Optimal Control in Binary Models with the Disorder

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***Abstract*—The paper considers the problem of optimal control in models with disorder and provides the computational scheme of its solution. The proposed method is based on replacing the disorder with its estimate by the stopping time. The calculations take place on the binary tree with nodes divided into two classes. The division of tree nodes into two classes is carried out using a decision rule associated either with testing two hypothesis, or with the optimal stopping of a Markov process developing on a binary tree. The estimate of the disorder is the moment at which the walk on the tree is placed in a node of the first class for the first time. The example of the optimal control problem is related to the problem of quantile hedging of a payoff.**

***Index Terms*—optimal control, model with disorder, quantile hedging, stopping time.**

I. Introduction

PROCESSES with disorder and processes with regime change are in the sphere of constant interests of researchers. First of all, it should be noted the work of Shiryaev and his colleagues [1]–[5], in which the problem of fast detection of disorder is studied. The results of these works will be further used. In [6] processes with disorder were used as a means of approximating the solution of the stochastic differential equation. In [6] disorders coincided with the moments of reaching of given levels by random processes.

The number of works devoted to the detection of the disorder is growing every year. Database arxiv.org search shows more than a hundred works for 2020 related to the problem of disorder detection in a random process. Undoubtedly, there is a need for algorithms for solving optimal control problems for processes with the disorder. There is apparently much less works associated with optimal control of processes with the disorder. None were found in the same database.

In the work [7] control problems were considered in models in which the disorder was a stopping time with a known distribution law.

In our opinion, research in the field of optimal control of processes with the disorder can develop in three directions.

In the first, a priori direction, the disorder is a stopping time and knowledge of the a priori distribution of the disorder is assumed. The control problem is solved as a parametric problem with the subsequent integration of the parametric solution by the disorder as a disturbing parameter. For details see work [7]. This approach does not use the disorder information that is in the controlled process.

The second direction use of an a posteriori estimate of the disorder by the stopping time with the replacement of the disorder by an estimate. The main problem lies in the complex dependence of the estimate on the controlled process.

The third direction is based on considering the disorder as an uncertain parameter of the model belonging to a given set of disorder values, with the transition to the minimax formulation of the control problem in order to obtain a guaranteed result. Now, there is a growing interest in robust formulations, so this line of research looks natural.

This paper refers to the second of the listed areas. In the paper the binary model is considered, since, in our opinion, a binary model is the unique model that provides the ability to obtain a computational result. In addition, in the paper a binary approximation of the Wiener process is considered. The trick is not new, it is enough to recall in this connection the Prokhorov-Donsker principle of the weak convergence of a random walk with the corresponding normalization to the standard Wiener process [8]. There are a large number of works devoted to the binary approximation, which estimate the rate of convergence for functionals calculated on the trajectories of the diffusion processes. See for example works [9]–[12].

The structure of the work is as follows. In the second section the problem of calculating the disorder for binary sequences is considered. The disorder estimate is considered as a dividing of tree nodes into two classes using the stopping time closest to the disorder. The nodes of the first class include nodes in which the disorder occurs. A binary sequence is considered as a random walk along nodes of a tree, and the disorder estimate is the minimum time at which a random walk is at a node of the first class. The proposed method differs from the classical methods in which different recurrently calculated statistics, for example, the cumulative sum, the Shiryaev-Roberts statistics, or the posterior probability of a disorder are compared with a time-dependent level. These methods require preliminary calculations to determine the edge of the stop. The proposed method performs comparable pre-computations to classify tree nodes and simple follow-up computations associated with a random walk. This method of detecting the disorder allows solving optimal control problems. In this section the binary approximation is considered.

In the third section, the Black-Scholes model, popular in financial mathematics, with the addition of disorder is considered, and its binary approximation is given. In the same section, one of the optimal control problems is solved.

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This problem in financial mathematics is called the quantile hedging problem.

The fourth section is devoted to computational examples.

