Stability of Nonlinear Delay Hybrid Urban Traffic Network Driven by Stable Noises

Peng Gao, Zhuan Liu, Chao Wei

Abstract—**This article is devoted to investigate the stability of nonlinear delay hybrid urban traffic network driven by α-stable noises. We derive that the unique global solution of urban traffic network exists and discuss the stability of solution by general Itô formula, Hölder inequality and Bolzano-Weierstrass. We provide an example to verify the results.**

Index Terms—**Delay hybrid urban traffic network; α-stable noises; existence; stability; unique global solution.**

I. Introduction

With the rapid development of society and economy, the process of global urbanization is still advancing in an unstoppable trend. China's urbanization rate will rise to 0.75, the scale of cities will continue to expand, the national income is rising rapidly, and the people's living standards are improving day by day. But, it also brings many challenges, such as air pollution, traffic congestion, and resource shortages. These phenomenon always present stochastic characteristics. Stochastic phenomenon especially systems have been described by stochastic differential equations. Furthermore, systems are always influenced by noises. Hence, some authors has modelled the actual systems by stochastic systems ([2], [12], [19], [26]). However, the majority of stochastic disturbances represent non-Gauss characteristic such as α-stable noise. In the last few years, α-stable noises were utilized in financial, biological and medical fields ([5], [24], [27]). Wei ([20]) obtained the estimators about CIR model with α-stable noises and analyzed their asymptotic properties. Liu et al. ([10]) discussed spectrum sensing problem with symmetric α-stable noise by maximum generalized correntropy. Ning and Sun ([13]) studied bifurcations about self-sustained system with α-stable noises.

In the last few years, some authors discussed the stochastic systems with Markovian switching. For example, Liu et al. ([8]) derived an event-based communication scheme to study the distributed filtering problem. Wang et al. ([17]) utilized aperiodically intermittent control to analyze the stabilization of hybrid delayed system. Xia et al. ([23]) considered dissipative method to discuss the hybrid neural networks. With deepening of human production practice, time lags have been noticed in biochemical, population, physics and engineering. It is found that the appearance of this phenomenon may be related to the connection of each sub-component of the system and the characteristics of sub-components. A system with time delay is called delay system because the change of its state is not only dependent on the current state, but also related to the previous state. In the past few decades, many authors investigated the delay system ([3], [4], [14], [16], [18]). Li et al. ([6]) constructed a new slack variable-dependent inequality involving double integrals of system state and derived an improved stability criterion. Qi et al. ([15]) used a new criterion to design controller for delay stochastic system with actuator saturation. Zhou et al. ([25]) studied the exponential synchronization for delay stochastic system.

In the last few years, some authors studied stability of systems ([1], [9], [21], [22]). Li et al. ([7]) used Lyapunov function to disscuss the stability of stochastic delay system and gave a new nonlinear growth condition. Ma et al. ([11]) utilized Itô formula to study practical stability about stochastic system driven by Lévy noise. But, the stability of delay hybrid urban traffic network with α-stable noises has not been studied by many authors. In this article, the existence, almost surely stability of unique global solution for delay hybrid urban traffic network with α-stable noise are investigated by general Itô formula, Hölder inequality and Bolzano-Weierstrass.

This paper is organized as follows: In Section 2, the delay hybrid urban traffic network with α-stable noises is introduced. In Section 3, we prove the existence, uniqueness and almost surely stability of the solution. In Section 4, we give an example. In Section 5, the conclusion is provided.

II. Problem Formulation and Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ-algebras $\{\mathscr{F}_t\}_{t\geq 0}$ and $Z = \{Z_t, t \geq 0\}$ be a strictly symmetric α-stable Lévy motion.

A random variable ω is said to have a stable distribution with index of stability $\alpha \in (0, 2]$, scale parameter $\sigma \in (0, \infty)$, skewness parameter $\beta \in [-1, 1]$ and location parameter $\mu \in (-\infty, \infty)$ if it has the following characteristic function:

$$\phi_\eta(u) = \begin{cases} \exp\{-\sigma^\alpha |u|^\alpha (1 - i\beta sgn(u) \tan \dfrac{\alpha\pi}{2}) + i\mu u\}, \\ \exp\{-\sigma|u|(1 + i\beta \dfrac{2}{\pi} sgn(u) \log |u|) + i\mu u\}. \end{cases}$$

We denote $\omega \sim S_\alpha(\sigma, \beta, \mu)$. When $\mu = 0$, we say η is strictly α-stable, if in addition $\beta = 0$, we call η symmetrical α-stable. Throughout this paper, it is assumed that α-stable motion is strictly symmetrical and $\alpha \in (1, 2)$.

