An Optimized Two-Step Block Hybrid Method with Symmetric Intra-Step Points for Second Order Initial Value Problems

Madubuchichi Gabriel Orakwelu, Sicelo Goqo and Sandile Motsa

Abstract—We derive a two-step block hybrid method for solving general second-order initial value problems by optimizing the local truncation errors. We test the optimized block method’s efficiency on both scalar and system initial value problems of the linear and non-linear type, and the results obtained compared with similar schemes.

Index Terms—Interpolation, collocation, hybrid methods.

I. INTRODUCTION

Researchers proposed the hybrid linear multistep methods to overcome the Dahlquist Barrier theorem by imposing intrastep points during formulation. These modified linear multistep schemes find applications in robotics, electric circuits, vibrating strings etc. This paper focuses on implicit block hybrid methods, specifically derived for second-order initial value problems (IVP) of the form.

\[ y''(x) = f(x, y, y'), \quad y(a) = \eta_0, \quad y'(a) = \eta_1. \]  

(1)

Block hybrid methods for the direct solution of IVPs have been of interest to researchers within the past decade (see Anake et al. [3], Jator and Adeyefa [10], Adeyeye and Omar [2]). Biala et al. [6] formulated a Simpson’s type, block hybrid method with two intra-step stiff systems points. Ramos and Singh [16] presented an A-stable two-step block hybrid method for first-order IVPs. Jator and Agyingi [11] derived a backward differentiation scheme specifically for large stiff problems. The two off-grid points imposed are zeros of the second degree Chebyshev polynomial of the first kind.


This study presents an optimized two-step block hybrid method (OTSBHM) with four distinct symmetric off-step points derived by optimizing the LTE for solving (1). For comparisons, we formulate specific Block Two-Step Hybrid Methods (BTSHM) based on Equispaced Points (BTSHM-EP) and Bhaskara Points (BTSHM-BP). We conduct numerical experiments to test the proposed methods’ accuracy and compare with the similar schemes derived from Table I’s points.

II. FORMULATION, SPECIFICATION AND ANALYSIS OF THE METHOD

A. Formulation of the OTSBHM for second order IVP’s

The IVP is solved over an interval \( x \in [a, b] \), where \( a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b \). The step length is given as \( h = x_{n+1} - x_n \) for \( n = 0, 1, \ldots, N \). The continuous method is based on approximating the exact solution \( y(x) \) at grid points \( x_i \) by a polynomial of the form

\[ y(x) \approx Y(x) = \sum_{i=0}^{n_x+n_y-1} c_i x^i \]  

(2)

where \( c_i \) are unknown coefficients, \( n_x \) is the number of interpolation points and \( n_y \) is the number of collocation points. The formulation of the continuous method for second order IVPs starts with the introduction of the basis function given by (2). For our two-step block hybrid method, we interpolate at \( x_n \) and \( x_{n+2} \). Through numerical experimentation, it was determined that optimal hybrid method can only be found through interpolating at these points. The collocation is imposed at \( x_n \) and \( x_{n+2} \) plus prescribed intra-step points defined as \( x_{p_\nu} = x_n + \nu h_{p_\nu} \). Here we assume that \( 0 < p_\nu < 2, \nu = 1, 2, \ldots, m \), where \( m \) is the number of intra-step points. Consequently, a system of \( n_x + n_y + 1 \) equations with \( n_x + n_y + 1 \) unknowns is obtained from

\[ Y(x_{n+i}) = y_{n+i}, \quad i = 0, 2, \]  

(3)

\[ Y''(x_{n+i}) = f_{n+i}, \quad i = 0, 1, 2, \]  

(4)
In addition, every intra-step point \( \tilde{p} \) is associated with another point \( \tilde{p} = 2 - p \). The assumption is based on the symmetric nature of collocation points. For the two-step methods considered in this section, the symmetry is about \( x_{n+1} \).