II. Disorder Estimate

A. The Stochastic Disorder Model

Let us consider the binary stochastic process X_t with $X_t \in \{0,1\}$ and discrete time $t \in \{1,\ldots,n,\ldots\}$ on natural stochastic basis—$\langle\Omega,(F_n)_{n\geq 0},F,P\rangle$. In the basis the set of elementary stochastic events Ω is the set of binary sequences $\omega=(\omega_1,\ldots,\omega_n,\ldots)$, the filtration elements are σ-subalgebras $F_n=\sigma(X_1,\ldots,X_n)$, $F_0=\sigma(\Omega,\emptyset)$, the σ-algebra $F=\sigma\left(\bigcup_{t\geq 0}F_t\right)$, P—probability. Consider the stochastic variable $\theta\in\{1,\ldots,n,\ldots\}$, which is the disorder. For any sequence ω the conditional elementary probability $p(\omega/\theta=n)$ is determined, and the probability distribution—$p(n)$ is determined for the stochastic variable θ. The probability P is determinante in the tremens of the elementary probabilities by the formula: $p(\omega)=\sum_{n\geq 1}p(\omega/\theta=n)p(n)$. We will use the following expression for the conditional probabilities:

$$p(\omega/\theta=n)=p_\infty(\omega_1)\cdot\ldots\cdot p_\infty(\omega_{n-1})\cdot p_0(\omega_n)\cdot\ldots\cdot p_0(\omega_m)\ldots,\quad n>1;\qquad (1)$$
$$p(\omega/\theta=1)=p_0(\omega_1)\cdot\ldots\cdot p_0(\omega_m)\ldots$$

For each factor in this formula the expression

$$p_0(\omega_i)=q_0^{\omega_i}(1-q_0)^{1-\omega_i},\quad p_\infty(\omega_i)=q_\infty^{\omega_i}(1-q_\infty)^{1-\omega_i}$$

is true.

B. The Information Tree and Stopping Times

In connection with a binary sequence, consider a binary tree with a set of nodes $[(A_i^j)_{j=0}^{2^i-1}]_{i=0}^N$, in which $A_i^j\longrightarrow\{A_{i+1}^{2j},A_{i+1}^{2j+1}\}$. It is easy to establish an isomorphism between the segments of the binary sequence ω_1^i and the nodes of the tree A_i^j. This isomorphism is specified in a standard way: the segment of the binary sequence ω_1^i is isomorphic to the node A_i^j if and only if $j=\sum_{k=1}^i 2^{i-k}\omega_k$.

C. The Problem of Disorder Estimate

The problem of disorder detection is to calculate the stopping time τ_*, located as closed as possible to the θ. In the above-mentioned works, the problem of disorder detecting is considered as the problem of optimal stopping of a Markov process [14]. The linear criteria in the optimal stopping problem has the form: $P(\theta>\tau)+\alpha E(\tau-\theta)^+$. The first term is the probability of “false alarm”, the second is the average delay. Shown as standard [13], that the optimal stopping problem for discrete time and a finite horizon with the given criteria reduces to the following optimal stopping problem:

$$V=\min_{1\leq\tau\leq T}E\left[1-\phi_\tau+\alpha\sum_{i=1}^{\tau-1}\phi_i\right]\qquad (2)$$

In (2) $\phi_n=P(\theta\leq n/F_n)=\sum_{j=1}^n P(\theta=j/F_n)$—the sequence of posterior probabilities. According to the Bayes formula, the elements of this sequence $\phi_n=\frac{\sum_{j=1}^n p(X_1,\ldots,X_n/\theta=j)P(\theta=j)}{p(X_1,\ldots,X_n)}$.

Simultaneously with this sequence consider the sequence of likelihood ratios:

$$\psi_n=\frac{\phi_n}{1-\phi_n}=\frac{\sum_{j=0}^n p(X_1,\ldots,X_n/\theta=j)P(\theta=j)}{p_\infty(X_1,\ldots,X_n)P(\theta>n)}.$$

Problem (2) is expressed in terms of this sequence as follows:

$$V=\min_{1\leq\tau\leq T}E\left[\frac{1}{1+\psi_\tau}+\alpha\sum_{i=1}^{\tau-1}\frac{\psi_i}{\psi_i+1}\right].\qquad (3)$$

Next, the adapted sequence will be used: $\beta_n=\sum_{i=1}^n\frac{\psi_i}{\psi_i+1}$. One of the suitable methods for solving problem (3) is dynamic programming. The use of dynamic programming involves the following actions:

1. Determine the sequence of Bellman functions:

$$V_n(\omega_1^n)=\min_{n\leq\tau\leq T}E\left[\frac{1}{1+\psi_\tau}+\alpha\beta_{\tau-1}/X_1^n=\omega_1^n\right]$$

or in terms of an information tree:

$$V_n(A_n^i)=\min_{n\leq\tau\leq T}E_{A_n^i}\left[\frac{1}{1+\psi_\tau}+\alpha\beta_{\tau-1}\right],$$
$$i=0,2,\ldots,2^n-1.$$

The mathematical expectation is calculated under the condition that the remainder of the random walk on the binary tree starts at the node A_n^i.