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We will study the following delay hybrid urban traffic network with α-stable noises:

$$\begin{aligned} dx(t) &= f(x(t), x(t-\tau(t)), t, r(t))dt \\ &\quad + g(x(t), x(t-\tau(t)), t, r(t))dZ(t), \end{aligned} \quad (1)$$

where $x(0) = \{x(\beta) : -\tau \leq \beta \leq 0\} = \xi \in \mathcal{C}^b_{\mathscr{F}_0}([-\tau, 0); \mathbb{R}^n)$, $r(0) = r_0 \in \mathbb{S}$, $Z(t)$ is an m-dimensional strictly symmetric α-stable motion with the index $\alpha \in (1, 2)$, $0 \leq \tau(t) \leq \tau$, $\dot{\tau}(t) \leq d_\tau < 1$, $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{n\times m}$, We assume that $Z(t)$ and $r(t)$ are independent.

The Lévy measure ν is:

$$\nu(dz) = \frac{C_\alpha}{|z|^{\alpha+1}} dz, \quad (2)$$

where

$$C_\alpha = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{(1+\alpha)}{2})}{\pi^{\frac{1}{2}}\Gamma(1-\frac{\alpha}{2})}, \quad (3)$$

where $\Gamma(\cdot)$ is a Gamma function.

Firstly, we provide some assumptions and definition.

Assumption 1: When $x = 0, y = 0$,

$$\sup_{t\geq 0, i\in S} \{|f(x, y, t, i)| \vee |g(x, y, t, i)| : t \geq 0, i \in S\} \leq K_0,$$

where K_0 is a constant.

Assumption 2: $\forall t \geq 0$, $|x| \vee |x^*| \vee |y| \vee |y^*| \leq K$ and $i \in S$,

$$\begin{aligned} &|f(x, y, t, i) - f(x^*, y^*, t, i)|^2| \\ &\vee |g(x, y, t, i) - g(x^*, y^*, t, i)|^2 \\ &\leq L_K(|x - x^*|^2 + |y - y^*|^2), \end{aligned}$$

where $L_K > 0$.

Assumption 3:

$$\lim_{|x|\to\infty} \inf_{t\geq 0, i\in S} V(x, t, i) = \infty,$$

$$\mathcal{L}V(x, y, t, i) \leq n(t) - \alpha_1 m_1(x) + \alpha_2 m_2(y)$$

where $V(x, t, i) \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$, $n \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, $m_1, m_2 \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}_+)$, $\alpha_1 > 0, \alpha_2 > 0$.

Definition 1: If

$$\mathbb{P}(\lim_{t\to\infty} x(t; \xi, r_0) = 0) = 1,$$

for $\forall \xi \in \mathcal{C}^b_{\mathscr{F}_0}([-\tau, 0); \mathbb{R}^n)$ and $r_0 \in \mathbb{S}$, the system is almost surely stability.

III. Main results and proofs

Theorem 1: Under the conditions of 1-3, the unique global solution $\{x(t), t \geq 0\}$ of system (1) exists.

Proof: Let $k > k_0$ where $k_0 > 0$ is large enough and $x_0 < k_0$. Define the stop time

$$\tau_k = \inf\{t \in [0, \tau_e) : x(t) \notin (\frac{1}{k}, k)\}, \quad (4)$$

where τ_e is the time of explosion.

Since the coefficients of system (1) satisfy the local Lipschitz conditions. Then, for $x_0 \in R$, the unique local solution $x(t)$ exists when $t \in [0, \tau_e)$. Let $\lim_{k\to\infty} \tau_k < \tau_e$ a.s. If $\tau_\infty = \infty$ a.s., then, for $\forall t \geq 0$, we obtain $x(t) \in R$. For constant $\gamma > 0$ and $0 < p < 1$, let

$$V_\gamma(x) = (x^2 + \gamma^2)^{\frac{p}{2}}, \quad x \in R. \quad (5)$$

According to Itô formula,

$$\begin{aligned} &V_\gamma(x(t\wedge\tau_k), t\wedge\tau_k, r(t\wedge\tau_k)) \\ &= V_\gamma(x_0, 0, r(0)) \\ &+ \int_0^{t\wedge\tau_k} LV_\gamma(x(s), s, r(s))ds + M_1(t\wedge\tau_k), \end{aligned} \quad (6)$$

where $0 \leq t \leq T$, $M_1(t\wedge\tau_k)$ is a local martingale and

$$\begin{aligned} &LV_\gamma = px(x^2+\gamma^2)^{\frac{p-2}{2}} f(x) \\ &+ \int_0^1 [V_\gamma(x+g(x)z) - V_\gamma(x) - V_\gamma^{'} g(x)z 1_{\{0\leq z\leq 1\}}] \quad (7) \\ &\frac{C_\alpha}{|z|^{\alpha+1}} dz. \quad (8) \end{aligned}$$

