On the Global Convergence of a New Super Halley's Family for Solving Nonlinear Equations

Mohammed Barrada, Reda Benkhouya, *Member, IAENG,* Mohammed Lahmer, and Idriss Chana

***Abstract*—In this work, we derive a one-parameter family of Super Halley's method for finding simple roots of nonlinear equations. The scheme is powerful since it regenerates an infinity interesting methods. The convergence analysis shows that the order of convergence of each method of the proposed family is at least three. The originality of the new family manifests in the fact that all these methods are governed by a recurring formula that depends on a natural integer parameter p. Moreover, under certain conditions, the convergence speed of these methods improves by increasing p. A fairly detailed study on their global convergence is carried out. To illustrate the abilities and performances of proposed family, numerical comparisons have been made with several other existing third order and higher order methods.**

***Index Terms*—Nonlinear equations, One-parameter family, Iterative methods, Order of convergence, Third order method, Super Halley's method**

I. Introduction

The design of iterative formulas for solving nonlinear equations is a very important and interesting task in engineering, scientific computing and applied mathematics in general [1], [2]. In this research, we are interested in finding simple roots of a nonlinear equation:

$$f(x) = 0 \tag{1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I is a scalar function. The zero α of f, assumed simple, can be determined as a fixed point of some Iteration Function (I.F.) by means of the one-point iteration method [3]–[12]:

$$x_{n+1} = F(x_n) \qquad for \quad n = 0, 1, 2, \cdots \tag{2}$$

where x_0 is starting value. A point α is called a fixed point of F if $F(\alpha) = \alpha$. The convergence of the sequence (x_n) to the root α can be guaranteed under certain conditions and by making a good choice of iterative function F.

The best known iterative method for determining a solution for this problem is Newton's method [13] given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, \ldots \tag{3}$$

Manuscript received December 24, 2020; revised June 30, 2021

Mohammed Barrada is a Professor of Ecole Supérieure de Technologie, and researcher with ISIC, LMMI ENSAM, Moulay Ismail University, Morocco (e-mail: barrada.med@gmail.com)

Reda Benkhouya is a Researcher of Faculty of Sciences, MISC, Ibn Tofail University, Morocco (phone: +212661666116; e-mail: reda.benkhouya@gmail.com)

Mohamed Lahmer is a Professor of Ecole Supérieure de Technologie, and researcher with ISIC, LMMI ENSAM, Moulay Ismail University, Morocco (e-mail: mohammed.lahmer@gmail.com)

Idriss Chana is a Professor of Ecole Supérieure de Technologie, and researcher with IMAGE, LMMI ENSAM, Moulay Ismail University, Morocco (e-mail: idrisschana@gmail.com)

a special case of (2) with $F(x) = x - \frac{f(x)}{f'(x)}$. This method converges quadratically to the simple root α, if x_0 is sufficiently close to α.

Recently, some new methods, with cubic convergence, have been developed. For example, Halley [5], [13]–[22], Chebyshev [1], [13], [19], [22]–[24], Hansen-Patrick [25], Ostrowski [24], Chun [26], Sharma [27]–[29], Jiang-Han [30], Barrada et al. [20], [31]–[33], Amat [19], Traub [13], Kou, Li and Wang [34], Chun and Neta [35], Torres et al. [36] have proposed some interesting and well-known methods. Among the methods, of order 3, most known in literature, we cite in particular Super-Halley's method [18], [19], [26], [27], [30], [35] given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} V_0(L_n) \tag{4}$$

where $V_0(L_n) = \dfrac{2 - L_n}{2(1 - L_n)}$

and $L_n = L_f(x_n) = \dfrac{f(x_n)f^{''}(x_n)}{f'(x_n)^2}$

A special case of (2) with I.F. :

$$F_0(x) = x - \frac{f(x)}{f'(x)} \left(\frac{2 - L_f(x)}{2(1 - L_f(x))} \right)$$

.

On the other note, several researches have been carried out with the aim to create multi-step iterative methods with improved convergence order. Fang et al. [37], Torres et al. [36] have constructed Some fifth-order convergent iterative methods. Wang and Zhang [38], Kou et al. [34], [39], Chun and Ham [40] have developed some families of sixth-order methods. Bi W. et al. [41] introduced some families of eighth-order convergence methods.

In articles [20], [31]–[33], we proposed some interesting new family of Halley's method and Chebyshev's method. In this paper, based on the Super Halley's method and second-order Taylor polynomial, we will construct a new family for finding simple roots of nonlinear equations with cubical convergence. The main characteristics of this family are that, on one hand, its methods can be derived from each other from a recurrent formula which depends on a natural integer parameter p and, on the other hand, under certain hypothesis, the speed of convergence of these methods improves by increasing p. The efficiency of this method will be tested on a number of numerical examples. A comparison with third, five and sixth order methods will be realized.