2. Obtain the Bellman equations:

$$V_n(A_n^i)=\min\Bigg(\frac{1}{1+\psi_n(A_n^i)}+\alpha\beta_{n-1}(A_{n-1}^j),\qquad (4)$$
$$V_{n+1}(A_{n+1}^{2i})P(X_{n+1}=1)$$
$$+V_{n+1}(A_{n+1}^{2i+1})P(X_{n+1}=0)\Bigg),\quad j=\lfloor i/2\rfloor.$$

The boundary condition for these equations: $V_T=\frac{1}{1+\psi_T}+\alpha\beta_{T-1}$. The probability

$$P(X_{n+1}=1)=q_0P(\theta\leq n+1)+q_\infty P(\theta>n+1).$$

3. Calculating the optimal stopping time:

$$\tau_n=\begin{cases}n, & V_n=\frac{1}{1+\psi_n}+\alpha\beta_{n-1}\\ \tau_{n+1}, & V_n<\frac{1}{1+\psi_n}+\alpha\beta_{n-1}\end{cases}$$

With the boundary condition: $\tau_T=T$. The optimal solution of the problem (3) $\tau^*=\tau_1,V^*=V_1$.

Consider a stopping time τ. The stopping time divides the tree nodes into two classes as follows. Consider a stochastic event $\{\tau=n\}$, that belongs to the σ-algebra $F_n=\sigma\left((A_n^j)_{j=0}^{2^n-1}\right)$, that is why $\{\tau=n\}=\bigcup_{j\in H_{1,n}}A_n^j$. Note that the set $H_{1,n}$ may be empty. In this case $\{\tau=n\}=\emptyset$. If the set $H_{1,n}\neq\emptyset$, then the nodes A_n^j, $j\in H_{1,n}$ belong to the first class. If the node A_n^j belongs to the first class, then the nodes A_{n+1}^{2j} and A_{n+1}^{2j+1} also belong to the first class. Remaining nodes of the tree belong to the second class (H_2). Now the sequence ω can be considered as a random walk on a binary tree; if the walk is in the node of the first class, then the rest of the walk will occur along the nodes of this class.

D. The Sequence of Likelihood Ratios

To implement the dynamic programming method, it is necessary to calculate the value of the sequence of likelihood ratios at each node of the tree. With independence and the same distribution

$$p_\infty(X_1,\dots,X_{k-1}) = (1-q_\infty)^{k-1}\left(\frac{q_\infty}{1-q_\infty}\right)^{\sum_{i=1}^{k-1} X_i},$$

$$p_0(X_k,\dots,X_n) = (1-q_0)^{n-k-1}\left(\frac{q_0}{1-q_0}\right)^{\sum_{i=k}^{n} X_i}$$

As a result, the sequence of likelihood ratios:

$$\psi_n(X_1,\dots,X_n) \tag{5}$$

$$= \frac{1}{P(\theta>n)}\sum_{k=1}^{n} L^{n-k+1}R^{\Sigma_{i=k}^n X_i}P(\theta=k).$$

In (5) $L=\frac{1-q_0}{1-q_\infty}$, $R=\frac{q_0(1-q_\infty)}{q_\infty(1-q_0)}$.

We obtain recurrent equations for the sequence of likelihood ratios. For this we expand ψ_{n+1} in the following way:

$$\psi_{n+1}(X_1,\dots,X_{n+1}) = \frac{P(\theta>n)}{P(\theta>n+1)}$$

$$\times\left[\frac{LR^{X_{n+1}}}{P(\theta>n)}\sum_{k=1}^{n+1} L^{n-k+1}R^{\sum_{i=k}^{n} X_i}P(\theta=k)\right]$$

$$= \frac{P(\theta>n)}{P(\theta>n+1)}$$

$$\times\left[LR^{X_{n+1}}\left(\psi_n(X_1,\dots,X_n)+\frac{P(\theta=n+1)}{P(\theta>n)}\right)\right].$$