Then, we have

$$\begin{aligned} &\int_0^1 [V_\gamma(x+g(x)z) - V_\gamma(x) - V_\gamma^{'} g(x)z] \frac{C_\alpha}{|z|^{\alpha+1}} dz \\ &\leq sgn(g(x))|g(x)|^\alpha \\ &\int_0^{g(x)} [V_\gamma(x+g(x)z) - V_\gamma(x) - V_\gamma^{'} g(x)z] \\ &\times \frac{C_\alpha}{|g(x)z|^{\alpha+1}} d(g(x)z) \\ &= sgn(g(x))|x|^{\frac{\alpha}{2}} \int_0^{g(x)} V_\gamma^{''}(\xi) \frac{C_\alpha}{|g(x)z|^{\alpha+1}} d(g(x)z) \end{aligned}$$

,

where $\xi \in (x, x+g(x)z)$ and

$$V_\gamma^{''}(\xi) = p(\xi^2+\gamma^2)^{\frac{p-2}{2}}((p-1)\xi^2+\gamma^2). \quad (9)$$

It is obviously that

$$-\xi^2 - \gamma^2 \leq (p-1)\xi^2 + \gamma^2 \leq \xi^2 + \gamma^2. \quad (10)$$

Then, we obtain

$$|(p-1)\xi^2 + \gamma^2| \leq \xi^2 + \gamma^2. \quad (11)$$

Substituting (10) into (8), it follows that

$$V_\gamma^{''}(\xi) \leq p(\xi^2+\gamma^2)^{\frac{p}{2}-1} \leq p\xi^{p-2}. \quad (12)$$

Then, it can be checked that

$$\begin{aligned} &\int_0^1 [V_\gamma(x+g(x)z) - V_\gamma(x) - V_\gamma^{'} g(x)z] \frac{C_\alpha}{|z|^{\alpha+1}} dz \\ &\leq sgn(g(x))|g(x)|^\alpha \\ &\int_0^{g(x)} g^2(x)z^2 p||x| - |g(x)z||^{p-2} \\ &\frac{C_\alpha}{|g(x)z|^{\alpha+1}} d(g(x)z) \\ &\leq sgn(g(x)) \frac{C_\alpha p|\varphi(x)|^{p-2}}{2(2-\alpha)} 4x, \end{aligned} \quad (13)$$

where $\varphi(x) = \min(||x| - |g(x)z||)$.

When $|x| = |g(x)z|$, we obtain

$$\begin{aligned}
&sgn(g(x))|g(x)|^{\alpha} \\
&\int_0^{g(x)} g^2(x)z^2 V_\gamma^{''}(\xi)\frac{C_\alpha}{|g(x)z|^{\alpha+1}}d(g(x)z) \\
&\leq sgn(g(x))|g(x)|^{\alpha} \\
&\int_0^{g(x)} g^2(x)z^2 p(\xi^2+\gamma^2)^{\frac{p-2}{2}}\frac{C_\alpha}{|g(x)z|^{\alpha+1}}d(g(x)z) \\
&\leq sgn(g(x))|g(x)|^{\alpha} \\
&\int_0^{g(x)} g^2 z^2 p(||x|-|g(x)z||+\gamma^2)^{\frac{p-2}{2}} \\
&\times\frac{C_\alpha}{|g(x)z|^{\alpha+1}}d(g(x)z) \\
&\leq \frac{C_\alpha p|\gamma|^{p-2}}{2(2-\alpha)}4x.
\end{aligned}$$

Since $0 < p < 1$, we have

$$\lim_{\gamma\to\infty}\frac{C_\alpha p|\gamma|^{p-2}}{2(2-\alpha)}4x = 0. \tag{14}$$

Therefore, when γ is big enough, $\int_0^1 [V_\gamma(x+g(x)z) - V_\gamma(x) - V_\gamma^{'}g(x)z]\frac{C_\alpha}{|z|^{\alpha+1}}dz$ is convergent.

Let $f(\gamma) = V_\gamma(x+g(x)z) - V_\gamma(x)$, then we obtain

$$f'(\gamma) = p\gamma[((x+g(x)z)^2+\gamma^2)^{\frac{p-2}{2}} - (x^2+\gamma^2)^{\frac{p-2}{2}}].$$

Since $|x+g(x)z| > x$, we obtain $f'(\gamma) < 0$, then $f(\gamma) < f(0)$.