The symbolic representation of the intra-step points and other similar points is provided in Table I. Solving (3)-(5) for the unknowns \( c_i, i = 0, 1, \ldots, n_i + n_o \), and on substituting in (2) gives a continuous approximation of the form

\[
Y''(x_{n+p}) = f_{n+p}, \quad \nu = 1, 2, \ldots, m. \tag{5}
\]

Since (1) contains the first derivative, the first derivative is given as,

\[
Y'(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+2} + h^2 \sum_{j=0}^{2} \beta_j(x)f_{n+j}
\]

\[
+ h^2 \sum_{\nu} \beta_{p_{\nu}}(x) f_n + p_{\nu}. \tag{6}
\]

where the following conditions are imposed on the starter equation (7)

\[
Y'(x) = \delta(x), \quad Y'(a) = \delta_0. \tag{8}
\]

**B. Specification of the OTSBHM for second order IVP’s**

We derive the OTSBHM by evaluating (6) and (7) at the main and intra-step points to yield methods (9)–(20)
\[
\begin{align*}
- & \frac{h^2(p - 2)^3}{1680(q - 2)q} f_n(13p^3 - 22p^2 - 6p(7q^2 - 14q + 2) + 14q^2 - 28q + 8) + h^2p^3 \quad f_n+1(-13p^3 + 56p^2 + 14p(3q^2 - 6q - 4) - 70(q - 2)q) \\
+ & \frac{h^2p(p - 2)}{1680(p - 1)^2(p - q)(p + q - 2)} (5p^4 - 20p^3 + p^2(-14q^2 + 28q + 12) + 4p(7q^2 - 14q + 4) + 4(7q^2 - 14q + 4)) f_n+p-2 \\
+ & \frac{h^2p(p - 2)}{840(p - 1)^2(q - 1)^2} f_{n+1}(13p^6 - 78p^5 + p^3(-42q^2 + 8q + 124) + 24p^3(7q^2 - 14q + 1) + 8p^2(7q^2 - 14q + 6) - 32p(14q^2 - 28q + 11) + 32(7q^2 - 14q + 6)) \\
+ & \frac{h^2}{1680(p - 1)^2(p - q)(p + q - 2)} (35p^6 - 210p^5 - 70p^4(q^2 - 2q - 6) + 280p^3(q^2 - 2q - 1) - 280p^2(q^2 + 2q - 2)q + 16(7q^2 - 14q + 4)) - 280p^2(q^2 - 2q - 16) + 16(7q^2 - 14q + 4) \\
+ & 4) f_{n+1} + \frac{h^2}{1680(q - 2)(q - 1)^2(q - 1)^2} f_{n+1}(13p^6 - 78p^5 + p^3(-42q^2 + 8q + 124) + 24p^3(7q^2 - 14q + 1) + 8p^2(7q^2 - 14q + 6) - 32p(14q^2 - 28q + 11) + 32(7q^2 - 14q + 6)) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8) \\
+ & \frac{h^2}{1680(p - 2)p} f_n(14p^2(3q - 1) + p(28 - 84q) - 13q^3 + 22q^2 - 14q - 8)
\end{align*}
\]
\[
\frac{h^2}{840(p - 2)p(q - 2)q} f_{n+2} (7p^2 (10q^2 - 20q + 7) - 14p (10q^2 - 20q + 7) + 49q^2 - 98q + 38)
\]
\[
\frac{h^2}{840(p - 2)p(q - 2)q} f_{n} (7p^2 (10q^2 - 20q + 7) - 14p (10q^2 - 20q + 7) + 49q^2 - 98q + 38)
\]
\[
+ \frac{h^2 (49q^2 - 98q + 38) f_{n+q+2}}{840(q - 2)(q - 1)(p - q)(p + q - 2)}
\]
\[
\frac{h^2 (49q^2 - 98q + 38) f_{n+q}}{840(q - 2)(q - 1)(p - q)(p + q - 2)}
\]
\[
+ \frac{h^2 (49q^2 - 98q + 38) f_{n+p}}{840(p - 2)(p - 1)(p - q)(p + q - 2)}
\]
\[
+ \frac{h^2 (49q^2 - 98q + 38) f_{n-p+2}}{840(p - 2)(p - 1)(p - q)(p + q - 2)} + h \delta_{n+1}
\]
\[
+ \frac{1}{2} (y_n - y_{n+2}) = 0.
\]