Use of symbols:

$$Z_{n+1}(x) = \frac{P(\theta>n)}{P(\theta>n+1)}LR^x,$$

$$U_{n+1} = \frac{P(\theta=n+1)}{P(\theta>n+1)}.$$

Simplifies the recurrent equation:

$$\psi_{n+1}(X_1,\dots,X_{n+1})$$

$$= Z_{n+1}(X_{n+1})\psi_n(X_1,\dots,X_n)+U_{n+1},$$

$$\psi_1 = LR^{X_1}\frac{P(\theta=1)}{P(\theta>1)}.$$

Recurrent equations (6) allow calculating the ψ for any node of the tree:

$$\psi_{n+1}(A_{n+1}^{2i}) \tag{6}$$

$$= \frac{P(\theta>n)}{P(\theta>n+1)}LR\psi_n(A_n^i)+U_{n+1},$$

$$\psi_{n+1}(A_{n+1}^{2i+1}) = \frac{P(\theta>n)}{P(\theta>n+1)}L\psi_n(A_n^i)+U_{n+1}.$$

Consider other variants of the disorder detection by the stopping time.

E. The Bayesian Model

At any time n the random events are considered: $H_{0,n}=\{\theta\le n\}, H_{\infty,n}=\{\theta>n\}$. If the θ is the stopping time, then that events belong to σ-algebra $F_n=\sigma\{X_1,\dots,X_n\}$. To make a decision in favor of one of them, we can use the previously defined statisticians: $\phi_n=P(\theta\le n/X_1,\dots,X_n)$ and $1-\phi_n=P(\theta>n/X_1,\dots,X_n)$. Let $d(X_1,\dots,X_n)\in\{H_{0,n};H_{\infty,n}\}$ is the decision rule. The decision rule is defined for all realizations of a segment X_1^n of a binary sequence or for each node A_n^i of a binary tree. The quality of the decision rule can be measured using Bayesian risk: $BR_n(d)=a(d,H_{0,n})\phi_n+a(d,H_{\infty,n})(1-\phi_n)$. An optimal solution is one that minimizes Bayesian risk: $d^*(X_1,X_2,\dots,X_n)=\arg\min_d[a(d,H_{0,n})\phi_n+a(d,H_{\infty,n})(1-\phi_n)]$. It is easy to see that the optimal decision rule will be:

$$d^*(X_1,\dots,X_n)$$

$$=\begin{cases} H_{0,n}, & \psi_n\ge\frac{a(H_{0,n},H_{\infty,n})-a(H_{\infty,n},H_{\infty,n})}{a(H_{\infty,n},H_{0,n})-a(H_{0,n},H_{0,n})}\\ H_{\infty,n}, & \psi_n<\frac{a(H_{0,n},H_{\infty,n})-a(H_{\infty,n},H_{\infty,n})}{a(H_{\infty,n},H_{0,n})-a(H_{0,n},H_{0,n})},\end{cases}$$

if $a(H_{\infty,n},H_{0,n})>a(H_{0,n},H_{0,n})$. For example, if the errors are equivalent (our case), then the natural choice for a is the following choice: $a(H_{0,n},H_{\infty,n})=a(H_{\infty,n},H_{0,n})=1; a(H_{0,n},H_{0,n})=a(H_{\infty,n},H_{\infty,n})=0$. For such a choice, the ratio $\frac{a(H_{0,n},H_{\infty,n})-a(H_{\infty,n},H_{\infty,n})}{a(H_{\infty,n},H_{0,n})-a(H_{0,n},H_{0,n})}=1$. Thus, for each node of the tree, the decision rule $d^*(A_n^i)=d^*(\omega_1,\omega_2,\dots,\omega_n)$ is determined. Here the segment of the sequence ω_1^n is isomorphic to the node A_n^i. The decision rule allows to assign an arbitrary node A_n^i of the tree to one of two classes as follows: the node belongs to the first class, if the parent node $A_{n-1}^{\lfloor i/2\rfloor}$ belongs to the first class, or the value of the decision rule $d(A_i^n)=H_{0,n}$. The rest of the nodes belong to the second class.