Thus,

$$\begin{aligned}
&|f(\gamma)| \\
&= |V_\gamma(x+g(x)z) - V_\gamma(x)| \\
&< |f(0)| = |(x+g(x)z)^p - x^p| < |g(x)z|^p.
\end{aligned} \tag{15}$$

Therefore,

$$\begin{aligned}
&\int_1^{+\infty}[V_\gamma(x+g(x)z) - V_\gamma(x)]\frac{C_\alpha}{|z|^{\alpha+1}}dz \\
&\leq sgn(g(x))|g(x)|^{\alpha} && (16)\\
&\int_{g(x)}^{+\infty}|V_\gamma(x+g(x)z) - V_\gamma(x)|\frac{C_\alpha}{|g(x)z|^{\alpha+1}}d(g(x)z) \\
&\leq sgn(g(x))|g(x)|^{\alpha} && (17)\\
&\int_{g(x)}^{+\infty}|x+g(x)z|^p\frac{C_\alpha}{|g(x)z|^{\alpha+1}}d(g(x)z) \\
&= sgn(g(x))|g(x)|^{\alpha+p}\frac{C_\alpha}{\alpha-p}. && (18)
\end{aligned}$$

When $x \in [0, 1)$, substituting (12) and (15) into (7), we get

$$\begin{aligned}
&LV_\gamma(x) \\
&\leq px(x^2+\gamma^2)^{\frac{p-2}{2}}(f(x)) \\
&+sgn(g(x))\frac{C_\alpha p|\varphi(x)|^{p-2}}{2(2-\alpha)}4x \\
&+sgn(g(x))|g(x)|^p\frac{C_\alpha}{\alpha-p} \\
&= px(x^2+\gamma^2)^{\frac{p-2}{2}}(f(x)) \\
&+sgn(g(x))\frac{C_\alpha}{2(2-\alpha)}|g(x)|^p(\frac{|g(x)|}{|\varphi(x)|})^{2-p} \\
&+sgn(g(x))\frac{C_\alpha}{\alpha-p}|g(x)|^p \\
&= px(x^2+\gamma^2)^{\frac{p-2}{2}}(f(x)) \\
&+sgn(g(x))C_\alpha 2^p x^{\frac{p}{2}}(\frac{2^{1-p}}{2-\alpha}(\frac{|g(x)|}{|\varphi(x)|})^{2-p} + \frac{1}{\alpha-p}).
\end{aligned}$$

It is assumed that $C_\alpha(\frac{2^{1-p}}{2-\alpha}(\frac{|g(x)|}{|\varphi(x)|})^{2-p} + \frac{1}{\alpha-p}) \leq K(x^2+\gamma^2)^{\frac{p-2}{2}}$ and $px(1+2\theta x) + K_1 2^p x^{\frac{p}{2}} \leq K_2(x^2+\gamma^2)$.

Then,

$$\begin{aligned}
&LV_\gamma(x) \\
&\leq px(x^2+\gamma^2)^{\frac{p-2}{2}}(f(x)) \\
&+Ksgn(g(x))2^p x^{\frac{p}{2}}(x^2+\gamma^2)^{\frac{p-2}{2}} \\
&\leq (x^2+\gamma^2)^{\frac{p-2}{2}}(px(f(x)) + K_1 2^p x^{\frac{p}{2}}) \\
&\leq (x^2+\gamma^2)^{\frac{p-2}{2}}K_2(x^2+\gamma^2) \\
&= K_2 V_\gamma(x).
\end{aligned}$$

When $x = 0$, $V_\gamma(x) = V_\gamma(0) = \gamma^p$. Then, $V_\gamma^{'}(x) = 0$ and

$$LV_\gamma(x) \leq K_2\gamma^p.$$

According to Itô formula, as $x(t\wedge\tau_k) \in R$, we have

$$\begin{aligned}
&\mathbb{E}V_\gamma(x(T\wedge\tau_k), T\wedge\tau_k, r(T\wedge\tau_k)) \\
&= V_\gamma(x_0, 0, r(0)) + \mathbb{E}\int_0^{T\wedge\tau_k} LV_\gamma(x(s), s, r(s))ds,
\end{aligned} \tag{19}$$

where $0 \leq t \leq T$.