II. Derivation of new iterative process

Newton's method is derivate by approximating the given function f at $x = x_n$ by the tangent line

$$y(x) = f(x_n) + f^{'}(x_n)(x - x_n)$$

to the graph of f at $(x_n, f(x_n))$. By solving $y(x_{n+1}) = 0$ for x_{n+1}, we find the sequence (3).

The linear approximation in Newton's method is simply the first-degree Taylor polynomial of f at x_n. Now let's use a second degree polynomial:

$$y(x) = f(x_n) + f^{'}(x_n)(x - x_n) + \frac{f^{''}(x_n)}{2}(x - x_n)^2 \quad (5)$$

Where x_n is again an approximate solution of $f(x) = 0$. The goal is to calculate a point $(x_{n+1}, 0)$ where the graph of y intersects the x-axis, that is, to solve of following equation for x_{n+1}:

$$0 = f(x_n)+f^{'}(x_n)(x_{n+1}-x_n)+\frac{f^{''}(x_n)}{2}(x_{n+1}-x_n)^2 \quad (6)$$

by replacing $(x_{n+1} - x_n)$ located on the right-hand side of (6) by Super Halley's correction given in (4), we get :

$$0 = f(x_n) + f^{'}(x_n)(x_{n+1} - x_n) + \frac{f^{''}(x_n)}{2}\left(-\frac{f(x_n)}{2f^{'}(x_n)}\left(\frac{2 - L_n}{1 - L_n}\right)\right)^2 \quad (7)$$

From which it follows that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}V_1(L_n) \quad (8)$$

where $V_1(L_n) = 1 + \frac{L_n}{2}V_0^2(L_n) = \dfrac{L_n^3 + 4L_n^2 - 12L_n + 8}{8(1 - L_n)^2}$

By repeating the above procedure p times and each time replace $(x_{n+1} - x_n)$ located on the right side of (6) with the last method found, we derive the following general family of Super Halley's method (Bp):

$$\begin{cases} x_{n+1} = x_n - V_p(L_n)\dfrac{f(x_n)}{f'(x_n)} \\ V_{p+1}(x) = 1 + \dfrac{x}{2}V_p^2(x) \qquad\qquad n = 0, 1, 2, \dots \\ V_0(x) = \dfrac{2 - x}{2(1 - x)} \end{cases} \quad (9)$$

where p is a non-zero natural integer parameter.

The iterative process (9), noted (Bp), represents a general family of Super Halley's method for finding simple roots of nonlinear equations. It is a special case of (2) with following (I.F.) :

$$F_p(x) = x - \frac{f(x)}{f'(x)}.V_p(L_f(x)) \quad (10)$$

The scheme (9) is powerful because it regenerates the Super-Halley method (B0), and several new methods such as (B1) , given by (8), and (B2) given by:

$$x_{n+1} = x_n - V_2(L_n)\frac{f(x_n)}{f'(x_n)}$$

Where V_2 is given by:

$$V_2(L_n) = \frac{L_n^7 + 8L_n^6 - 8L_n^5 + 48L_n^4 - 304L_n^3 + 576L_n^2 - 448L_n + 128}{128(1 - L_n)^4}$$

III. Analysis of convergence

A. Order of convergence

The order of convergence of sequence (9) is given by the following theorem.

Theorem 1. *Let p be a parameter where p is a non-negative integer. We Suppose that the function f has at least two continuous derivatives in the neighborhood of a zero, α. Further, we assume that $f^{'}(\alpha) \neq 0$ and x_0 is sufficiently close to α. Then, the sequences* (9), *converge cubically to α, for any natural integer parameter p, and satisfy the error equation*

$$e_{n+1} = -\frac{f^{(3)}(\alpha)}{3!f^{'}(\alpha)}e_n^3 + \mathcal{O}(e_n^4) \quad (11)$$

where $e_n = x_n - \alpha$ is the error at n^{th} iteration

Proof: Let α be a simple root, i.e. $f(\alpha) = 0$ and $f^{'}(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$. We use the following Taylor expansions about α :

$$\begin{cases} f(x_n) = f^{'}(a)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \mathcal{O}(e_n^5)] \\ f^{'}(x_n) = f^{'}(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \mathcal{O}(e_n^4)] \\ f''(x_n) = f^{'}(\alpha)[2c_2 + 6c_3e_n + 12c_4e_n^2 + \mathcal{O}(e_n^3)] \end{cases} \quad (12)$$

$$\text{where} \qquad c_i = \frac{f^{(i)}(\alpha)}{i!f^{'}(\alpha)}, \qquad i = 2, 3, \dots \quad (13)$$