F. Minimization of the Maximum Possible Probability of Error

Let us consider another popular assessment of the quality of a solution—the maximum possible error probability. If $d(X_1,X_2,\dots,X_n)=H_{0,n}$, then the probability of error is $P_0=p(X_1,X_2,\dots,X_n/\theta>n)$. If $d(X_1,X_2,\dots,X_n)=H_{\infty,n}$, then the maximum possible error probability is $P_\infty=\max_{1\le j\le n}\{p(X_1,X_2,\dots,X_n)/\theta=j\}$. The optimal decision rule for this criteria will be:

$$d^*(X_1,\dots,X_n)=\begin{cases} H_{0,n}, & \max\limits_{1\le j\le n}\frac{P(X_1,X_2,\dots,X_n/\theta=j)}{P(X_1,X_2,\dots,X_n/\theta>n)}\ge1\\ H_{\infty,n}, & \max\limits_{1\le j\le n}\frac{P(X_1,X_2,\dots,X_n/\theta=j)}{P(X_1,X_2,\dots,X_n/\theta>n)}<1\end{cases}$$

Using of the logarithm allows you to write this decision rule as follows:

$$d^*(X_1,\dots,X_n)$$

$$=\begin{cases} H_{0,n}, & \max\limits_{1\le j\le n}\ln\frac{P(X_1,X_2,\dots,X_n/\theta=j)}{P(X_1,X_2,\dots,X_n/\theta>n)}\ge0\\ H_{\infty,n}, & \max\limits_{1\le j\le n}\ln\frac{P(X_1,X_2,\dots,X_n/\theta=j)}{P(X_1,X_2,\dots,X_n/\theta>n)}<0.\end{cases}$$

Let us denote by

$$\zeta_n = \max_{1\le j\le n}\ln\frac{P(X_1,X_2,\dots,X_n/\theta=j)}{P(X_1,X_2,\dots,X_n/\theta>n)}$$

$$= \max_{1\le j\le n}[(n-j+1)\ln C+\sum_{k=j}^{n}X_k\ln B].$$

Here $\ln C = \ln \frac{1-q_0}{1-q_\infty}$, $\ln B = \ln \frac{q_0(1-q_\infty)}{q_\infty(1-q_0)}$.

Recurrent equations for the ζ_n will be:

$$\zeta_1(A_1^0) = \ln C, \quad \zeta_1(A_1^1) = \ln C + \ln B,$$

$$\zeta_{n+1}(A_{n+1}^{2i}) = \ln C + \max\{\zeta_n(A_n^i), 0\},$$

$$\zeta_{n+1}(A_{n+1}^{2i+1}) = \ln(CB) + \max\{\zeta_n(A_n^i), 0\}.$$

Similarly to how it was done earlier, for each node of the tree, a decision rule is calculated, which splits the nodes of the tree into two classes.

G. The Continuous Problem. The Binary Solution

Let us consider the stochastic process $X(t)$, satisfying the stochastic differential equation:

$$dX(t) = (\mu_1 I(t < \theta) + \mu_2 I(t \geq \theta))dt + \sigma dW(t),$$

$$X(0) = 0,$$

θ—stochastic variable, $W(t)$—standard Wiener process.

Let us consider the centered and normalized process $Y(t) = \frac{X(t)-\mu_1 t}{\sigma}$, for which the equation is true:

$$dY(t) = \mu I(t \geq \theta)dt + dW(t), \tag{7}$$

$$Y(0) = 0, \quad \mu = \frac{\mu_2 - \mu_1}{\sigma}.$$

Simultaneously with the process Y consider two processes $Y^0(t)$ and $Y^\infty(t)$:

$$dY^0(t) = \mu dt + dW(t), \quad Y^0(0) = 0,$$

$$dY^\infty(t) = dW(t), \quad Y^\infty(0) = 0.$$

Next, we turn to the discrete approximation. Consider a uniform partition of the interval $[0,1]$ by points $\eta_i = \frac{i-1}{N}$, $i = 1, \ldots, N+1$ and two binary processes:

$$X_i^0 = \frac{1}{2}(\text{sign}(Y_{\eta_i}^0 - Y_{\eta_{i-1}}^0) + 1) \tag{8}$$

$$= \frac{1}{2}\left(\text{sign}\left(\mu\frac{1}{N} + \varepsilon_i\sqrt{\frac{1}{N}}\right) + 1\right),$$

$$X_i^\infty = \frac{1}{2}(\text{sign}(Y_{\eta_i}^\infty - Y_{\eta_{i-1}}^\infty) + 1) = \frac{1}{2}(\text{sign}(\varepsilon_i) + 1).$$

The ε_i is i.i.d random variables with standard normal distribution. The probability $q_\infty = P(X_i^\infty = 1) = 1/2$. The probability $q_0 = P(X_i^0 = 1) = \Phi\left(\mu/\sqrt{N}\right)$, the function Φ is the Laplace function.

The next step is to use the disorder estimation technique described in the previous section, matching the discrete and continuous a priory distributions of the disorder.

