Numerical Solutions for Unsteady Modified Helmholtz Type Problems of Anisotropic Trigonometrically Graded Materials

Moh. Ivan Azis*, Ikha Magdalena, Widowati

Abstract—**An LT-BEM is used to solve numerically a class of variable coefficient unsteady modified Helmholtz equation. The variable coefficients equation is transformed to a constant coefficients equation which is then Laplace-transformed (LT) so that the time variable vanishes. A boundary-only integral equation involving a time-free fundamental solution can then be derived and employed to find numerical solutions using a boundary element method (BEM). The results obtained are inversely transformed numerically using the Stehfest formula. Some problems considered show that the combined LT-BEM is easy to implement, efficient and accurate for solving numerically the problems.**

Index Terms—**Anisotropic functionally graded materials, modified Helmholtz equation, Laplace transform, boundary element method**

I. Introduction

We will consider initial boundary value problems governed by a modified Helmholtz type equation with variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial\mu(\mathbf{x},t)}{\partial x_j}\right]-\beta^2(\mathbf{x})\,\mu(\mathbf{x},t)=\alpha(\mathbf{x})\frac{\partial\mu(\mathbf{x},t)}{\partial t} \tag{1}$$

The coefficients $[\kappa_{ij}]$ $(i,j=1,2)$ is a real symmetric positive definite matrix. Also, in (1) the summation convention for repeated indices holds. Therefore equation (1) may be written explicitly as

$$\frac{\partial}{\partial x_1}\left(\kappa_{11}\frac{\partial\mu}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\kappa_{12}\frac{\partial\mu}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(\kappa_{12}\frac{\partial\mu}{\partial x_1}\right)$$
$$+\frac{\partial}{\partial x_2}\left(\kappa_{22}\frac{\partial\mu}{\partial x_2}\right)-\beta^2\mu=\alpha\frac{\partial c}{\partial t}$$

Equation (1) is usually used to model infiltration problems (see for examples [1]–[3]).

During the last decade functionally graded materials (FGMs) have become an important topic, and numerous studies on them for a variety of applications have been reported. FGMs are materials possessing characteristics which vary (with time and position) according to a mathematical function. Therefore equation (1) is relevant for FGMs. FGMs are mainly artificial materials which are produced to meet a preset practical performance (see for example [4], [5]). This constitutes relevancy of solving equation (1).

A number of studies on the modified Helmholtz equation had been done for finding its numerical solutions. However the studies mainly focus on the case of homogeneous media isotropic equation (see for example [1]–[3]). For such kind of materials, the boundary element method (BEM) and other methods had been successfully used to find the numerical solutions of problems associated to them.

But this is not the case for inhomogeneous materials, due to the unavailability of fundamental solutions for equations of variable coefficients which govern problems of inhomogeneous media. Some progress of solving problems for inhomogeneous media using various techniques has been done. Timpitak and Pochai [6] investigated finite difference solutions of unsteady diffusion-convection problems for heterogeneous media. Noda et al. [7] studied the analytical solutions to a transient heat conduction equation of variable coefficients with a source term for a functionally graded orthotropic strip (FGOS). In this study, the inhomogeneity of the FGOS is simplified to be functionally graded in the x variable only. In [8] Azis and Clements worked on finding numerical solutions to nonlinear transient heat conduction problems for anisotropic quadratically graded materials using a boundary domain element method. The quadratically varying coefficient in the governing equation considered by Azis and Clements [8] can certainly be represented as a sum of constant and variable coefficients. Some later studies on the class of constant-plus-variable coefficients equations had been done a number of authors. Samec and Škerget [9] considered a non-steady diffusive–convective transport equation with variable velocity which is represented as a sum of constant and variable terms. Ravnik and Škerget in [10] studied steady state diffusion-convection problems with inhomogeneous isotropic diffusivity, variable velocity and incompressible fluid using a domain boundary integral equation method (DBIEM). In this work both the diffusivity and the velocity take a constant-plus-variable form. Ravnik and Škerget in [11] considered an unsteady state diffusion-convection problems with sources, inhomogeneous isotropic conductivity, variable velocity and incompressible fluid using a DBIEM. In this study both the diffusivity and the velocity are again taken to be of constant-plus-variable form. AL-Bayati and Wrobel [12], [13] focused on convection–diffusion–reaction equation of incompressible flow with constant diffusivity and variable velocity taking the form of constant-plus-variable terms. Ravnik and Tibuat [14] also considered an unsteady diffusion-convection equation

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with variable diffusivity and velocity. The diffusivity is of the constant-plus-variable form. By taking the variable coefficients as a sum of constant and variable coefficients, the derived integral equation will then involve both boundary and domain integrals. The constant coefficient term will contribute boundary integrals as the fundamental solutions are available, and the variable coefficient term will give domain integrals.