Using (12) we get

$$\begin{cases} [f^{'}(x_n)]^2 = [f^{'}(\alpha)]^2[1 + 4c_2e_n + 2(2c_2^2 + 3c_3)e_n^2 \\ \qquad + 4(3c_2c_3 + 2c_4)e_n^3 + \mathcal{O}(e_n^4)] \\ \dfrac{f(x_n)}{f^{'}(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + \mathcal{O}(e_n^4) \end{cases} \quad (14)$$

and

$$\begin{aligned} L_n = \frac{f(x_n)f^{''}(x_n)}{[f^{'}(x_n)]^2} &= 2c_2e_n - 6(c_2^2 - c_3)e_n^2 \\ &\quad +4(4c_2^3 - 7c_2c_3 \\ &\quad +3c_4)e_n^3 + \mathcal{O}(e_n^4) \end{aligned} \quad (15)$$

Using the Taylor's series expansion [29] of $V_p(L_n)$ about $L(\alpha)$ leads to

$$\begin{aligned} V_p(L_n) = V_p(L(\alpha)) &+ (L_n - L(\alpha))V_p^{'}(L(\alpha)) \\ &+\frac{1}{2}(L_n - L(\alpha))^2V_p^{''}(L(\alpha)) \\ &+\mathcal{O}\left((L_n - L(\alpha))^3\right) \end{aligned}$$

Where p is a natural integer parameter.
Taking into account that $L(\alpha) = 0$, we obtain

$$V_p(L_n) = V_p(0) + L_nV_p^{'}(0) + \frac{1}{2}L_n^2V_p^{''}(0) + \mathcal{O}\left(L_n^3\right) \quad (16)$$

We have: $V_0(x) = \frac{2-x}{2(1-x)}$ *and* $V_{p+1}(x) = 1 + \frac{x}{2}V_p^2(x)$

We obtain

$$\begin{cases} V_0^{'}(x) = \dfrac{1}{2(1 - x)^2} \\ V_{p+1}^{'}(x) = \dfrac{1}{2}V_p^2(x) + x.V_p(x)V_p^{'}(x) \\ V_0^{''}(x) = \dfrac{1}{(1 - x)^3}, \\ V_{p+1}^{''}(x) = 2V_p(x)V_p^{'}(x) + x\left(V_p^{'2}(x) + V_p(x)V_p^{''}(x)\right) \end{cases} \quad (17)$$

It is easy to prove that function V_p check following conditions:

$$For\,all\,p \in \mathbb{N}, V_p(0) = 1, V_p^{'}(0) = \frac{1}{2}\;\; and\;\; V_p^{''}(0) = 1 \quad (18)$$

Thus, the Formula (16) becomes

$$For\,all\,p \in \mathbb{N}, \quad V_p(L_n) = 1 + \frac{1}{2}L_n + \frac{1}{2}L_n^2 + \mathcal{O}(L_n^3) \quad (19)$$

Using (15), we get

$$\begin{aligned} &For\;\; all\;\; p \in \mathbb{N}, \\ &V_p(L_n) = 1 + c_2 e_n + [-c_2^2 + 3c_3]e_n^2 + \mathcal{O}(e_n^3) \end{aligned} \quad (20)$$

Substituting (14) and (20) in formula (9), we obtain the error equation

$$e_{n+1} = -c_3 e_n^3 + \mathcal{O}(e_n^4)$$

which completes the proof of the theorem. ■

B. Global Convergence of the super Halley's family

We will make a first study of the global convergence of some selected methods from the proposed family (Bp), in the case where they converge towards the root in a monotone way [3], [27], [42]–[44]. But before, we give two elementary lemmas, which will be used to this study.

Lemma 1. *Let us write the iterative function of f, from the sentences (Bp) :*

$$F_p(x) = x - \frac{f(x)}{f^{'}(x)}.V_p(L_f(x))$$

Then, the derivative of F_p is given by:

$$\begin{aligned} F_p^{'}(x) = 1 - L_f(x)[1 + L_f(x)(L_{f^{'}}(x) - 2)]V_p^{'}(L_f(x)) \\ -V_p(L_f(x))(1 - L_f(x)) \end{aligned} \quad (21)$$

Lemma 2. *Let x a real number such as $0 \leqslant x < 1$ and (a_p) the sequence defined by:*

$$a_0 = \frac{2-x}{2(1-x)}, \quad a_{p+1} = 1 + \frac{x}{2}a_p^2, \qquad for \quad p = 0, 1, 2 \ldots$$

then (a_p) is an increasing sequence with strictly positive terms.

Proof: As $0 \leqslant x < 1$, it is easy to prove by induction that $a_p > 0$ for all $p \in \mathbb{N}$.