III. The Black-Scholes Model with the Disorder

Consider a popular model of the evolution of the stock price with an additional element—the disorder:

$$dS(t) = S(t)dX(t), \quad S(0) = S_0. \tag{9}$$

In (9) we change the process $X(t)$ by the process $\overline{X}(t)$:

$$d\overline{X}(t) = (\mu_1 I(t < \tau^*) + \mu_2 I(t \geq \tau^*))dt + \sigma dW(t),$$

τ^* is the stopping time. Binary approximation leads to the discrete Cox-Ross-Rubinstein approximation with the built-in disorder:

$$S_n(A_n^{2i}) \tag{10}$$

$$= S_{n-1}(A_{n-1}^i)\left(1 + \mu(A_{n-1}^i)\frac{1}{N} + \frac{\sigma}{\sqrt{N}}\right),$$

$$S_n(A_n^{2i+1}) = S_{n-1}(A_{n-1}^i)\left(1 + \mu(A_{n-1}^i)\frac{1}{N} - \frac{\sigma}{\sqrt{N}}\right).$$

With transit probabilities

$$P(A_{n-1}^i, A_n^{2i}) = P(A_{n-1}^i, A_n^{2i+1}) = \frac{1}{2}.$$

This formula uses: $S_n = S(\eta_n)$ and

$$\mu(A_k^m) = \begin{cases} \mu_1, & A_k^m \in H_1 \\ \mu_2, & A_k^m \in H_2 \end{cases},$$

the tree nodes are divided into classes by the discrete stopping time closest to the discrete disorder, see the first section.

A. The Quantile Hedging Problem for Model with the Disorder

Next, consider the optimal control problem:

$$\min_\gamma E(f(S_N) - Y_N)^+, \tag{11}$$

$$\Delta Y_n = \gamma_n \Delta S_n, Y_0 \leq m.$$

In (11) γ_n are defined on the nodes A_{n-1}^i of the information tree. Basic process S_n is determined by the equation (10). This problem is a special case of the class of optimal control problems presented in the work [7]. The features of this class of problems are as follows. The goal is determined at the right end of the time interval, the control enters the equation for the controlled process linearly, there is only one measure for which the base process is a martingale, and the constraint is set at the left end of the time interval. In this paper the general scheme for solving the optimal control problems for this class of problems is proposed.

For the problem (11) this scheme consists of the following steps.

1. To solve the problem

$$\max_\xi E\xi f_N, \quad 0 \leq \xi \leq 1, \quad E^*\xi f_N \leq m.$$

The presence of an asterisk means calculating the average over the martingale measure, the existence and uniqueness of which is assumed.

To determine the martingale measure, we define the density process W_n: $dP_n^* = W_n dP_n$, P_n^*—narrowing the martingale measure to F_n, P_n—narrowing the original measure to F_n. Let us note the characteristic properties of the density process. The process density is not-negative process $W_n \geq 0$, the expectation $EW_n = 1$, and the process density is the martingale. It follows from these properties that the following equations are valid for the density process:

$$W_n(A_n^{2i}) = W_n(A_{n-1}^i)(1 + \alpha(A_{n-1}^i)), \qquad (12)$$
$$W_n(A_n^{2i+1}) = W_n(A_{n-1}^i)(1 - \alpha(A_{n-1}^i)),$$
$$W_0 = 0, \quad |\alpha| < 1.$$

From Bayes formula for the conditional expectation follows that process $W_n S_n$ is the martingale, if P^* is the martingale measure. From this $\alpha(A_{n-1}^i) = -\frac{\mu(A_{n-1}^i)}{\sigma\sqrt{N}}$, and $|\alpha| < 1$, with N is enough large. Thus, for a sufficiently N large martingale measure exists and it is unique.

Lets return to the optimization problem:

$$\max_{\xi} E\xi f_N, \quad 0 \le \xi \le 1, \quad E\xi W_N f_N \le m.$$

For the finite set Ω, this optimization problem is a linear programming problem:

$$\max \sum_{i=0}^{2^N-1} \xi_i f_N^i, \quad \sum_{i=0}^{2^N-1} \xi_i \varphi_N^i \le M, \quad 0 \le \xi_i \le 1. \qquad (13)$$

In (13) the following notation is used:

$$\xi_i = \xi(A_N^i), \quad f_N^i = f_N(A_N^i),$$
$$\varphi_N^i = W_N(A_N^i) f_N(A_N^i), \quad M = 2^N m.$$

Suppose the f_N is bounded function on a finite set of possible values of S_N. The problem has solution because the goal function is bounded and the set of feasible design is not-empty.