Reduction to constant coefficients equation is another technique that can be used to transform a variable coefficients equation to a constant coefficients equation. Therefore the technique will preserve the boundary-only integral equation. Recently Azis and co-workers had been working on steady state problems of anisotropic inhomogeneous media for several types of governing equations, for examples [15]–[20] for Helmholtz equation, [21]–[24] for the modified Helmholtz equation, [25] for elasticity problems, [26]–[30] for the diffusion convection equation, [31]–[34] for the Laplace type equation, [35]–[41] for the diffusion convection reaction equation. Some other classes of inhomogeneity functions for FGMs that differ from the class of constant-plus-variable coefficients are reported from these papers. Azis et al. also had been working on *unsteady state* problems of *anisotropic inhomogeneous* media for some types of governing equations (see [42]–[46]).

This paper is intended to extend the recently published works in [21]–[24] for steady anisotropic modified Helmholtz type equation with spatially variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\kappa_{ij}(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial x_j}\right]-\beta^2(\mathbf{x})\,\mu(\mathbf{x},t)=0$$

to unsteady anisotropic modified Helmholtz type equation with spatially variable coefficients of the form (1).

Equation (1) will be transformed to a constant coefficient equation from which a boundary integral equation will derived. It is necessary to place some constraint on the class of coefficients κ_{ij} and β^2 for which the solution obtained is valid. The analysis of this paper is purely formal; the main aim being to construct effective BEM for class of equations which falls within the type (1).

II. The initial-boundary value problem

Referred to a Cartesian frame Ox_1x_2 solutions $\mu(\mathbf{x},t)$ and its derivatives to (1) are sought which are valid for time interval $t \geq 0$ and in a region Ω in R^2 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves. On $\partial\Omega_1$ the dependent variable $\mu(\mathbf{x},t)$ ($\mathbf{x}=(x_1,x_2)$) is specified and on $\partial\Omega_2$

$$P(\mathbf{x},t)=\kappa_{ij}(\mathbf{x})\frac{\partial \mu(\mathbf{x},t)}{\partial x_i}n_j \qquad (2)$$

is specified where $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ and $\mathbf{n}=(n_1,n_2)$ denotes the outward pointing normal to $\partial\Omega$. The initial condition is taken to be

$$\mu(\mathbf{x},0)=0 \qquad (3)$$

The method of solution will be to transform the variable coefficient equation (1) to a constant coefficient equation, and then taking a Laplace transform of the constant coefficient equation, and to obtain a boundary integral equation in the Laplace transform variable s. The boundary integral equation is then solved using a standard boundary element method (BEM). An inverse Laplace transform is taken to get the solution c and its derivatives for all $(\mathbf{x},t)$ in the domain. The inverse Laplace transform is implemented numerically using the Stehfest formula.

The analysis is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (1) take the form $\kappa_{11}=\kappa_{22}$ and $\kappa_{12}=0$ and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

III. The boundary integral equation

The coefficients $\kappa_{ij},\beta^2,\alpha$ are required to take the form

$$\kappa_{ij}(\mathbf{x}) = \overline{\kappa}_{ij}g(\mathbf{x}) \qquad (4)$$

$$\beta^2(\mathbf{x}) = \overline{\beta}^2 g(\mathbf{x}) \qquad (5)$$

$$\alpha(\mathbf{x}) = \overline{\alpha}g(\mathbf{x}) \qquad (6)$$

where the $\overline{\kappa}_{ij},\overline{\beta}^2,\overline{\alpha}$ are constants and g is a differentiable function of $\mathbf{x}$. Further we assume that the coefficients $\kappa_{ij}(\mathbf{x})$, $\beta^2(\mathbf{x})$ and $\alpha(\mathbf{x})$ are trigonometrically graded by taking $g(\mathbf{x})$ as an trigonometric function

$$g(\mathbf{x}) = \left[A\cos(c_0+c_ix_i) + B\sin(c_0+c_ix_i)\right]^2 \qquad (7)$$

where A,B,c_0 and c_i are constants. Therefore if

$$\overline{\kappa}_{ij}c_ic_j+\lambda=0 \qquad (8)$$

then (7) satisfies

$$\overline{\kappa}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i\partial x_j}-\lambda g^{1/2}=0 \qquad (9)$$

Use of (4)-(6) in (1) yields

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g\frac{\partial\mu}{\partial x_j}\right)-\overline{\beta}^2 g\mu=\overline{\alpha}g\frac{\partial\mu}{\partial t} \qquad (10)$$