Let us show by induction that (a_p) is increasing sequence, for a given p. We have:

$a_1 - a_0 = \dfrac{x^3}{8(1-x)^2}$. As $x \geqslant 0$, then $a_1 \geqslant a_0$. Now we assume that for an integer p, we have $a_{p+1} \geqslant a_p$. Since $a_p > 0$ and $a_{p+1} > 0$, then $a_{p+1}^2 \geqslant a_p^2$, and as $x \geqslant 0$, we deduce that $a_{p+2} \geqslant a_{p+1}$ and the induction is completed. ■

C. Monotonic Convergence of the Sequences (Bp)

Theorem 2. *Let $p \in \mathbb{N}$, $f \in C^m[a\,,b], m \geqslant 4, f^{'} \neq 0$, $f^{''} \neq 0$, $0 \leqslant L_f < 1$ and the iterative function F_p of f, defined by* (10)*, is increasing function on an interval $[a\,,b]$ containing the root α of f. Then the sequence given by (9) is decreasing (resp. increasing) and converges to α from any point $x_0 \in [a\,,b]$ checking $f(x_0)f^{'}(x_0) > 0$ (resp. $f(x_0)f^{'}(x_0) < 0$)*

Proof: Let us consider the case where $f(x_0)f^{'}(x_0) > 0$, then $x_0 > \alpha$. Applying Mean Value Theorem to the function F_p, where p is a natural integer, we obtain:

$$x_1 - \alpha = F_p(x_0) - F_p(\alpha) = F_p^{'}(s)(x_0 - \alpha)$$

for some $s \in (\alpha\,, x_0)$. As F_p is an increasing function on $[a\,,b]$, then derivative of F_p given by (21) checks $F_p^{'}(x) \geqslant 0$ in $[\alpha\,,b]$, we deduce that $x_1 \geqslant \alpha$. By induction, we obtain $x_n \geqslant \alpha$ for all $n \in \mathbb{N}$.

On the other note, according to (9), we have:

$$x_1 - x_0 = -V_p(L_0)\frac{f(x_0)}{f^{'}(x_0)}$$

As $0 \leqslant L_0 < 1$ then, from Lemma 2, we have :

$$a_p = V_p(L_0) > 0 \;\; for \;\; all \;\; p \in \mathbb{N}$$

Since $\frac{f(x_0)}{f^{'}(x_0)} > 0$, we deduce that $x_1 \leqslant x_0$. Now it is easy to prove by induction that $x_{n+1} \leqslant x_n$ for all $n \in \mathbb{N}$.

Thereby, the sequence (9) is decreasing and converges to a limit $\lambda \in [a\,,b]$ where $\lambda \geqslant \alpha$. So, by taking the limit in (9) we obtain:

$$\lambda = \lambda - \frac{f(\lambda)}{f^{'}(\lambda)}V_p(L_f(\lambda))$$

We have $V_p(L_f(\lambda)) > 0$ for all $p \in \mathbb{N}$ and for every real $L_f(\lambda) \in [0\,,1)$, so $V_p(L_f(\lambda)) \neq 0$ and consequently $f(\lambda) = 0$. As α is the unique root of f in $[a\,,b]$, therefore $\lambda = \alpha$. This completes the proof of theorem.

Analogously, we prove that the sequences (9) are increasing and converges to α under the same assumptions of Theorem 2, but for $f(x_0)f^{'}(x_0) < 0$.

■

IV. Principal advantage of new family

As the family (Bp) is governed by formula (9), depending on the parameter p, where p is a nonnegative integer, it would be interesting to look for which p values, and under which conditions, the convergence is faster.

Theorem 3. *Let $p \in \mathbb{N}^*$. Let (u_n) and (w_n) be defined, respectively, by the sequences (x_n^{p+1}) and (x_n^p) given by equation* (9)*, $f \in C^m[a,b], m \geqslant 4, f^{'}(x) \neq 0$, $f^{''}(x) \neq 0$, $0 \leqslant L_f(x) < 1$ and the iterative functions F_p and F_{p+1} of f, defined by* (10)*, be increasing functions on an interval $[a\,,b]$ containing the root α of f. Starting from the same initial point $x_0 \in [a\,,b]$, the rate of convergence of sequence (x_n^{p+1}) is higher than one of sequence (x_n^p).*

Proof: Supposing that the initial value x_0 checks $f(x_0)f^{'}(x_0) > 0$, so $x_0 > \alpha$. According to Theorem 2, we know that if $f^{'}(x) \neq 0$, $f^{''}(x) \neq 0$, $0 \leq L_f < 1$, F_p and F_{p+1} are increasing functions an interval $[a\,,b]$, the sequences (x_n^p) and (x_n^{p+1}), given by (9), are decreasing and converge to α from any point $x_0 \in [a\,,b]$

Let (u_n) and (w_n) be defined, respectively, by (x_n^{p+1}) and (x_n^p). Since $u_0 = w_0 = x_0$ and the two sequences are decreasing, we expect that $u_n \leqslant w_n$ for all $n \in \mathbb{N}$. This can be proved by induction. Let $n = 1$, then:

$$u_1 - w_1 = -\frac{f(x_0)}{f'(x_0)}(V_{p+1}(L_0) - V_p(L_0))$$

As $0 \leq L_0 = L_f(x_0) < 1$, then from Lemma 2: $V_{p+1}(L_0) \geqslant V_p(L_0)$. As $\frac{f(x_0)}{f'(x_0)} > 0$, we deduce that : $u_1 \leq w_1$.