(15)

\[
\frac{2h^2 (p - 1)}{105(q - 1)^2} f_{n+1} (3p^4 - 12p^3 + p^2 (-7q^2 + 14q + 4))
\]
\[
+ 2p (7q^2 - 14q + 8) + 4(7q^2 - 14q + 6))
\]
\[
\frac{h^2 p^2}{840(p - 2)(q - 2)q} f_{n} (24p^4 - 133p^3 + 14p^2 (4q^2 - 8q - 17) + 35p (5q^2 - 10q - 4) - 140(q - 2)q)
\]
\[
- h^2 (2p - 2)^2
\]
\[
\frac{840(p - 2)(p - 1)p(q - 2)q}{40(q - 2)q} f_{n+2} (24p^4 - 59p^3 - 8p^2 (7q^2 - 14q - 2) + p (49q^2 - 98q + 16) - 14q^2 + 28q - 8)
\]
\[
- \frac{h^2}{40(p - 2)(p - 1)p(q - 2)q} f_{n+p} (105p^6 - 780p^5
\]
\[
+ p^4 (-189q^2 + 378q + 1572) + 4p^3 (189q^2 - 378q - 272)
\]
\[
- 16p^2 (49q^2 - 98q - 2) + 8p (7q^2 - 14q + 4)) + 16 (7q^2 - 14q + 4)) f_{n-p+2}
\]
\[
+ \frac{h^2}{840(p - 2)(q - 2)q} f_{n-2} (24p^4 - 59p^3 - 8p^2 (7q^2 - 14q - 2) + p (49q^2 - 98q + 16) - 14q^2 + 28q - 8)
\]
\[
- \frac{h^2}{840(p - 2)(p - 1)p(q - 2)q} f_{n+2} (24p^4 - 59p^3 - 8p^2 (7q^2 - 14q - 2) + p (49q^2 - 98q + 16) - 14q^2 + 28q - 8)
\]

(16)
We optimize the supplementary method (20) with LTE
\[
\begin{align*}
\frac{h^9}{793600} & \left(-3p_1^2(7p_2^2 - 14p_2 + 4) + 6p_1(7p_2^2 - 14p_2 + 4) \\
& - 4(3p_2^2 - 6p_2 + 2))y^{(9)}(x_n) \right) \\
+ \frac{h^{10}}{907200} & \left(-3p_1^2(7p_2^2 - 14q + 4) + 6p(7q^2 - 14q + 4) \\
& - 4(3q^2 - 6q + 2))y^{(10)}(x_n) \right) \\
+ \frac{h^{11}}{2099563200} & \left(-11p_1^2(7q^2 - 14q + 4) + 44p^3(7q^2 - 14q + 4) \\
& - 11p_1^2(7q^2 - 28q^3 + 277q^4 - 498q + 140) \\
& + 22p(7q^4 - 28q^5 + 249q^6 - 442q + 124) - 4(11q^4 - 44q^5) \\
& + 385q^2 - 682q + 222) \right) y^{(11)}(x_n) + \cdots .
\end{align*}
\]

The first and second term in LTE (21) have the same coefficient given as
\[
(-3p_1^2(4 - 14p_2 + 7p_2^2) + 6p_1(4 - 14p_2 + 7p_2^2) - 4(2 - 6p_2 + 3p_2^2)).
\]

The third coefficient in the LTE (21) is also
\[
-11p_1^2(7p_2^2 - 14p_2 + 4) + 44p^3(7p_2^2 - 14p_2 + 4) - 11p_1^2(7p_2^2 - 28p_2^3 + 403p_2 - 750p_2 + 212) + 22p_1(7p_2^2 - 28p_2^3 + 375p_2 - 694p_2 + 196) - 4(11p_2^4 - 44p_2^5 + 583p_2^2 - 1078p_2 + 354).
\]

We equate the coefficients (22) and (23) to zero and simultaneously solve to obtain values of \(p_1\) and \(p_2\) given as
\[
p_1 = 1 - \sqrt{\frac{1}{33} \left(15 - 2\sqrt{15}\right)},
\]
\[
p_2 = 1 - \sqrt{\frac{1}{33} \left(15 + 2\sqrt{15}\right)}.
\]

The optimized LTE is obtained by substituting the values \(p_1\) and \(p_2\) in (21), and given as
\[
\frac{h^{14}y^{(14)}(x_n)}{927102069450} + \cdots .
\]