Then, we obtain

$$\begin{aligned}
&\mathbb{E}V_\gamma(x(T\wedge\tau_k), T\wedge\tau_k, r(T\wedge\tau_k)) \\
&\leq V_\gamma(x_0, 0, r(0)) + K_2\gamma^p\mathbb{E}(T\wedge\tau_k) \\
&\leq V_\gamma(x_0, 0, r(0)) + K_2\gamma^p T.
\end{aligned}$$

Note that for all $\delta \in \{\tau_k \leq T\}$, there exists a constant k large enough satisfying $x(\tau_k, \delta) \geq k$ or $x(\tau_k, \delta) \leq \frac{1}{k}$.

Hence,

$$V_\gamma(x(\tau_k, \delta)) \geq (k^2+\gamma^2)^{\frac{p}{2}} \wedge (\frac{1}{k^2}+\gamma^2)^{\frac{p}{2}}. \tag{20}$$

Therefore,

$$\begin{aligned}
&(k^2+\gamma^2)^{\frac{p}{2}} \wedge (\frac{1}{k^2}+\gamma^2)^{\frac{p}{2}}P(\tau_k \leq T) \\
&\leq \mathbb{E}(V_\gamma(x(\tau_k, \delta))1_{\tau_k\leq T}) \leq V_\gamma(x_0) + K_2\gamma^p T.
\end{aligned} \tag{21}$$

Let $k \to \infty$, we obtain

$$P(\tau_\infty \leq T) = 0. \tag{22}$$

Then,

$$P(\tau_\infty = \infty) = 1. \tag{23}$$

The unique global solution exists. ■

Theorem 2: Under the conditions of 1-3, for $\forall i \in \mathbb{S}$, if there exists function $V \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$, $n \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, $m_1, m_2 \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}_+)$, $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$ satisfy

$$\mathcal{L}V(x, y, t, i) \le n(t) - \alpha_1 m_1(x) + \alpha_2 m_2(y),$$

$$m_1(x) > m_2(x), x \neq 0,$$

$$\lim_{|x|\to\infty} \inf_{t\ge 0, i\in S} V(x, t, i) = \infty,$$

the system (1) is almost surely stability.

Proof: Since

$$\begin{aligned}
&V(x(t), t, r(t)) \\
&= V(\xi(0), 0, r_0) \\
&+ \int_0^t \mathcal{L}V(x(s), x(s-\tau(s)), s, r(s))ds \\
&+ \int_0^t V_x(x(s), s, r(s))g(x(s), x(s-\tau(s), s, r(s))dZ(s) \\
&\le V(\xi(0), 0, r_0) + \int_0^t n(s)ds - \int_0^t \alpha_1 m_1(x(s))ds \\
&+ \int_0^t \alpha_2 m_2(x(s-\tau(s)))ds \\
&+ \int_0^t V_x(x(s), s, r(s))g(x(s), x(s-\tau(s), s, r(s))dZ(s) \\
&\le V(\xi(0), 0, r_0) + \int_0^t n(s)ds - \alpha_1 \int_0^t m_1(x(s))ds \\
&+ \alpha_1 \int_{-\tau}^0 m_2(x(s))ds \\
&+ \int_0^t V_x(x(s), s, r(s))g(x(s), x(s-\tau(s), s, r(s))dZ(s).
\end{aligned}$$

As

$$\int_0^\infty n(s)ds < \infty, \tag{24}$$

we obtain

$$\lim_{t\to\infty} \int_0^t m_1(x(s))ds < \infty \tag{25}$$

and

$$\lim_{t\to\infty} \sup V(x(t), t, r(t)) < \infty. \tag{26}$$

Then, we obtain

$$\sup_{0\le t<\infty} \inf_{|x|\ge|x(t)|, 0\le t<\infty, i\in S} V(x, t, i) < \infty. \tag{27}$$

and

$$\sup_{0\le t<\infty} |x(t)| < \infty. \tag{28}$$

Since $\xi \in \mathcal{C}^b_{\mathscr{F}_0}([-\tau, 0); \mathbb{R}^n)$, there exists a positive k_0 and $|\xi| < k_0$. For $k > k_0$, we define stopping time

$$\varsigma_k = \inf\{t \ge 0 : |x(t)| \ge k\}, \tag{29}$$

where $\inf \phi = \infty$.

It is obviously that when $k \to \infty$, $\varsigma_k \to \infty$ $a.s.$.