Now we assumed that $u_n \leq w_n$. Since, under above hypotheses, F_{p+1} is increasing function in $[a\,,b]$, we obtain $F_{p+1}(u_n) \leq F_{p+1}(w_n)$.

On the other hand, we have :

$$F_{p+1}(w_n) - F_p(w_n) = -\frac{f(w_n)}{f'(w_n)}\left(V_{p+1}(L_n) - V_p(L_n)\right) \leq 0$$

We deduce that $F_{p+1}(u_n) \leq F_p(w_n)$. So $u_{n+1} \leq w_{n+1}$ and induction is completed. The case $f(x_0)f'(x_0) < 0$ is similar to the previous one.

Consequently, the originality and the power of Super Halley's Family is illustrated analytically by justifying that, under certain conditions, the convergence speed of these methods increases with the parameter p. Since the famous Super Halley's method is a particular case of this family whose parameter ($p = 0$) is the smallest, its convergence speed is lower than other new methods of the same family, having higher parameters. ■

V. Numerical results

In this section we exhibit numerical results showing the behavior of some methods in the new family for some arbitrary chosen equations.

All results have been carried out in MATLAB R2015b and the stopping criterion has been taken as $|x_{n+1} - x_n| \leq 10^{-15}$ and $|f(x_n)| \leq 10^{-15}$. We give the number of iterations (N) or/and the number of function evaluations (NOFE) required to satisfy the stopping criterion, CU denotes that the method converges to an undesired root, F denotes that the method fails and D denotes divergence. The tests functions, used in Table II, III and IV, and their roots α, are displayed in Table I.

A. Numerical Comparison between some methods of new family

Let us consider the function f_{13} defined in Table I. Tacking $x_0 = 14$, we have $f(x_0)f'(x_0) > 0$. Table II presents a numerical comparison between some methods from the proposed family (Bp) obtained for $p = 1, 3, 4, 15$ and 21.

We show, in Table II, that :

- All the sequences (B1, B3, B4, B15 and B21) defined by (9) is decreasing and converges to the solution $\alpha = 6$ of equation $f(x) = 0$ in I;
- By increasing parameter p, the convergence speed of the methods (Bp) increases and their number of iterations decreases;

Table I: Test functions and their roots.

Test functions	Root (α)
$f_1(x) = x^2 - 5x + 6$	2.000000000000000 3.000000000000000
$f_2(x) = (\sin x)^2 - x^2 + 1$	1.404491648215341
$f_3(x) = x^3 - 5x^2 + 10x + 12$	-0,814380855386419
$f_4(x) = (x-3)e^x + 1$	2,947530902542285
$f_5(x) = x \ln x$	1.000000000000000
$f_6(x) = (x-1)^3 - 1$	2.000000000000000
$f_7(x) = \cos x - x$	0,7390851332151607
$f_8(x) = \frac{1}{2}x^3 + \frac{3}{4}x^2 - 3x - 1$	1,570796326794897
$f_9(x) = (\sin x)^2 - \sqrt{3}\sin x$	-3,141592653589793
$f_{10}(x) = x^3 + 4x^2 - 10$	1,365230013414097
$f_{11}(x) = \frac{2}{3}x^3 + 2x^2 - x + 1$	-3,54288610445217
$f_{12}(x) = \frac{2x^2}{x^2+1} - \ln(1+x^2)$	1,98029130043221
$f_{13}(x) = x^2 - 11x + 30$	6.000000000000000

- The convergence rate of Super Halley's method (B1) is lower than that of the other new methods which have higher values of parameter p (B3, B4, B15 and B21).

B. Comparison with other third order methods

In Table III, we shall present numerical results obtained by employing classical Newton's method (N) defined by formula (3), and some third order methods: Chebyshev's method (C) defined by (13) in [26], Sharma's method (S1) defined by equation (17) with $\alpha = 0.5$ in [29], Chun's method (CH) defined by (23) with $a_n = 1$ in [25], Jiang-Han's rational method (JH) defined by (19) with parameter $\alpha = 1$ in [30], Sharma's method (S2) defined by (20) with $a_n = 1$ in [27], Hansen and Patrick's method (HP) defined by (2.13) in [25] and Halley's method (H) defined in [14], [17], [18]. To represent the new Super Halley's family (9), we choose five formulas designated as B2, B5, B7, B11 and B19.

In Table III, all the methods converge cubically and require three function evaluations per step. Consequently, they have the same efficiency index $E = \sqrt[3]{3}$. Thus, the comparison can be made on the basis of the number of iterations (N). We see that the five proposed methods B2, B5, B7, B11 and B19 of the new family are better or similar to other used third-order methods, as they converge often to the root much faster and take lower number of iterations.