Let

$$\mu(\mathbf{x},t)=g^{-1/2}(\mathbf{x})\,\psi(\mathbf{x},t) \qquad (11)$$

therefore substitution of (4) and (11) into (2) gives

$$P(\mathbf{x},t)=-P_g(\mathbf{x})\,\psi(\mathbf{x},t)+g^{1/2}(\mathbf{x})\,P_\psi(\mathbf{x},t) \qquad (12)$$

where

$$P_g(\mathbf{x})=\overline{\kappa}_{ij}\frac{\partial g^{1/2}}{\partial x_j}n_i \qquad P_\psi(\mathbf{x})=\overline{\kappa}_{ij}\frac{\partial\psi}{\partial x_j}n_i$$

Also, (10) may be written in the form

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left[g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial x_j}\right]-\overline{\beta}^2 g^{1/2}\psi=\overline{\alpha}g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial t}$$

which can be simplified

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g^{1/2}\frac{\partial\psi}{\partial x_j}+g\psi\frac{\partial g^{-1/2}}{\partial x_j}\right)-\overline{\beta}^2 g^{1/2}\psi=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

Use of the identity

$$\frac{\partial g^{-1/2}}{\partial x_i}=-g^{-1}\frac{\partial g^{1/2}}{\partial x_i}$$

implies

$$\overline{\kappa}_{ij}\frac{\partial}{\partial x_i}\left(g^{1/2}\frac{\partial\psi}{\partial x_j}-\psi\frac{\partial g^{1/2}}{\partial x_j}\right)-\overline{\beta}^2 g^{1/2}\psi=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

Rearranging and neglecting the zero terms yield

$$g^{1/2}\overline{\kappa}_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j}-\psi\overline{\kappa}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i\partial x_j}-\overline{\beta}^2 g^{1/2}\psi=\overline{\alpha}g^{1/2}\frac{\partial\psi}{\partial t}$$

Equation (9) then implies

$$\overline{\kappa}_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j}-\left(\overline{\beta}^2+\lambda\right)\psi=\overline{\alpha}\frac{\partial\psi}{\partial t} \tag{13}$$

Taking the Laplace transform of (11), (12), (13) and applying the initial condition (3) we obtain

$$\psi^*(\mathbf{x},s)=g^{1/2}(\mathbf{x})\,\mu^*(\mathbf{x},s) \tag{14}$$

$$P_{\psi^*}(\mathbf{x},s)=\left[P^*(\mathbf{x},s)+P_g(\mathbf{x})\,\psi^*(\mathbf{x},s)\right]g^{-1/2}(\mathbf{x}) \tag{15}$$

$$\overline{\kappa}_{ij}\frac{\partial^2\psi^*}{\partial x_i\partial x_j}-\left(\overline{\beta}^2+\lambda+s\overline{\alpha}\right)\psi^*=0 \tag{16}$$

where s is the variable of the Laplace-transformed domain.

A boundary integral equation for the solution of (16) is given in the form

$$\begin{aligned}\eta(\mathbf{x}_0)\,\psi^*(\mathbf{x}_0,s) &= \int_{\partial\Omega}\left[\Gamma(\mathbf{x},\mathbf{x}_0)\,\psi^*(\mathbf{x},s)\right.\\ &\quad \left.-\Phi(\mathbf{x},\mathbf{x}_0)\,P_{\psi^*}(\mathbf{x},s)\right]dS(\mathbf{x})\end{aligned} \tag{17}$$

where $\mathbf{x}_0=(a,b)$, $\eta=0$ if $(a,b)\notin\Omega\cup\partial\Omega$, $\eta=1$ if $(a,b)\in\Omega$, $\eta=\frac{1}{2}$ if $(a,b)\in\partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at (a,b). The so called fundamental solution Φ in (17) is any solution of the equation

$$\overline{\kappa}_{ij}\frac{\partial^2\Phi}{\partial x_i\partial x_j}-\left(\overline{\beta}^2+\lambda+s\overline{\alpha}\right)\Phi=\delta(\mathbf{x}-\mathbf{x}_0)$$

and the Γ is given by

$$\Gamma(\mathbf{x},\mathbf{x}_0)=\overline{\kappa}_{ij}\frac{\partial\Phi(\mathbf{x},\mathbf{x}_0)}{\partial x_j}n_i$$

where δ is the Dirac delta function. For two-dimensional problems Φ and Γ are given by