Thus, for any $\varepsilon > 0$, there exists $k_\varepsilon \ge k_0$ and when $k \ge k_\varepsilon$,

$$\mathbb{P}(\varsigma_k < \infty) \le \varepsilon. \tag{30}$$

According to (9), we have

$$\lim_{t\to\infty} \inf m_1(x(t)) = 0. \tag{31}$$

Next, the following results will be proved:

$$\lim_{t\to\infty} m_1(x(t)) = 0. \tag{32}$$

Suppose (29) dose not hold, we can obtain

$$\mathbb{P}\{\lim_{t\to\infty} \sup m_1(x(t) > 0\} > 0. \tag{33}$$

Then, there exists the following stopping time sequence:

$$\eta_1 = \inf\{t \ge 0 : m_1(x(t) \ge 2\varepsilon_1\},$$

$$\eta_{2j} = \inf\{t \ge \eta_{2j-1} : m_1(x(t) \le \varepsilon_1\}, \quad j = 1, 2, \cdots,$$

$$\eta_{2j+1} = \inf\{t \ge \eta_{2j} : m_1(x(t) \ge 2\varepsilon_1\}, \quad j = 1, 2, \cdots,$$

and $\varepsilon_0 > 0$, $\varepsilon > \varepsilon_1 > 0$ satisfy

$$\mathbb{P}(\varsigma_{2j} < \infty : j \in \mathrm{Z}) \ge \varepsilon_0. \tag{34}$$

According to local Lipschitz condition, $\forall k > 0$, there exists $L_k > 0$ satisfy

$$|f(x, y, t, i)| \vee |g(x, y, t, i)| \vee |H(x, y, t, i, \nu)| \le L_k,$$

for any $t \ge 0$, $i \in S$ and $|x| \vee |y| \le k$.

For any $j \in Z$ and the indicative function $\mathbb{I}_A$, when $T < \eta_{2j} - \eta_{2j-1}$, we obtain

$$\begin{aligned}
&\mathbb{E}[\mathbb{I}_{\{\eta_{2j}<\eta_k\}} \sup_{0\le t\le T} |x(\eta_{2j-1}+t) - x(\eta_{2j-1})|^2] \\
&= \mathbb{E}[\mathbb{I}_{\{\eta_{2j}<\eta_k\}} \\
&\sup_{0\le t\le T} |\int_{\eta_{2j-1}}^{\eta_{2j-1}+t} f(x(s), x(s-\tau(s)), s, r(s))ds \\
&+ \int_{\eta_{2j-1}}^{\eta_{2j-1}+t} g(x(s), x(s-\tau(s)), s, r(s))dZ(s)|^2 \\
&\le 4\mathbb{E}[\mathbb{I}_{\{\eta_{2j}<\eta_k\}} \\
&\sup_{0\le t\le T} |\int_{\eta_{2j-1}}^{\eta_{2j-1}+t} f(x(s), x(s-\tau(s)), s, r(s))ds|^2] \\
&+ 4\mathbb{E}[\mathbb{I}_{\{\eta_{2j}<\eta_k\}} \\
&\sup_{0\le t\le T} (\int_{\eta_{2j-1}}^{\eta_{2j-1}+t} |g(x(s), x(s-\tau(s)), s, r(s))|^{2\alpha} ds)^{\frac{1}{\alpha}} \\
&\le 4L_k^2 T^2 + 4L_k^2 T^{\frac{1}{\alpha}},
\end{aligned}$$

Since $m_1(x)$ is uniformly continuous in $\overline{S}_k = \{x \in \mathbb{R}^n : |x| \le k\}$. Then, $\forall p > 0$, when $x, y \in \overline{S}_k$ and $|x - y| < c_p$, $|m_1(x) - m_1(y)| < p$. Let $\varepsilon = \frac{\varepsilon_0}{3}$, $k \ge k_\varepsilon$ and $p = \varepsilon_1$.

We obtain

$$\begin{aligned}
&\mathbb{P}(\{\varsigma_k \le \eta_{2j}\}) + \mathbb{P}(\{\eta_{2j} < \varsigma_k\} \\
&\cap \{\sup_{0\le t\le T} |n_1(x(\eta_{2j-1}+t)) - n_1(x(\eta_{2j-1}))| \ge \varepsilon_1\}) \\
&\le \mathbb{P}(\{\eta_k \le \eta_{2j}\} \cap \{\eta_{2j} = \infty\}) \\
&+ \mathbb{P}(\{\eta_k \le \eta_{2j}\} \cap \{\eta_{2j} < \infty\}) \\
&+ \mathbb{P}(\{\eta_{2j} < \eta_k\} \\
&\cap \{\sup_{0\le t\le T} |x(\eta_{2j-1}+t) - x(\eta_{2j-1})| \ge c_{\eta_1}\}) \\
&\le \frac{4L_k^2 T^2 + 4L_k^2 T^{\frac{1}{\alpha}}}{c_{\eta_1}^2} + 1 - 2\varepsilon.
\end{aligned}$$