C. Comparison with higher order methods

In Table IV, we compared four methods of the proposed family (B2, B7, B13 and B19), with some higher order methods : (K) a sixth-order method denotes for Kou [39]; (F), a fifth-order method, denotes for Fang et al. (formula (2) in [37]). (CA) a sixth-order method, denotes for Chun and Ham (formulas (10), (11), (12) in [40]); (W) a fourth-order iterative method, denotes of Wang and Zhang (formula (19) with ($\gamma = \beta = -0.6$) in [38]. (T) a sixth-order method, denotes for Fernandez-Torres and al. (formulas (14) and (15) in [36]).

Table 4 shows the number of iterations (N) and the number of function evaluations (NOFE) required to approximate the root α. The efficiency and power of the new family is also

Table II: Numerical comparison between some methods of the proposed family.

B1	B3	B4	B15	B21
14.0	14.0	14.0	14.0	14.0
7.401499276471242	7.047518717194924	6.921353177190888	6.316280960753437	6.199090390091074
6.09218540988839	6.021178939757613	6.009191392373784	6.00000000220852	6.0
6.000007898965875	6.000000000012418	6.0	6.0	
6.0	6.0			

Table III: Comparison with other third order methods.

		N: Number of iterations											
Test function	x_0	N	C	S1	CH	JH	S2	HP	H	B2	B5	B7	B11	B19
f_1	-1,5	7	5	4	5	6	5	4	5	3	3	3	2	2
f_1	5	7	5	4	5	5	5	4	4	3	3	2	2	2
f_2	1	6	5	4	4	5	4	3	4	3	3	3	3	4
f_3	0.1	6	5	4	5	5	5	3	4	4	4	3	3	3
f_4	2.6	6	5	4	5	4	5	3	4	4	3	3	3	3
f_5	0.63	5	5	4	4	4	4	3	4	3	3	3	3	3
f_5	2.2	5	4	4	4	4	4	4	4	3	3	3	3	3
f_6	1.65	6	5	4	4	5	4	3	4	3	3	3	4	4
f_7	-0.1	5	4	4	4	4	4	3	4	3	3	3	3	3
f_7	1,6	5	4	3	4	4	4	4	4	3	3	4	3	3
f_8	1,93	5	4	4	4	4	4	4	4	3	3	3	3	3
f_9	-3.5	5	4	4	4	4	4	3	3	3	3	3	3	3
f_{10}	0.8	6	4	4	4	4	4	3	4	3	3	3	3	3

Table IV: Comparison with some higher order methods.

		N: Number of iterations									NOFE: Number of functions evaluations								
Test function	x_0	K	F	CA	W	T	B2	B7	B13	B19	K	F	CA	W	T	B2	B7	B13	B19
f_1	-1.5	3	4	3	d	3	3	3	2	2	12	16	12	D	12	9	9	6	6
f_1	5	3	3	3	5	19	3	2	2	2	12	12	12	15	76	9	6	6	6
f_2	1	3	3	3	4	5	3	3	3	3	12	12	12	12	20	9	9	9	9
f_4	2.6	2	3	3	4	D	4	3	3	3	8	12	12	12	D	12	9	9	9
f_4	1.6	2	3	3	4	D	3	3	3	3	8	12	12	12	D	9	9	9	9
f_5	2.2	2	3	2	4	D	3	3	3	3	8	12	8	12	D	9	9	9	9
f_6	1.65	2	3	3	4	4	3	3	4	4	8	12	12	12	16	9	9	12	12
f_7	1.6	3	3	2	4	3	3	3	3	3	12	12	8	12	12	9	9	9	9
f_9	-3.52	CU	3	3	3	3	3	3	3	3	CU	12	12	9	12	9	9	9	9
f_{10}	2.3	2	3	3	4	5	3	3	3	3	8	12	12	12	20	9	9	9	9
f_{11}	-4.4	2	3	3	3	5	3	3	3	3	8	12	12	9	20	9	9	9	9
f_{12}	1.4	2	3	3	4	3	3	4	3	3	8	12	12	12	12	9	12	9	9

confirmed by Table 4 which shows that, for the considered examples, our new four methods (B2, B7, B13 and B19) require a smaller number of function evaluations than most of the selected methods of higher order (fifth and sixth order). However, these results lead us to ask a big question: how can our methods, which are of order three, converge more quickly than other methods of higher order? A very likely answer is the good quality of our methods but there are also other factors such as the choice of the initial point. Indeed, we know that, to find the theoretical order of convergence of a method, we suppose that the initial point x_0 is sufficiently close to the root of the function α. But, if x_0 is too far from α (and x_0 in the basin of attraction of α), the order of convergence changes particularly for the first iterations. Thus, we calculate the computational order of convergence (ρ) at the step n, given by :

$$\rho \cong \frac{\ln |(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln |(x_n-\alpha)/(x_{n-1}-\alpha)|}$$

where x_{n-1}, x_n, are three consecutive iterations.