$$\begin{aligned}\Phi(\mathbf{x},\mathbf{x}_0) &= \begin{cases}\frac{K}{2\pi}\ln R & \text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}=0\\ \frac{\imath K}{4}H_0^{(2)}(\omega R) & \text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}<0\\ \frac{-K}{2\pi}K_0(\omega R) & \text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}>0\end{cases}\\ \Gamma(\mathbf{x},\mathbf{x}_0) &= \begin{cases}\frac{K}{2\pi}\frac{1}{R}\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i\\ \frac{-\imath K\omega}{4}H_1^{(2)}(\omega R)\,\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i\\ \frac{K\omega}{2\pi}K_1(\omega R)\,\overline{\kappa}_{ij}\frac{\partial R}{\partial x_j}n_i\end{cases}\\ &\quad \begin{cases}\text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}=0\\ \text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}<0\\ \text{if } \overline{\beta}^2+\lambda+s\overline{\alpha}>0\end{cases}\end{aligned} \tag{18}$$

where

$$\begin{aligned}K &= \ddot{\tau}/D\\ \omega &= \sqrt{|\overline{\beta}^2+\lambda+s\overline{\alpha}|/D}\\ D &= \left[\overline{\kappa}_{11}+2\overline{\kappa}_{12}\dot{\tau}+\overline{\kappa}_{22}\left(\dot{\tau}^2+\ddot{\tau}^2\right)\right]/2\\ R &= \sqrt{(\dot{x}_1-\dot{a})^2+(\dot{x}_2-\dot{b})^2}\\ \dot{x}_1 &= x_1+\dot{\tau}x_2\\ \dot{a} &= a+\dot{\tau}b\\ \dot{x}_2 &= \ddot{\tau}x_2\\ \dot{b} &= \ddot{\tau}b\end{aligned}$$

where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic

$$\overline{\kappa}_{11}+2\overline{\kappa}_{12}\tau+\overline{\kappa}_{22}\tau^2=0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively. K_0, K_1 denote the modified Bessel function of order zero and order one respectively, $\imath$ represents the square root of minus one. The derivatives $\partial R/\partial x_j$ needed for the calculation of the Γ in (18) are given by

$$\begin{aligned}\frac{\partial R}{\partial x_1} &= \frac{1}{R}(\dot{x}_1-\dot{a})\\ \frac{\partial R}{\partial x_2} &= \dot{\tau}\left[\frac{1}{R}(\dot{x}_1-\dot{a})\right]+\ddot{\tau}\left[\frac{1}{R}\left(\dot{x}_2-\dot{b}\right)\right]\end{aligned}$$

Use of (14) and (15) in (17) yields

$$\begin{aligned}\eta g^{1/2}\mu^* &= \int_{\partial\Omega}\left[\left(g^{1/2}\Gamma-P_g\Phi\right)\mu^*\right.\\ &\quad \left.-\left(g^{-1/2}\Phi\right)P^*\right]dS\end{aligned} \tag{19}$$

This equation provides a boundary integral equation for determining μ^* and its derivatives at all points of Ω.

Knowing the solutions $\mu^*(\mathbf{x},s)$ and its derivatives $\partial\mu^*/\partial x_1$ and $\partial\mu^*/\partial x_2$ which are obtained from (19), the numerical Laplace transform inversion technique using the Stehfest formula is then employed to find the values of $\mu(\mathbf{x},t)$ and its derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$. The Stehfest formula is

$$\begin{aligned}\mu(\mathbf{x},t) &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\mu^*(\mathbf{x},s_m)\\ \frac{\partial\mu(\mathbf{x},t)}{\partial x_1} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\frac{\partial\mu^*(\mathbf{x},s_m)}{\partial x_1}\\ \frac{\partial\mu(\mathbf{x},t)}{\partial x_2} &\simeq \frac{\ln 2}{t}\sum_{m=1}^{N}V_m\frac{\partial\mu^*(\mathbf{x},s_m)}{\partial x_2}\end{aligned} \tag{20}$$

where

$$\begin{aligned}s_m &= \frac{\ln 2}{t}m\\ V_m &= (-1)^{\frac{N}{2}+m}\times\\ &\sum_{k=\left[\frac{m+1}{2}\right]}^{\min\left(m,\frac{N}{2}\right)}\frac{k^{N/2}(2k)!}{\left(\frac{N}{2}-k\right)!k!(k-1)!(m-k)!(2k-m)!}\end{aligned}$$

IV. Numerical Examples

In order to verify the analysis derived in the previous sections, we will consider several problems either as test examples of analytical solutions or problems without simple analytical solutions.