C. Analysis of the OTSBCM for second order IVP’s

We can write the formula for the block method for the second-order IVP equation in matrix-vector form as
\[
AY_n = hBF_n + \Delta_n + h^2CF_n,
\]
where \(A\), \(B\) and \(C\) are coefficient matrices and the vectors \(Y_n\), \(\Delta_n\) and \(F_n\) are defined as
\[
Y_n = (y_n, y_{n+p_1}, y_{n+p_2}, \ldots, y_{n+1}, \ldots, y_{n+2-p_2}, y_{n+2-p_1}, y_{n+2})^T,
\]
\[
\Delta_n = (\delta_n, \delta_{n+p_1}, \delta_{n+p_2}, \ldots, \delta_{n+1}, \ldots, \delta_{n+2-p_2}, \delta_{n+2-p_1})^T,
\]
\[
F_n = (f_n, f_{n+p_1}, f_{n+p_2}, \ldots, f_{n+1}, \ldots, f_{n+2-p_1}, f_{n+2-p_2})^T.
\]
1) Local truncation errors and order of the OTSBHM: To analyze the truncation errors, we define the linear operator \( \mathcal{L} \) as

\[
\mathcal{L}[z(x_n); h] = \sum_{\nu} [\hat{\alpha}_\nu z(x_n + \nu h) - h^2 \hat{\beta}_\nu z''(x_n + \nu h)]
\]

where \( \hat{\alpha}_\nu, \hat{\beta}_\nu, \) and \( \hat{\gamma}_\nu \) are columns of the matrices \( A, B \) and \( C \). The orders and error constants of the methods can be obtained through Taylor series about \( x_n \). The \( \text{LTE}'s \) of the OTSBHM are given as

\[
A^{(1)} = \begin{bmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & p_{m-1} \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 - p_1 \\
0 & \cdots & 0 & -1
\end{bmatrix}
\]

and

\[
\bar{A}^{(0)} = \begin{bmatrix}
0 & \cdots & 0 & -2 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & -p_1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & -p_2 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -(2 - p_m) & 1 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & -2 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & -p_1 & 0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

The first characteristic polynomial for the OTSBHM is \( \rho(R) = R^{11} + (1 + R) \). Accordingly, the OTSBHM is zero stable, consistent and convergent.

**Problem 1**

The first example is an IVP of Bessel type given as

\[
x^2 y'' + xy' + (x^2 - 0.25)y = 0,
\]

\[
y(1) = \sin \sqrt{2}/\pi, \quad y'(1) = (2 \cos 1 - \sin 1)/\sqrt{2}\pi,
\]

\[
\text{Exact: } y(x) = \sin x \sqrt{2}\pi
\]

This problem is solved using the three similar block hybrid methods with different steps \( N = 67, 82, 97, 112 \) and 125.

<table>
<thead>
<tr>
<th>( N )</th>
<th>BTSHA-EP</th>
<th>BTSHA-BP</th>
<th>OTSBHM</th>
</tr>
</thead>
<tbody>
<tr>
<td>67</td>
<td>2.7897 \times 10^{-12}</td>
<td>1.4856 \times 10^{-13}</td>
<td>6.6160 \times 10^{-19}</td>
</tr>
<tr>
<td>82</td>
<td>5.6879 \times 10^{-13}</td>
<td>3.0245 \times 10^{-14}</td>
<td>8.6679 \times 10^{-20}</td>
</tr>
<tr>
<td>97</td>
<td>1.5064 \times 10^{-13}</td>
<td>8.0032 \times 10^{-15}</td>
<td>5.7713 \times 10^{-21}</td>
</tr>
<tr>
<td>112</td>
<td>4.8152 \times 10^{-14}</td>
<td>2.5568 \times 10^{-15}</td>
<td>2.1993 \times 10^{-21}</td>
</tr>
<tr>
<td>125</td>
<td>2.0119 \times 10^{-13}</td>
<td>1.0679 \times 10^{-15}</td>
<td>1.6398 \times 10^{-22}</td>
</tr>
</tbody>
</table>