Dual problem is:

$$\min_{\lambda \ge 0} \max_{0 \le \xi_i \le 1} \qquad (14)$$
$$\left(\sum_{i=0}^{2^N-1} \varphi_i^N \left(\frac{f_i^N}{\varphi_i^N} - \lambda \right) \xi_i + \lambda M \right).$$

The problem (15) has the solution too. We will assume that the sequence of likelihood ratios is ordered in ascending order and consider the internal problem:

$$F(\lambda) = \max_{0 \le \xi_i \le 1} \varphi_i^N \xi_i \left(\frac{f_i^N}{\varphi_i^N} - \lambda \right) + \lambda M.$$

The function $F(\lambda)$ is the convex function. Let $\lambda \in \left(\frac{f_N^{k-1}}{\varphi_N^{k-1}}, \frac{f_N^k}{\varphi_N^k} \right)$, on that interval

$$F(\lambda) = \sum_{i=k}^{2^N-1} f_i^N + \lambda \left(M - \sum_{i=k}^{2^N-1} \varphi_i^N \right).$$

If

$$M - \sum_{i=k}^{2^N-1} \varphi_i^N \ge 0,$$

then

$$F(\lambda) \ge F\left(\frac{f_N^{k-1}}{\varphi_N^{k-1}} \right), \quad \lambda \in \left[\frac{f_N^{k-1}}{\varphi_N^{k-1}}, \frac{f_N^k}{\varphi_N^k} \right).$$

From this and the convexity of the function $F(\lambda)$ it follows that to solve the problem it is necessary to find a number k^* for which $M - \sum_{i=k^*}^{2^N-1} \varphi_i^N \ge 0$ and $M - \sum_{i=k^*-1}^{2^N-1} \varphi_i^N < 0$. The solution of the problem (13) is:

$$\overline{\xi}_i = \begin{cases} 1, & i \ge k^*, \\ \frac{M - \sum_{i=k^*}^{2^N-1} \varphi_N^i}{\varphi_N^{k^*-1}}, & i = k^* - 1, \\ 0, & i < k^* - 1. \end{cases} \qquad (15)$$

Comment. A random variable ξ can be viewed as a function of membership in a fuzzy set. Therefore, the problem (12) is called the fuzzy Neumann—Pearson problem. The solution of the fuzzy Neumann—Pearson problem is given in [14].

2. Next, the closed martingale is calculated: $Y_n = E^*(\overline{\xi} f_N / F_n)$, here $\overline{\xi}$– the solution of the optimization problem from the first point. Conditional mathematical expectations are calculated using the unique martingale measure. To calculate the conditional mathematical expectations, we will use the already mentioned Bayes formula, as a result we get the following equality:

$$Y_{n-1}(A_{n-1}^i) = \frac{1}{2}\left(Y_n(A_n^{2i}) \left(1 - \frac{\mu(A_{n-1}^i)}{\sigma\sqrt{N}} \right) \right.$$
$$\left. + Y_n(A_n^{2i+1}) \left(1 + \frac{\mu(A_{n-1}^i)}{\sigma\sqrt{N}} \right) \right)$$

with boundary condition $Y_N = \overline{\xi} f_N$.

Next, we calculate γ from the equality: $\Delta Y_n(A_n^{2i}) = \gamma_n(A_{n-1}^i) \Delta S_n(A_n^{2i})$. This completes the solution of the considered control problem.

Comment. With the same result, one could use the equality: $\Delta Y_n(A_n^{2i+1}) = \gamma_n(A_{n-1}^i) \Delta S_n(A_n^{2i+1})$.

IV. Calculation

This section contains two computational examples. The first example examines the problem of detecting the disorder using the two statistics presented in the “Disorder estimate” section. The stability of the method is investigated with respect to changes in the distribution law of independent and identically distributed random variables ε_i. The statistics were calculated on the assumption that ε_i are distributed according to the normal law with zero mathematical expectation and variance equal to one. In the simulation, a different distribution laws was used, also with zero mean and unit variance. The table shows one of the results of experiments, namely, for random variables $\varepsilon_i = \frac{1}{\sqrt{2}}(\varepsilon_i^1 + \varepsilon_i^2)$. The density of the distribution law of the first term is $p_1(x) = \begin{cases} \exp(-x), & x \ge 0 \\ 0, & x < 0 \end{cases}$, the second term – $p_2(x) = \begin{cases} \exp(x), & x \le 0 \\ 0, & x > 0 \end{cases}$. We recall that for the normal law the parameters $q_\infty = 0.5$, $q_0 = \Phi\left(\mu/\sqrt{N}\right)$. Here N is the number of dividing points of the interval $[0, 1]$.