Let $T = T(\varepsilon, \varepsilon_1, k)$ is small enough to satisfy

$$\frac{4L_k^2T^2 + 4L_k^2T^{\frac{1}{\alpha}}}{c_{\varepsilon_1}^2} \leq \varepsilon. \tag{35}$$

Then, it can be checked that

$$\begin{aligned}&\mathbb{P}(\{\eta_{2j} < \varsigma_k\} \\ &\cap\{\sup_{0\leq t\leq T} |n_1(x(\eta_{2j-1}+t)) - n_1(x(\eta_{2j-1}))| < \varepsilon_1\}) \\ &\geq \varepsilon.\end{aligned} \tag{36}$$

Hence, we obtain

$$\begin{aligned}&\Sigma_{j=1}^{\infty} T\varepsilon_1\varepsilon = \frac{1}{2}\Sigma_{j=1}^{\infty} T\varepsilon_0\varepsilon_1 = \infty \\ &\leq \Sigma_{j=1}^{\infty} T\varepsilon_1 \mathbb{P}(\{\eta_{2j} < \varsigma_k\} \\ &\cap\{\sup_{0\leq t\leq T} |n_1(x(\eta_{2j-1}+t)) - n_1(x(\eta_{2j-1}))| < \varepsilon_1\}) \\ &\leq \Sigma_{j=1}^{\infty} \varepsilon_1 \mathbb{E}[\mathbb{I}_{\eta_{2j}<\varsigma_k}(\eta_{2j} - \eta_{2j-1})] \\ &\leq \Sigma_{j=1}^{\infty} \varepsilon_1 \mathbb{E}[\mathbb{I}_{\eta_{2j}<\eta_k} \int_{\eta_{2j-1}}^{\eta_{2j-1}+t} n_1(x(t))dt] \\ &\leq \mathbb{E}[\int_0^{\infty} n_1(x(t))dt] \\ &< \infty.\end{aligned}$$

Obviously, the above result is contradictory. Then, there exists $\overline{\Omega} \in \Omega$ such that $\mathbb{P}(\overline{\Omega}) = 1$ and

$$\lim_{t\to\infty} m_1(x(t,\omega)) = 0, \quad \sup_{0\leq t<\infty} |x(t,\omega)| < \infty, \quad \forall\omega \in \overline{\Omega}. \tag{37}$$

Therefore, for $\forall\omega \in \overline{\Omega}$, $\{x(t,\omega)\}_{t\geq 0} \in \mathbb{R}^n$ is bounded. There exists a increasing sequence $\{t_i\}_{i\geq 1}$ such that $\{x(t_i,\omega)\}_{i\geq 1}$ is convergent. Since $m_1(x) > 0$ as $x \neq 0$, it is known that $n_1(x) = 0$ when $x = 0$.

Therefore, the solution of system is almost surely stability. ■

IV. Example

Let $Z(t)$ be a one-dimensional α-stable motion with $\alpha = 1.5$, $r(t) \in \mathbb{S} = \{1, 2\}$ and $\Gamma = (\gamma_{ij})_{2\times 2} =$

$$\begin{pmatrix} -0.6 & 0.6 \\ 0.3 & -0.3 \end{pmatrix}$$

Consider the nonlinear delay hybrid stochastic system as follows:

$$\begin{aligned}&dx(t) = f(x(t), x(t-\tau(t)), t, r(t))dt \\ &+g(x(t), x(t-\tau(t)), t, r(t))dZ(t),\end{aligned}$$

where

$$f(x,y,t,1) = -2x^{\frac{1}{4}} + y^{\frac{3}{4}},$$

$$g(x,y,t,1) = -x^{\frac{3}{4}} + 2y^{\frac{3}{4}},$$

$$f(x,y,t,2) = (1+t)^{-\frac{1}{4}} - x^{\frac{1}{4}},$$

$$g(x,y,t,2) = 3x^{\frac{3}{4}}\sin(t) + \frac{4}{3}y^{\frac{3}{4}}\cos(t),$$

where $\tau(t) = 0.3 + 0.3\cos(t)$.