In article [32], we have shown that, in general, if we choose an initial point x_0 far from the root α, we find values of computational order of convergence (ρ) almost equal to the theoretical value ($\rho \approx d = 3$) for the one-point methods, of order 3, unlike the multi-points methods, of high order, which show values of ρ clearly lower than the theoretical value. This leads us to suppose that, for a choice of the initial point x_0 far from α, the high order methods would start the first iterations with a low speed; then, as the iterations advances, they would progressively regain speed to reach their maximum in the last iteration. Thus, the delay in the first iterations could lead to a decrease in the average speed of convergence, and consequently to an increase in the number of iterations. This would explain why, in several cases, our methods, which are of order 3, converge faster than other methods of much higher order, contrary to predictions.

VI. Conclusion

In this paper, we have developed a new family of third-order iterative methods for solving nonlinear equations with simple roots. The proposed scheme regenerates Super Halley's method and many new interesting methods. The originality of this family lies in the fact that these methods are linked by a recurring formula depending on a natural integer parameter p. Moreover, in case where certain hypotheses are satisfied, the sequences converge more rapidly when the value of parameter p increases. As parameter p can take very large values, the convergence speed can be largely improved with p. To test the new methods, several numerical examples were presented. The performances of our methods are compared with known methods of similar or higher order. Numerical results have confirmed the efficiency and speed of the techniques of the new family built in this article.

References

[1] J. M. Ortega and W. C. Rheinboldt, "Iterative solution of nonlinear equations in several variables," *SIAM*, vol. 30, 1970.

[2] S. C. Chapra, R. P. Canale *et al.*, "Numerical methods for engineers," *McGraw-Hill Higher Education, Boston*, 1988.

[3] S. Amat, S. Busquier, J. Gutiérrez, and M. Hernández, "On the global convergence of chebyshev's iterative method," *Journal of Computational and Applied Mathematics*, vol. 220, no. 1-2, pp. 17-21, 2008.

[4] G. Liu, C. Nie, and J. Lei, "A novel iterative method for nonlinear equations," IAENG International Journal of Applied Mathematics, vol. 48, no. 4, pp444-448, 2018.

[5] T. Scavo and J. Thoo, "On the geometry of halley's method," *The American mathematical monthly*, vol. 102, no. 5, pp. 417-426, 1995.

[6] A. Azam and B. Rhoades, "Some fixed point theorems for a pair of selfmaps of a cone metric space," IAENG International Journal of Applied Mathematics, vol. 42, no. 3, pp193-197, 2012.

[7] I. A. Moghrabi, "A rational model for curvature quasi-newton methods," Lecture Notes in Engineering and Computer Science: Proceedings of the World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Fransisco, USA, pp136-139.

[8] Y. Khongtham, "Contraction on some fixed point theorem in bv (s)-metric spaces," Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2018, 23-25 October, 2018, San Fransicso, USA, pp29-32.

[9] D. Shah and M. Sahni, "Dms way of finding the optimum number of iterations for fixed point iteration method," Lecture Notes in Engineering and Computer Science: Proceedings of the World Congress on Engineering 2018, 4-6 July, 2018, London, U.K., pp87-89.

[10] H. Fukhar-ud din, A. Khan, and M. Khan, "A new implicit algorithm of asymptotically quasi-nonexpansive maps in uniformly convex banach spaces," IAENG International Journal of Applied Mathematics, vol. 42, no. 3, pp171-175, 2012.

[11] M. V. Kumar and S. Subhani, "Application of common fixed point theorem on fuzzy metric space," Lecture Notes in Engineering and Computer Science: Proceedings of the World Congress on Engineering and Computer Science 2017, 25-27 October, 2017, San Fransisco, USA, pp484-489.

[12] Y. Khongtham, "New iterative method for variational inclusion and fixed point problems," Lecture Notes in Engineering and Computer Science: Proceeding of the World Congress on Engineering 2014, 2-4 July, 2014, London, U.K., pp865-869.

[13] J. Traub, "Iterative methods for the solution of equations prentice-hall," *Englewood Cliffs, New Jersey*, 1964.

[14] E. Halley, "A new, exact, and easy method of finding the roots of any equations generally, and that without any previous reduction," *Philosophical Transactions of the Royal Society of London*, vol. 18, pp. 136-145, 1694.

[15] W. Gander, "On halley's iteration method," *The American Mathematical Monthly*, vol. 92, no. 2, pp. 131-134, 1985.

[16] G. Salehov, "On the convergence of the process of tangent hyperbolas," in *Dokl. Akad. Nauk SSSR*, vol. 82, pp. 525-528, 1952.

[17] M. Davies and B. Dawson, "On the global convergence of halley's iteration formula," *Numerische Mathematik*, vol. 24, no. 2, pp. 133-135, 1975.

[18] A. Melman, "Classroom note: geometry and convergence of euler's and halley's methods," *SIAM review*, vol. 39, no. 4, pp. 728-725, 1997.