In the Tables II and III, we provide the absolute errors at the endpoint \( x = 8 \), and the maximum errors respectively. From the results displayed in Tables II and III, the OTSBHM outperforms the BTSHA-EP and BTSHA-BP.
Consider the inhomogeneous second order IVP given by

$$y'' + 100y = 99 \sin x, \quad y(0) = 1, \quad y'(0) = 11,$$

with the exact solution given as

$$y(x) = \cos (10x) + \sin (10x) + \sin (x).$$

Problem 2

The second example is a non-linear IVP given as

$$y'' + x(y')^2 = 0,$$

$$y(1) = 1, \quad y' = \frac{1}{2},$$

$$\text{Exact} : y(x) = 1 + \frac{1}{2} \ln \left( \frac{2 - x}{2 + x} \right).$$

Problem 2 is solved using similar three-block hybrid methods considered. Table IV displays the absolute errors at the main points. From the results presented, it is clear that the OTSBHM outperforms the BTSHA-EP and BTSHA-BP methods.

### Table IV

**Absolute error for Problem 2**

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$2.2205 \times 10^{-16}$</td>
<td>$2.2205 \times 10^{-16}$</td>
<td>$2.2205 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$8.8818 \times 10^{-16}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>$1.5543 \times 10^{-15}$</td>
<td>$2.2205 \times 10^{-16}$</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>$2.8866 \times 10^{-15}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>$4.6629 \times 10^{-15}$</td>
<td>$4.4409 \times 10^{-16}$</td>
<td>$2.2205 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$8.4377 \times 10^{-15}$</td>
<td>$2.2205 \times 10^{-16}$</td>
<td>$2.2205 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.7</td>
<td>$1.5321 \times 10^{-14}$</td>
<td>$6.6613 \times 10^{-16}$</td>
<td>$2.2205 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.8</td>
<td>$2.8644 \times 10^{-14}$</td>
<td>$1.5543 \times 10^{-15}$</td>
<td>$2.2205 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$5.8398 \times 10^{-14}$</td>
<td>$3.1086 \times 10^{-15}$</td>
<td>0</td>
</tr>
<tr>
<td>1.0</td>
<td>$1.2679 \times 10^{-13}$</td>
<td>$6.6613 \times 10^{-15}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Despite sharing an equal number of function evaluation per block, the OTSBHM is superior to the other similar methods. Table V displays the maximum errors using different step lengths.

### Table V

**Maximum errors for Problem 1 with different step-lengths $N$**

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>67</td>
<td>$6.1828 \times 10^{-12}$</td>
<td>$3.2946 \times 10^{-13}$</td>
<td>$4.2255 \times 10^{-15}$</td>
</tr>
<tr>
<td>82</td>
<td>$1.2694 \times 10^{-12}$</td>
<td>$6.7517 \times 10^{-14}$</td>
<td>$6.5778 \times 10^{-16}$</td>
</tr>
<tr>
<td>97</td>
<td>$3.3602 \times 10^{-13}$</td>
<td>$1.7857 \times 10^{-14}$</td>
<td>$1.3726 \times 10^{-16}$</td>
</tr>
<tr>
<td>112</td>
<td>$1.0773 \times 10^{-13}$</td>
<td>$5.7211 \times 10^{-15}$</td>
<td>$3.5442 \times 10^{-17}$</td>
</tr>
<tr>
<td>125</td>
<td>$4.5011 \times 10^{-14}$</td>
<td>$2.3896 \times 10^{-15}$</td>
<td>$1.2513 \times 10^{-17}$</td>
</tr>
</tbody>
</table>
Problem 4

Consider the stiff IVP system

$$y'' = (\epsilon - 2)y_1 + (2\epsilon - 2)y_2, \quad y_1(0) = 2, \quad y_1'(0) = 0,$$

$$y_2' = (1 - \epsilon)y_1 + (1 - 2\epsilon)y_2, \quad y_2(0) = -1, \quad y_2'(0) = 0,$$

with the exact solutions $y_1(x) = 2\cos x$, $y_2(x) = -\cos x$, and $\epsilon = 2500$.