We assume each problem belongs to a system which is valid in given spatial and time domains and governed by equation (1) and satisfying the initial condition (3) and some boundary conditions as mentioned in Section II. The characteristics of the system which are represented by the coefficients $\kappa_{ij}(\mathbf{x}),\beta^2(\mathbf{x}),\alpha(\mathbf{x})$ in equation (1) are assumed to be of the form (4), (5) and (6) in which $g(\mathbf{x})$ is a trigonometric function of the form (7). The coefficients

TABLE I
VALUES OF V_m OF THE STEHFEST FORMULA

V_m	$N=6$	$N=8$	$N=10$	$N=12$
V_1	1	$-1/3$	$1/12$	$-1/60$
V_2	-49	$145/3$	$-385/12$	$961/60$
V_3	366	-906	1279	-1247
V_4	-858	$16394/3$	$-46871/3$	$82663/3$
V_5	810	$-43130/3$	$505465/6$	$-1579685/6$
V_6	-270	18730	-236957.5	1324138.7
V_7		$-35840/3$	$1127735/3$	$-58375583/15$
V_8		$8960/3$	$-1020215/3$	$21159859/3$
V_9			164062.5	-8005336.5
V_{10}			-32812.5	5552830.5
V_{11}				-2155507.2
V_{12}				359251.2

$\kappa_{ij}(\mathbf{x}), \beta^2(\mathbf{x}), \alpha(\mathbf{x})$ may represent respectively the diffusivity or conductivity, the wave number and the change rate of the unknown $\mu(\mathbf{x}, t)$.

Standard BEM with constant elements is employed to obtain numerical results. For a simplicity, a unit square will be taken as the geometrical domain for all problems. A number of 320 elements of equal length, namely 80 elements on each side of the unit square, are used. And the time interval is chosen to be $0 \le t \le 5$. A FORTRAN script is developed to compute the solutions and a specific FORTRAN command is imposed to calculate the elapsed CPU time for obtaining the results. A simple script is also embedded to calculate the values of the coefficients $V_m, m = 1, 2, \ldots, N$ for any even number N. Table I shows the values of V_m for several values of N.

For all problems the inhomogeneity function is taken to be

$$\begin{aligned} g^{1/2}(\mathbf{x}) &= \cos(0.7 - 0.4x_1 - 0.3x_2) \\ &\quad + \sin(0.7 - 0.4x_1 - 0.3x_2) \end{aligned}$$

and the constant anisotropy coefficient $\overline{\kappa}_{ij}$

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

so that 8 implies

$$\lambda = -0.28$$

We set the constant coefficient $\overline{\beta}^2$

$$\overline{\beta}^2 = 1$$

A. Examples with analytical solutions

1) Problem 1:: Other aspects that will be justified are the convergence (as N increases) and time efficiency for obtaining the numerical solutions. The analytical solutions are assumed to take a separable variables form

$$\mu(\mathbf{x}, t) = g^{-1/2}(\mathbf{x})\, h(\mathbf{x})\, f(t)$$

where $h(\mathbf{x}), f(t)$ are continuous functions. The boundary conditions are assumed to be (see Figure 1)

P is given on side AB
P is given on side BC
μ is given on side CD
P is given on side AD

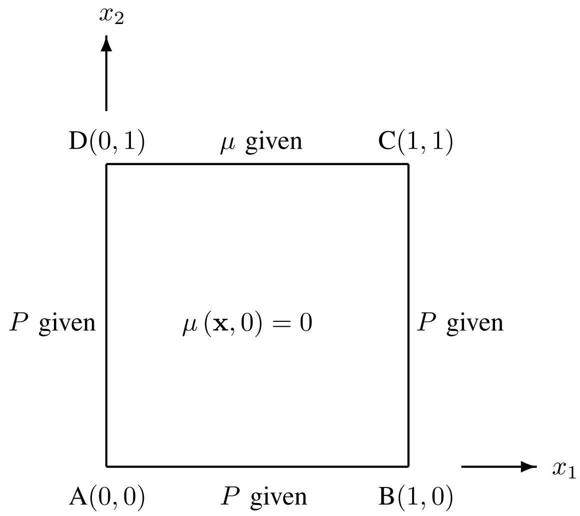

Fig. 1. The boundary conditions for the problems in Section IV-A

For each N, numerical solutions for μ and the derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$ at 19×19 points inside the space domain which are

$$\begin{aligned} (x_1, x_2) &= \{0.05, 0.1, 0.15, \ldots, 0.9, 0.95\} \\ &\quad \times \{0.05, 0.1, 0.15, \ldots, 0.9, 0.95\} \end{aligned}$$

and 11 time-steps which are

$$t = 0.0005, 0.5, 1, 1.5, \ldots, 4, 4.5, 5$$

are computed. The aggregate relative error E is calculated using the norm

$$E = \left[\frac{\sum_t \sum_{i=1}^{19\times19} (\varsigma_{n,i} - \varsigma_{a,i})^2}{\sum_t \sum_{i=1}^{19\times19} \mu_{a,i}^2} \right]^{\frac{1}{2}}$$

where ς_n and ς_a represent respectively the numerical and analytical solutions μ or the derivatives $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$. The elapsed CPU time τ (in seconds) is also computed and the time efficiency number ε for obtaining the numerical solutions of error E is defined as

$$\varepsilon = E\tau$$

This formula explains that the smaller time τ with smaller error E, the more efficient the procedure (smaller ε).