[19] S. Amat, S. Busquier, and J. Gutiérrez, "Geometric constructions of iterative functions to solve nonlinear equations," *Journal of Computational and Applied Mathematics*, vol. 157, no. 1, pp. 197-205, 2003.

[20] M. Barrada, R. Benkhouya, and I. Chana, "A new halley's family of third-order methods for solving nonlinear equations." IAENG International Journal of Applied Mathematics, vol. 50, no. 1, pp58-65, 2020.

[21] M. A. H. Verón, "A note on halley's method," *Extracta mathematicae*, vol. 3, no. 3, pp. 104-106, 1988.

[22] M. Hernández and M. Salanova, "A family of chebyshev-halley type methods," *International Journal of Computer Mathematics*, vol. 47, no. 1-2, pp. 59-63, 1993.

[23] ——, "Chebyshev method and convexity," *Applied mathematics and computation*, vol. 95, no. 1, pp. 51-62, 1998.

[24] A. Ostrowski, "Solution of equations and systems of equations," American Press, New York, 1973.

[25] E. Hansen and M. Patrick, "A family of root finding methods," *Numerische Mathematik*, vol. 27, no. 3, pp. 257-269, 1977.

[26] C. Chun, "A one-parameter family of third-order methods to solve nonlinear equations," *Applied mathematics and computation*, vol. 189, no. 1, pp. 126-130, 2007.

[27] J. Sharma, "A family of third-order methods to solve nonlinear equations by quadratic curves approximation," *Applied mathematics and computation*, vol. 184, no. 2, pp. 210-215, 2007.

[28] ——, "A one-parameter family of second-order iteration methods," *Applied mathematics and computation*, vol. 186, no. 2, pp. 1402-1406, 2007.

[29] J. Sharma, R. Guha, and R. Sharma, "A unified approach to generate weighted newton third order methods for solving nonlinear equations," *J. Numer. Mathe. Stoch*, vol. 4, no. 1, pp. 48-58, 2012.

[30] D. Jiang and D. Han, "Some one-parameter families of third-order methods for solving nonlinear equations," *Applied mathematics and computation*, vol. 195, no. 2, pp. 392-396, 2008.

[31] M. Barrada, M. L. Hasnaoui, and M. Ouaissa, "On the global convergence of improved halley's method," Engineering Letters, vol. 28, no. 2, pp609-615, 2020.

[32] M. Barrada, M. Ouaissa, Y. Rhazali, and M. Ouaissa, "A new class of halley's method with third-order convergence for solving nonlinear equations," *Journal of Applied Mathematics, Article ID 3561743*, vol. 2020, pp. 1-13, 2020.

[33] M. Barrada, H. Bennis, M. Kabbaj, and C. Ziti, "On the global convergence of a fast halley's family to solve nonlinear equations," *Journal of King Saud University-Science*.

[34] K. Jisheng, L. Yitian, and W. Xiuhua, "Third-order modification of newton's method," *Journal of computational and applied mathematics*, vol. 205, no. 1, pp. 1-5, 2007.

[35] C. Chun and B. Neta, "Some modification of newton's method by the method of undetermined coefficients," *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2528-2538, 2008.

[36] G. Fernandez-Torres, F. R. Castillo-Soria, and I. Algredo-Badillo, "Fifth and sixth-order iterative algorithms without derivatives for solving non-linear equations," *International Journal of Pure and Applied Mathematics*, vol. 83, no. 1, pp. 111-119, 2013.

[37] L. Fang, L. Sun, and G. He, "An efficient newton-type method with fifth-order convergence for solving nonlinear equations," *Computational & Applied Mathematics*, vol. 27, no. 3, pp. 269-274, 2008.

[38] X. Wang and T. Zhang, "High-order newton-type iterative methods with memory for solving nonlinear equations," *Mathematical Communications*, vol. 19, no. 1, pp. 91-109, 2014.

[39] J. Kou, "The improvements of modified newton's method," *Applied Mathematics and Computation*, vol. 189, no. 1, pp. 602-609, 2007.

[40] C. Chun and Y. Ham, "Some sixth-order variants of ostrowski root-finding methods," *Applied Mathematics and Computation*, vol. 193, no. 2, pp. 389-394, 2007.

[41] W. Bi, H. Ren, and Q. Wu, “Three-step iterative methods with eighth-order convergence for solving nonlinear equations,” *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 105-112, 2009.

[42] J. Ezquerro and M. Hernández, “Different acceleration procedures of newton’s method,” *Novi Sad J. Math*, vol. 27, no. 1, pp. 1-17, 1997.

[43] M. A. Hernández Verón, “A note on halley’s method,” *Numerische Mathematik*, vol. 59, no. 1, pp. 273-276, 1991.

[44] J. M. Gutierrez and M. A. Hernández, “An acceleration of newton’s method: Super-halley method,” *Applied Mathematics and Computation*, vol. 117, no. 2-3, pp. 223-239, 2001.