n}\varpi_i\varsigma_i)]}{2} \\ \min & \dfrac{\frac{1}{10\sqrt{3}}(\sum_{i=1}^{n}\vartheta_i\kappa_i)+\frac{1}{10\sqrt{3}}(\sum_{i=1}^{n}\varpi_i\varsigma_i)}{2} \\ \text{s.t.} & \dfrac{\frac{1}{6}\sum_{i=1}^{n}(l\vartheta_i\kappa_i+l\varpi_i\varsigma_i)+\frac{1}{3}\sum_{i=1}^{n}(lh_i\kappa_i+t_il\vartheta_i-lf_i\varsigma_i-e_il\varpi_i)+\sum_{i=1}^{n}lf_ie_i+\sum_{i=1}^{n}lh_it_i}{2}\ge M(l_0) \\ & \sum_{i=1}^{n}\tilde{x}_i \underset{\sim}{\le} 1 \\ & 0\le\varsigma_i,\kappa_i\le u_i \end{cases} \tag{60}$$

$$\begin{cases} \max & \frac{1}{6}\sum_{i=1}^{n}(\vartheta_i\kappa_i+\varpi_i\varsigma_i)+\frac{1}{3}\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i-f_i\varsigma_i-e_i\varpi_i)+\sum_{i=1}^{n}f_ie_i+\sum_{i=1}^{n}h_it_i \\ \text{s.t.} & \frac{1}{3\sqrt{2}}[\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i)+\frac{4}{5}(\sum_{i=1}^{n}\vartheta_i\kappa_i)]+\frac{1}{3\sqrt{2}}[\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)-\frac{4}{5}(\sum_{i=1}^{n}\varpi_i\varsigma_i)]\le m \\ & \sum_{i=1}^{n}\vartheta_i\kappa_i+\sum_{i=1}^{n}\varpi_i\varsigma_i\le n \\ & \frac{1}{6}\sum_{i=1}^{n}(l\vartheta_i\kappa_i+l\varpi_i\varsigma_i)+\frac{1}{3}\sum_{i=1}^{n}(lh_i\kappa_i+t_il\vartheta_i-lf_i\varsigma_i-e_il\varpi_i)+\sum_{i=1}^{n}lf_ie_i+\sum_{i=1}^{n}lh_it_i\ge M(l_0) \\ & \sum_{i=1}^{n}\tilde{x}_i \underset{\sim}{\le} 1 \\ & 0\le\varsigma_i,\kappa_i\le u_i \end{cases} \tag{61}$$

$$\begin{cases} \max & \lambda \\ \text{s.t.} & \frac{1}{6}\sum_{i=1}^{n}(\vartheta_i\kappa_i+\varpi_i\varsigma_i)+\frac{1}{3}\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i-f_i\varsigma_i-e_i\varpi_i)+\sum_{i=1}^{n}f_ie_i+\sum_{i=1}^{n}h_it_i\ge\mu-(1-\lambda)w \\ & \frac{1}{3\sqrt{2}}[\sum_{i=1}^{n}(h_i\kappa_i+t_i\vartheta_i)+\frac{4}{5}(\sum_{i=1}^{n}\vartheta_i\kappa_i)]+\frac{1}{3\sqrt{2}}[\sum_{i=1}^{n}(f_i\varsigma_i+e_i\varpi_i)-\frac{4}{5}(\sum_{i=1}^{n}\varpi_i\varsigma_i)]\le m \\ & \sum_{i=1}^{n}\vartheta_i\kappa_i+\sum_{i=1}^{n}\varpi_i\varsigma_i\le n \\ & \frac{1}{6}\sum_{i=1}^{n}(l\vartheta_i\kappa_i+l\varpi_i\varsigma_i)+\frac{1}{3}\sum_{i=1}^{n}(lh_i\kappa_i+t_il\vartheta_i-lf_i\varsigma_i-e_il\varpi_i)+\sum_{i=1}^{n}lf_ie_i+\sum_{i=1}^{n}lh_it_i\ge M(l_0) \\ & \sum_{i=1}^{n}t_i\le 1+(1-\lambda)v \\ & 0\le\varsigma_i,\kappa_i\le u_i \\ & 0\le\lambda\le 1 \end{cases} \tag{62}$$

According to the fuzzy linear programming, (61) is finally transformed into (62), where λ represents the satisfactory level of investors; μ represents the desired return value of investors; w and v represent the tolerance values. w represents the degree to which the expected return is allowed to be less than the desired value μ, v represents the maximum extent that the sum of central values t_i exceeds the desired value 1.