Case 1:: We take

$$\begin{aligned} h(\mathbf{x}) &= 0.4 - 0.3x_1 - 0.1x_2 \\ f(t) &= 1 - \exp(-1.75t) \end{aligned}$$

Thus for $h(\mathbf{x})$ to satisfy (16)

$$\overline{\alpha} = -0.72/s$$

Table II shows the error E and efficiency number ε for solutions $\mu, \partial\mu/\partial x_1, \partial\mu/\partial x_2$ as N increases from $N = 6$ to $N = 12$. For the solutions $\mu, \partial\mu/\partial x_2$ the error E and efficiency number ε gets smaller as N moves up to $N = 10$ and for the solution $\partial\mu/\partial x_1$ the error E and efficiency number ε decrease as N moves up to $N = 12$ and $N = 10$ respectively. As shown in Table III, the optimized value of N for solutions $\mu, \partial\mu/\partial x_2$ to achieve their smallest error E and efficiency number ε is $N = 10$, but for the solution $\partial\mu/\partial x_1$ to reach its smallest error E and efficiency number ε the optimized value of N is $N = 12$ and $N = 10$

TABLE II
THE TOTAL ELAPSED CPU TIME τ, THE GLOBAL AVERAGE ERROR E, THE EFFICIENCY NUMBER $\varepsilon = \tau E$ FOR CASE 1

N		6	8	10	12
τ		249.859	332.906	418.594	498.781
μ	E	0.00456413	0.00158719	0.00095128	0.00097223
	ε	1.140390	0.528384	0.398200	0.484928
$\frac{\partial\mu}{\partial x_1}$	E	0.00413618	0.00138737	0.00076626	0.00066876
	ε	1.033463	0.461865	0.320751	0.333567
$\frac{\partial\mu}{\partial x_2}$	E	0.00446496	0.00149382	0.00081896	0.00082135
	ε	1.115611	0.497304	0.342810	0.409674

TABLE III
THE OPTIMIZED VALUE OF N FOR OBTAINING THE NUMERICAL SOLUTIONS $\mu, \partial\mu/\partial x_1, \partial\mu/\partial x_2$ OF BEST ERROR E AND EFFICIENCY NUMBER ε FOR CASE 1

	μ	$\frac{\partial\mu}{\partial x_1}$	$\frac{\partial\mu}{\partial x_2}$
E	$N = 10$	$N = 12$	$N = 10$
ε	$N = 10$	$N = 10$	$N = 10$

respectively. According to Hassanzadeh and Pooladi-Darvish [47] increasing N will increase the accuracy up to a point, and then the accuracy will decline due to round-off errors.

Case 2:: For the analytical solution we take

$$h(\mathbf{x}) = \sin(0.4 - 0.3x_1 - 0.1x_2)$$
$$f(t) = t/5$$

So that in order for $h(\mathbf{x})$ to satisfy (16)

$$\overline{\alpha} = -0.83/s$$

Tables IV and V show that for solution μ the smallest error E and efficiency number ε are achieved when $N = 12$ and $N = 8$ respectively, whereas for the solutions $\partial\mu/\partial x_1, \partial\mu/\partial x_2$ they are reached when $N = 8$.

Case 3:: We take

$$h(\mathbf{x}) = \exp(-0.4 + 0.3x_1 + 0.1x_2)$$
$$f(t) = 0.16t(5 - t)$$

Therefore (16) gives

$$\overline{\alpha} = -0.61/s$$

Tables VI and VII show that for solutions μ the smallest error E and efficiency number ε are achieved when $N = 12$, for solutions $\partial\mu/\partial x_1$ and $\partial\mu/\partial x_2$ the smallest error E and efficiency number ε are achieved when $N = 10$.

TABLE IV
THE TOTAL ELAPSED CPU TIME τ, THE GLOBAL AVERAGE ERROR E, THE EFFICIENCY NUMBER $\varepsilon = \tau E$ FOR CASE 2

N		6	8	10	12
τ		399.906	530.906	653.391	765.234
μ	E	0.00168221	0.00071267	0.00060798	0.00057896
	ε	0.672728	0.378362	0.397250	0.443040
$\frac{\partial\mu}{\partial x_1}$	E	0.00228414	0.00040042	0.00040509	0.00041696
	ε	0.913440	0.212583	0.264681	0.319074
$\frac{\partial\mu}{\partial x_2}$	E	0.00335038	0.00114237	0.00125029	0.00128123
	ε	1.339838	0.606494	0.816928	0.980439