Let $V(x,i) = x^2$. Then, we obtain

$$\mathcal{L}V(x,y,t,1) \leq -4x^{\frac{3}{2}} + 4y^{\frac{3}{2}},$$

$$\mathcal{L}V(x,y,t,2) \leq 3x(1+t)^{-\frac{1}{2}} - x^{\frac{3}{2}} + \frac{16}{9}y^{\frac{3}{2}}.$$

Since for any $\kappa > 0$,

$$\begin{aligned}&3x(1+t)^{-\frac{1}{2}} \\ &= (\frac{3}{2}\kappa x^{\frac{3}{2}})^{\frac{2}{3}}(3(\frac{\kappa}{2})^{-2}(1+t)^{-\frac{3}{2}})^{\frac{1}{3}} \\ &\leq \kappa x^{\frac{3}{2}} + (\frac{\kappa}{2})^{-2}(1+t)^{-\frac{3}{2}}.\end{aligned}$$

Thus, for all $t \geq 0$, $i \in S$, it is easy to check that

$$\mathcal{L}V(x,y,t,i) \leq (\frac{\kappa}{2})^{-2}(1+t)^{-\frac{3}{2}} - (4-\kappa)x^{\frac{3}{2}} + 4y^{\frac{3}{2}}.$$

Therefore, the solution of system is almost surely stability.

Remark 1: If the system is driven by Lévy noise, let $W(t)$ be a one-dimensional Brownian motion, The character measure π of Poisson jump satisfies $\pi(d\nu) = \lambda\phi(d\nu)$, where $\lambda = 1.6$ is the intensity of Poisson distribution and ϕ is the probability intensity of the standard normal distributed variable ν, $r(t) \in \mathbb{S} = \{1, 2\}$ and $\Gamma = (\gamma_{ij})_{2\times 2} =$

$$\begin{pmatrix} -0.5 & 0.5 \\ 0.2 & -0.2 \end{pmatrix}$$

Consider the nonlinear delay hybrid stochastic system driven by Lévy noises as follows:

$$\begin{aligned}&dx(t) = f(x(t), x(t-\tau(t)), t, r(t))dt \\ &+g(x(t), x(t-\tau(t)), t, r(t))dW(t) \\ &+\int_Z H(x(t-), x(t-\tau(t)), t, r(t-), \nu)N(dt, d\nu),\end{aligned}$$

where

$$f(x,y,t,1) = -5x^{\frac{1}{5}} + 3y^{\frac{4}{5}},$$

$$g(x,y,t,1) = -x^{\frac{1}{3}} + y^{\frac{1}{3}},$$

$$f(x,y,t,2) = 2(1+t)^{-\frac{1}{5}} - 2x^{\frac{1}{5}},$$

$$g(x,y,t,2) = 2x^{\frac{4}{5}}\cos(t) + \frac{5}{4}y^{\frac{4}{5}}\sin(t),$$

$$H(x,y,t,1,\nu) = -2x^{\frac{1}{5}} + 2y^{\frac{4}{5}},$$

$$H(x,y,t,2,\nu) = 3x^{\frac{1}{5}} + y^{\frac{4}{5}},$$

where $\tau(t) = 0.2 + 0.2\sin(t)$.

Let $V(x,i) = x^2$. Then, we obtain

$$\mathcal{L}V(x,y,t,1) \leq -25x^{\frac{6}{5}} + 5y^{\frac{6}{5}},$$

$$\mathcal{L}V(x,y,t,2) \leq 4x(1+t)^{-\frac{1}{5}} - 4x^{\frac{6}{5}} + \frac{25}{16}y^{\frac{6}{5}}.$$

Since for any $\kappa > 0$,

$$\begin{aligned}&4x(1+t)^{-\frac{1}{5}} \\ &= (\frac{6}{5}\kappa x^{\frac{6}{5}})^{\frac{5}{6}}(6(\frac{\kappa}{5})^{-5}(1+t)^{-\frac{6}{5}})^{\frac{1}{6}} \\ &\leq \kappa x^{\frac{6}{5}} + (\frac{\kappa}{5})^{-5}(1+t)^{-\frac{6}{5}}.\end{aligned}$$

Thus, for all $t \geq 0$, $i \in S$, it is easy to check that

$$\mathcal{L}V(x,y,t,i) \leq (\frac{\kappa}{5})^{-5}(1+t)^{-\frac{6}{5}} - (6-\kappa)x^{\frac{6}{5}} + 5y^{\frac{6}{5}}.$$

V. CONCLUSION

This article has studied the existence, stability of unique global solution of nonlinear delay hybrid urban traffic network with α-stable noises. The existence, uniqueness and stability of the solution for urban traffic network have been analyzed by general Itô formula, Doob martingale inequality and Bolzano-Weierstrass. We will consider the stability of nonlinear hybrid system driven by fractional Lévy noise in further research topics.

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