V. Numerical Examples

Five assets are chosen from Shanghai Exchange Market to illustrate the effectiveness of the model. Combined with the historical data and expert advice, the possibility distributions and fuzzy turnover rates of these assets are given in Table I and Table II.

Let $w=0.05$, $v=0.5$, $m=0.25$, $n=0.1$, $M(l_o)=0.02$ and $u_i=0.1$ be fixed when μ changes. The optimal results of (62) are shown in Table III.

In Table III, we can see that when the values of w, v, m, n, $M(l_0)$ and u_i are fixed and the desired value of investors takes different values from 0.20 to 0.40, the fuzzy investment proportion $\tilde{x}_i=(e_i,t_i;\varsigma_i,\kappa_i)$ will also be different. For example, when $\mu=0.20$, the investment proportions of all assets are $\tilde{x}_1=(0.0043,\ 0.1258;\ 0.0756,\ 0.0789)$, $\tilde{x}_2=(0.0304,\ 0.1229;\ 0.0817,\ 0.0660)$, $\tilde{x}_3=(0.0347,\ 0.0810;\ 0.0728,\ 0.0120)$, $\tilde{x}_4=(0.0520,\ 0.1142;\ 0.0866,\ 0.0000)$, $\tilde{x}_5=(0.0260,\ 0.1504;\ 0.0803,\ 0.0467)$. The sum of these fuzzy investment proportions for five assets is $\sum_{i=1}^{n} x_i=(0.1475,\ 0.592;\ 0.3900,\ 0.2036)$. When $\mu=0.40$, the investment proportions of all assets are $\tilde{x}_1=(0.0260,\ 0.1224;\ 0.0643,\ 0.0637)$, $\tilde{x}_2=(0.0500,\ 0.1881;\ 0.0715,\ 0.0686)$, $\tilde{x}_3=(0.0740,\ 0.1860;\ 0.0565,\ 0.0244)$, $\tilde{x}_4=(0.0142,\ 0.2418;\ 0.0949,\ 0.0094)$, $\tilde{x}_5=(0.0074,\ 0.1652;\ 0.0844,\ 0.0288)$. The sum of these fuzzy investment proportions for five assets is $\sum_{i=1}^{n} x_i=(0.0216,\ 0.9035;\ 0.3716,\ 0.1949)$.

When the desired value μ of investors varies from 0.20 to 0.40, that is, investors want higher returns, the fuzzy investment proportions of Asset 1 and Asset 5 decrease first and then increase, the fuzzy investment proportions of the other assets continue to increase. When μ varies from 0.20 to 0.26, the investment proportion of asset 5 is greater than the other assets. When μ varies from 0.30 to 0.40, the investment proportion of asset 3 is greater than the other assets.

Next, we will discuss how the investment proportion changes when the desired value μ and tolerance value v change. Let $w=0.02$, $m=0.25$, $n=0.1$, $M(l_o)=0.02$ and $u_i=0.1$, the optimal results of (62) are shown in Table IV.

When $\mu=0.40$ and $v=0.95$, the investment proportion of each asset can be obtained as $\tilde{x}_1=(0.0257,\ 0.1214;\ 0.0646,\ 0.0641)$, $\tilde{x}_2=(0.0496,\ 0.1865;\ 0.0718,\ 0.0690)$, $\tilde{x}_3=(0.0734,\ 0.1845;\ 0.0569,\ 0.0252)$, $\tilde{x}_4=(0.0141,\ 0.2398;\ 0.0949,\ 0.0104)$, $\tilde{x}_5=(0.0073,\ 0.1638;\ 0.0846,\ 0.0296)$.

In Table IV, we can see that when μ varies from 0.26 to 0.40 and v varies from 0.80 to 0.95, the fuzzy investment proportion $\tilde{x}_i=(e_i,t_i;\varsigma_i,\kappa_i)$ will also change. When μ and v increase at the same time, that is, investors want higher returns and are more tolerant of central values, the fuzzy investment proportions of Asset 1 and Asset 5 decrease first and then increase, the fuzzy investment proportion of the other assets continue to increase.

Compared with Table III, the investment proportion of Asset 1 remains unchanged, while the investment proportion of the other four assets are relatively reduced. These changes between Table III and Table IV are more pronounced when μ and v increase at the same time.