TABLE V
THE OPTIMIZED VALUE OF N FOR OBTAINING THE NUMERICAL SOLUTIONS $\mu, \partial\mu/\partial x_1, \partial\mu/\partial x_2$ OF BEST ERROR E AND EFFICIENCY NUMBER ε FOR CASE 2

	μ	$\frac{\partial\mu}{\partial x_1}$	$\frac{\partial\mu}{\partial x_2}$
E	$N = 12$	$N = 8$	$N = 8$
ε	$N = 8$	$N = 8$	$N = 8$

TABLE VI
THE TOTAL ELAPSED CPU TIME τ, THE GLOBAL AVERAGE ERROR E, THE EFFICIENCY NUMBER $\varepsilon = \tau E$ FOR CASE 3

N		6	8	10	12
τ		333.359	384.359	555.516	661.828
μ	E	0.16858197	0.01090963	0.00033883	0.00021239
	ε	56.198382	4.193218	0.188226	0.140563
$\frac{\partial\mu}{\partial x_1}$	E	0.16865011	0.01100402	0.00026815	0.00028978
	ε	56.221095	4.229498	0.148964	0.191786
$\frac{\partial\mu}{\partial x_2}$	E	0.16870872	0.01108307	0.00033121	0.00064502
	ε	56.240633	4.259882	0.183991	0.426895

TABLE VII
THE OPTIMIZED VALUE OF N FOR OBTAINING THE NUMERICAL SOLUTIONS $\mu, \partial\mu/\partial x_1, \partial\mu/\partial x_2$ OF BEST ERROR E AND EFFICIENCY NUMBER ε FOR CASE 3

	μ	$\frac{\partial\mu}{\partial x_1}$	$\frac{\partial\mu}{\partial x_2}$
E	$N = 12$	$N = 10$	$N = 10$
ε	$N = 12$	$N = 10$	$N = 10$

B. Examples without analytical solutions

The aim is to show the effect of inhomogeneity and anisotropy of the considered material on the solution μ.

1) Problem 2:: The material is supposed to be either inhomogeneous or homogeneous and either anisotropic or isotropic. If the material is homogeneous then

$$g(\mathbf{x}) = 1$$

and if it is isotropic then

$$\overline{\kappa}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So that there are four cases regarding the material, namely anisotropic inhomogeneous, anisotropic homogeneous, isotropic inhomogeneous and isotropic homogeneous material. We set $\overline{\alpha} = 1$ and the boundary conditions are (see Figure 2)

$$\begin{aligned} &P = P(t) \text{ on side AB} \\ &P = 0 \text{ on side BC} \\ &\mu = 0 \text{ on side CD} \\ &P = 0 \text{ on side AD} \end{aligned}$$

where $P(t)$ takes four forms

$$P(t) = P_1(t) = 1$$
$$P(t) = P_2(t) = 1 - \exp(-1.75t)$$
$$P(t) = P_3(t) = t/5$$
$$P(t) = P_4(t) = 0.16t(5 - t)$$

Therefore the system is geometrically symmetric about $x_1 = 0.5$. We use $N = 12$ for all cases of this problem.

The results are shown in Table VIII, Figures 3 and 4. Table VIII shows the solution μ at points $(0.2, 0.5), (0.8, 0.5)$ when the material under consideration is an isotropic homogeneous material. It can be seen that the values of μ at point $(0.2, 0.5)$ coincide with those at point $(0.8, 0.5)$. This is to be expected as the system is symmetrical about $x_1 = 0.5$ when the material is isotropic homogeneous. However, if the material is anisotropic homogeneous the values of μ at point $(0.2, 0.5)$ do not coincide with those at point $(0.8, 0.5)$. See Figure 3. This means anisotropy gives effect on the values of μ. Similarly, if the material is isotropic inhomogeneous (see Figure 4) the values of μ at point $(0.2, 0.5)$ also differ from

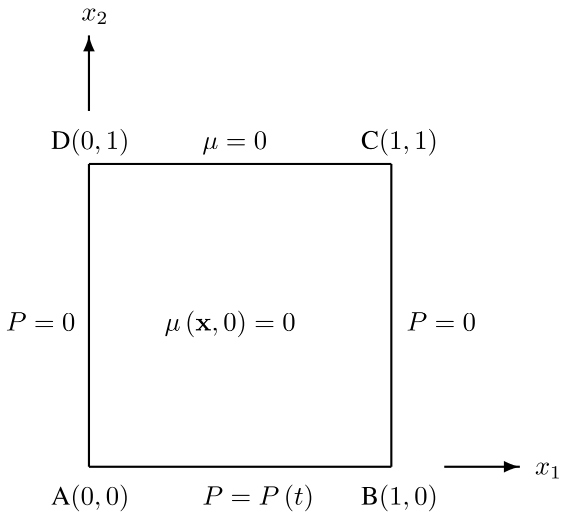

Fig. 2. The boundary conditions for Problem 2.