The results presented in Tables VI, VII, VIII, IX and X are the absolute errors generated by solving Problem 4 using the three similar methods with step lengths $h = \pi/2$, $\pi/3$, $\pi/4$, $\pi/5$ and $\pi/12$. As with previous comparisons, both algorithms share equal number of function evaluations per block. The results presented in Tables VI–X underscore the importance of systematically selecting collocation points that minimize the approximation error.
Problem 5

We consider the nonlinear Fehlberg problem

\[ y'' = -4x^2 y_1 - \frac{2y_2}{\sqrt{y_1^2 + y_2^2}}, \]
\[ y'' = \frac{2y_1}{\sqrt{y_1^2 + y_2^2}} - 4x^2 y_2, \]

subject to the conditions

\[ y_1 \left( \frac{\pi}{2} \right) = 0, \quad y_1' \left( \frac{\pi}{2} \right) = -2 \frac{\pi}{2}, \]
\[ y_2 \left( \frac{\pi}{2} \right) = 1, \quad y_2' \left( \frac{\pi}{2} \right) = 0, \]

with the exact solutions \( y_1(x) = 2 \cos(x^2), \ y_2(x) = \sin(x^2). \)

This problem was chosen to demonstrate the three similar block methods’ performance on a nonlinear system with variable coefficients and was solved in [12] and [17].

### Table X

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_1 )</th>
<th>BTSHM-EP</th>
<th>BTSHM-BP</th>
<th>OTSBHM</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2\pi )</td>
<td>( y_1(x) )</td>
<td>( 6.16 \times 10^{-21} )</td>
<td>( 1.69 \times 10^{-23} )</td>
<td>( 7.48 \times 10^{-33} )</td>
</tr>
<tr>
<td>( \quad y_2(x) )</td>
<td>( 3.08 \times 10^{-21} )</td>
<td>( 8.44 \times 10^{-24} )</td>
<td>( 3.74 \times 10^{-33} )</td>
<td></td>
</tr>
<tr>
<td>( 4\pi )</td>
<td>( y_1(x) )</td>
<td>( 2.47 \times 10^{-20} )</td>
<td>( 6.75 \times 10^{-23} )</td>
<td>( 2.99 \times 10^{-32} )</td>
</tr>
<tr>
<td>( \quad y_2(x) )</td>
<td>( 1.23 \times 10^{-20} )</td>
<td>( 3.38 \times 10^{-23} )</td>
<td>( 1.50 \times 10^{-32} )</td>
<td></td>
</tr>
<tr>
<td>( 6\pi )</td>
<td>( y_1(x) )</td>
<td>( 5.50 \times 10^{-20} )</td>
<td>( 1.52 \times 10^{-22} )</td>
<td>( 6.73 \times 10^{-32} )</td>
</tr>
<tr>
<td>( \quad y_2(x) )</td>
<td>( 2.77 \times 10^{-20} )</td>
<td>( 7.60 \times 10^{-23} )</td>
<td>( 3.37 \times 10^{-32} )</td>
<td></td>
</tr>
<tr>
<td>( 8\pi )</td>
<td>( y_1(x) )</td>
<td>( 9.86 \times 10^{-20} )</td>
<td>( 2.70 \times 10^{-22} )</td>
<td>( 1.20 \times 10^{-31} )</td>
</tr>
<tr>
<td>( \quad y_2(x) )</td>
<td>( 4.93 \times 10^{-20} )</td>
<td>( 1.55 \times 10^{-22} )</td>
<td>( 5.98 \times 10^{-32} )</td>
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</tr>
<tr>
<td>( 10\pi )</td>
<td>( y_1(x) )</td>
<td>( 1.54 \times 10^{-19} )</td>
<td>( 4.22 \times 10^{-22} )</td>
<td>( 1.87 \times 10^{-31} )</td>
</tr>
<tr>
<td>( \quad y_2(x) )</td>
<td>( 7.71 \times 10^{-20} )</td>
<td>( 2.11 \times 10^{-22} )</td>
<td>( 9.35 \times 10^{-32} )</td>
<td></td>
</tr>
</tbody>
</table>

We have presented an optimized two-step block hybrid method and implemented simultaneously to solve (1). We tested the schemes on both scalar and system initial value problems of the linear and non-linear type. Tables II-XI show the details of the numerical results. It is clear from the results that the type of intra-step points imposed during formulation affects the accuracy of implicit two-step hybrid block methods.

### References