TABLE I
THE POSSIBILITY DISTRIBUTION OF FIVE ASSETS

Asset	f_i	h_i	ϖ_i	ϑ_i
1	0.0550	0.1986	0.1217	0.3040
2	0.1060	0.2987	0.0530	0.3525
3	0.1568	0.2500	0.0845	0.5171
4	0.0301	0.3051	0.0017	0.6233
5	0.0157	0.2127	0.0590	0.4708

TABLE II
THE FUZZY TURNOVER RATE OF FIVE ASSETS

Asset	lf_i	lh_i	$l\varpi_i$	$l\vartheta_i$
1	0.001	0.025	0.008	0.020
2	0.007	0.023	0.006	0.022
3	0.008	0.015	0.006	0.017
4	0.012	0.020	0.006	0.025
5	0.006	0.028	0.004	0.024

TABLE III
THE OPTIMAL INVESTMENT PROPORTION WITH DIFFERENT μ

Desired value	e_i	t_i	ς_i	κ_i
$\mu=0.20$	0.0043	0.1258	0.0756	0.0789
	0.0304	0.1229	0.0817	0.0660
	0.0347	0.0810	0.0728	0.0120
	0.0520	0.1142	0.0866	0.0000
	0.0260	0.1504	0.0803	0.0467
$\mu=0.26$	0.0047	0.1248	0.0755	0.0784
	0.0305	0.1233	0.0816	0.0659
	0.0352	0.0825	0.0726	0.0123
	0.0509	0.1159	0.0869	0.0002
	0.0255	0.1495	0.0805	0.0463
$\mu=0.30$	0.0190	0.0897	0.0692	0.0619
	0.0367	0.1379	0.0771	0.0637
	0.0543	0.1364	0.0649	0.0250
	0.0104	0.1773	0.0962	0.0101
	0.0054	0.1211	0.0863	0.0300
$\mu=0.35$	0.0225	0.1060	0.0668	0.0628
	0.0433	0.1629	0.0743	0.0661
	0.0641	0.1612	0.0608	0.0247
	0.0123	0.2094	0.0955	0.0097
	0.0064	0.1431	0.0854	0.0294
$\mu=0.40$	0.0260	0.1224	0.0643	0.0637
	0.0500	0.1881	0.0715	0.0686
	0.0740	0.1860	0.0565	0.0244
	0.0142	0.2418	0.0949	0.0094
	0.0074	0.1652	0.0844	0.0288

TABLE IV
THE OPTIMAL INVESTMENT PROPORTION WITH DIFFERENT μ AND v

Parameter	e_i	t_i	ς_i	κ_i
$\mu=0.26$, $v=0.80$	0.0047	0.1247	0.0755	0.0784
	0.0305	0.1232	0.0816	0.0659
	0.0352	0.0825	0.0726	0.0124
	0.0509	0.1159	0.0869	0.0003
	0.0254	0.1495	0.0805	0.0463
$\mu=0.30$, $v=0.85$	0.0189	0.0893	0.0694	0.0622
	0.0365	0.1371	0.0773	0.0640
	0.0540	0.1356	0.0652	0.0256
	0.0104	0.1763	0.0962	0.0108
	0.0054	0.1204	0.0864	0.0306
$\mu=0.35$, $v=0.90$	0.0223	0.1053	0.0670	0.0631
	0.0430	0.1618	0.0745	0.0665
	0.0637	0.1601	0.0611	0.0254
	0.0122	0.2080	0.0956	0.0106
	0.0064	0.1421	0.0855	0.0301
$\mu=0.40$, $v=0.95$	0.0257	0.1214	0.0646	0.0641
	0.0496	0.1865	0.0718	0.0690
	0.0734	0.1845	0.0569	0.0252
	0.0141	0.2398	0.0949	0.0104
	0.0073	0.1638	0.0846	0.0296

VI. CONCLUSION

We regard the investment proportion as a triangular fuzzy number or a trapezoidal fuzzy number and consider the portfolio with the uncertain investment proportion in this paper. The results show that the conclusion of [28] is a special case of subsection A, and the conclusion of subsection A is a special case of subsection B. Then, we construct a portfolio model with return, risk, liquidity and fuzzy constraints, and simplify it by the fuzzy linear programming. Finally, we not only use numerical examples to solve the model, but also discuss the influence and the sensitivity analysis of different parameters.

In the future work, we will not only study models which will consider other constraints of real markets, but also include multi-period portfolio models. This will provide more pertinent investment advice to the majority of investors.

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