TABLE VIII
SOLUTION μ AT POINTS $(0.2, 0.5), (0.8, 0.5)$ FOR PROBLEM 2 OF ISOTROPIC HOMOGENEOUS MATERIAL.

t	μ	
	$(0.2, 0.5)$	$(0.8, 0.5)$
0.0005	-0.000000	-0.000000
0.5	0.066071	0.066071
1.0	0.198064	0.198064
1.5	0.321578	0.321578
2.0	0.414356	0.414356
2.5	0.469854	0.469854
3.0	0.486198	0.486198
3.5	0.462821	0.462821
4.0	0.399551	0.399551
4.5	0.296340	0.296340
5.0	0.153175	0.153175

those at point $(0.8, 0.5)$. This indicates that inhomogeneity also gives effect on the values of μ.

In addition, Figures 3 and 4 show that the trends of μ values (as the time t changes) follow the time variation of $P(t)$ except for the form of $P(t) = 1$. This is to be expected as $P(t)$, acting as the boundary condition on side AB, is the only time-dependent quantity for the system, and the coefficients $\kappa_{ij}(\mathbf{x}), \alpha(\mathbf{x})$ are time independent. Moreover, as shown in Figure 4, it is also expected that the values of μ for the cases of $P_1(t) = 1$ and $P_2(t) = 1 - \exp(-1.75t)$ tend to approach same steady state solution as t increases. Both functions $P_1(t) = 1$ and $P_2(t) = 1 - \exp(-1.75t)$ will converge to 1 as t gets bigger.

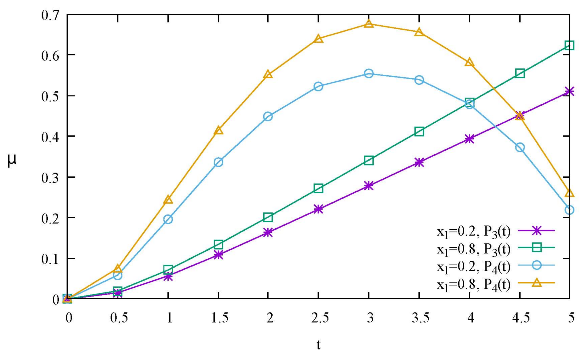

Fig. 3. Solution μ at points $(0.2, 0.5), (0.8, 0.5)$ for Problem 2 of anisotropic homogeneous material.

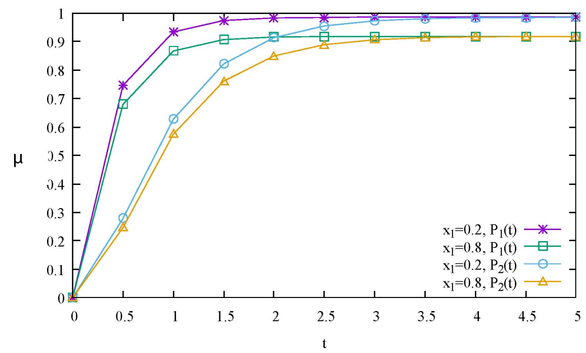

Fig. 4. Solution μ at points $(0.2, 0.5), (0.8, 0.5)$ for Problem 2 of isotropic inhomogeneous material.

V. CONCLUSION

A combined Laplace transform and standard BEM has been used to find numerical solutions to initial boundary value problems for anisotropic functionally graded materials which are governed by the parabolic equation (1). It is easy to implement and accurate. It involves a time variable free fundamental solution and therefore that is why it would be more accurate. Unlikely, the methods with time variable fundamental solution may produce less accurate solutions as the fundamental solution usually has singular time points.

It has been applied to a class of functionally graded materials, namely trigonometrically graded materials. As the coefficients $\kappa_{ij}(\mathbf{x}), \beta(\mathbf{x}), \alpha(\mathbf{x})$ do depend on the spatial variable $\mathbf{x}$ only and on the same inhomogeneity or gradation function $g(\mathbf{x})$, it is interesting to extend the study in the future to the case when the coefficients depend on different gradation functions varying also with the time variable t.

In order to use the boundary integral equation (19), the values $\mu(\mathbf{x}, t)$ or $P(\mathbf{x}, t)$ of the boundary conditions as stated in Section (II) of the original system in time variable t have to be Laplace transformed first. This means that from the beginning when we set up a problem, we actually put a set of approached boundary conditions. Therefore it is really important to find a very accurate technique of numerical Laplace transform inversion. Based on the results of problems in Section IV-A, the Stehfest formula is quite accurate.

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