The number of dividing points of the interval $N = 100$, the number of experiments was one hundred. In all experiments, the disorder was a random variable with a geometric distribution law (a priori distribution). The first statistic turned out to be, as expected, better, since it uses a priori information. The results of modeling using the normal distribution law are given in parentheses. It follows from the table 1 that the replacement of the distribution law did not lead to a significant distortion of the results.

TABLE I
THE ABSOLUTE VALUE OF THE MEAN ERROR IN THE DETECTION OF THE DISORDER

μ	1	1.5	2
The absolute value of the mean error, the first method	0.0042 (0.0039)	0.0032 (0.0033)	0.0022 (0.0019)
The absolute value of the mean error, the second method	0.0069 (0.0067)	0.0049 (0.0046)	0.0047 (0.0043)

TABLE II
DEPENDENCE OF THE OPTIMAL VALUE OF THE OBJECTIVE FUNCTION ON THE RIGHT SIDE OF THE CONSTRAINT

ζ	0	0.2	0.4	0.6	0.8
N=10	0.178	0.139	0.102	0.068	0.033
N=20	0.183	0.143	0.105	0.069	0.034

In the second example we will consider a quantile hedging problem in order to illustrate the efficiency of the method. Events will develop in the interval $[0, 1]$, although they could develop on any other time interval. The interval is split evently into N parts. The disorder can occur at any discrete moment of time $\eta_i = (i-1)/N$, $i = 1, \dots, N+1$. As apriori probability distribution for disorder we use the geometric probability distribution: $P(\theta = \eta_i) = \lambda^{i-1}(1-\lambda)$, $i = 1, 2, \dots$ For a given prior distribution the elements necessary for calculating likelihood ratios are calculated as follows: $Z_{n+1}(A^{2i}_{n+1}) = LR/\lambda$, $Z_{n+1}(A^{2i+1}_{n+1}) = L/\lambda$, $U_n = 1-\lambda$.

The probability used in calculating Bellman functions $P(X_{n+1} = 1) = q_0(1-\lambda^{n+1}) + \lambda^{n+1}/2$. Elements of the optimal control problem: the function $f(S_N) = (S_N - K)^+$, $m = \zeta E^* f(S_N)$. Parameters of the model: variable drift $\mu_1 = 0.1$, $\mu_2 = -0.1$, volatility $\sigma = 0.1$. Calculations made for $N = 10$ and $N = 20$. The result is shown in table II.

Let us comment on the results obtained. Firstly, for $\zeta = 1$ the optimal solution $\overline{\xi} = 1$, secondly, when $\zeta = 0$ the optimal solution $\overline{\xi} = 0$, and thirdly, the dependence of the optimal value of the objective functional on the parameter ζ is linear. Comparison of the first and second lines of the table, especially in the "realistic" range of parameter values, shows a good agreement of the results.

Here is one of the results from a series of experiments with different ways of assessing disorder. In the experiments carried out, the methods for assessing the disorder either did not affect the classification of the nodes or had little effect on the classification of the tree nodes. In the experiments carried out, the trends presented in table II, coincided.

V. CONCLUSION

The main result is a computational scheme for solving the problem of stochastic optimal control of processes with disorder. The disorder estimation by the stopping time divides the tree nodes into two classes. The disorder estimate is the first time, at which the walk is at the node of the first class. In the nodes of the first class the properties of the random process change.

A binary tree with nodes divided into two classes can be used to solve the stochastic optimal control problem, since the tree contains information about the disorder for each binary trajectory.

The computational scheme requires serious preliminary computational costs. However, the authors see the possibility of using fast computational algorithms on trees.

An important positive characteristic of the proposed computational scheme is its robustness, comparable to the robustness of sign analysis.

The application of the computational scheme for continuous models requires additional research, related to the answer of two questions. The first question is the accuracy of the approximation, the second is the sensitivity to the disorder. The answer to the first question exists, the second question requires additional research.

The computational example given in the paper can only be considered as a preliminary positive result.

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