Color Laplacian Energy of Generalised Complements of a Graph

Swati Nayak, Sabitha D'Souza* and Pradeep G. Bhat

***Abstract*—The color energy of a graph is defined as sum of absolute color eigenvalues of graph, denoted by $E_c(G)$. Let $G_c = (V, E)$ be a color graph and $P = \{V_1, V_2, \dots, V_k\}$ be a partition of V of order $k \geq 1$. The k -color complement $\{G_c\}_k^P$ of G_c is defined as follows: For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j and add the edges which are not in G_c such that end vertices have different colors. For each set V_r in the partition P, remove the edges of G_c inside V_r, and add the edges of $\overline{G_c}$ (the complement of G_c) joining the vertices of V_r. The graph $\{G_c\}_{k(i)}^P$ thus obtained is called the $k(i)-$ color complement of G_c with respect to the partition P of V. In this paper, we compute color Laplacian energy of generalised complements of few standard graphs. Color Laplacian energy depends on assignment of colors to the vertices and the partition of $V(G)$.**

***Index Terms*—$k-$color complement, $k(i)-$color complement, color Laplacian energy, color Laplacian spectrum.**

I. Introduction

Graphs considered in this paper are simple, undirected and without self loops. In an attempt to generalize the concept of complement of a graph G, Sampathkumar et al. [7] have introduced the concept of G_k^P and $G_{k(i)}^P$ with respect to a partition P of $V(G)$. Several results appeared in literature about these complements recently. For all notations and terminologies we refer [1], [2]. Now we give definitions of G_k^P and $G_{k(i)}^P$.

Definition 1. *[6] Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \dots, V_k\}$ be a partition of V of order $k \geq 1$. The $k-$complement G_k^P of G is defined as follows: For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j and add the edges which are not in G.*

The graph G is $k-$ self complementary $(k-s.c)$ with respect to P if $G_k^P \cong G$. Further, G is $k-$co$-$self complementary $(k-co-s.c.)$ if $G_k^P \cong \overline{G}$.

Definition 2. *[7] For each set V_r in the partition P, remove the edges of G inside V_r and add the edges of $\overline{G}$ joining the vertices of V_r. The graph $G_{k(i)}^P$ thus obtained is called the $k(i)-$complement of G with respect to the partition P of V.*

The graph G is $k(i)-$self complementary $(k(i)-s.c)$ if $G_{k(i)}^P \cong G$ for some partition P of order k. Further, G is $k(i)-$co$-$self complementary $(k(i)-co-s.c.)$ if $G_k^P \cong \overline{G}$.

The energy of a graph is the sum of absolute eigenvalues of the adjacency matrix of G. This concept was defined in 1978 and originated from theoretical Chemistry. It is related to the total π-electron energy in a molecule represented by a molecular graph. More on graph energy, one can refer [3], [4], [5], [11], [12], [13].

A coloring of graph G is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring G is called chromatic number, denoted by $\chi(G)$.

The color matrix $A_c(G) = a_{ij}$ of a colored graph is defined as follows. If $c(v_i)$ is the color of vertex v_i, then

$$a_{ij} = \begin{cases} 1, & if\ v_i \sim v_j\ \ with\ c(v_i) \neq c(v_j), \\ -1, & if\ v_i \nsim v_j\ \ with\ c(v_i) = c(v_j), \\ 0, & otherwise. \end{cases}$$

The set of eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of $A_c(G)$ is called the color eigenvalues of G. Color energy of graph is the sum of absolute colored eigenvalues.

i.e.,

$$E_c(G) = \sum_{i=1}^{n} \lambda_i.$$

The concept of color energy was introduced by Adiga et al. [8] and the origin of the color Laplacian energy [9] and color signless Laplacian energy [10] were established by Bhat et al. in the succeeding years. These introductory papers deal with the investigation of these three energies of null graph, star graph, complete graph, complete bipartite graph, crown graph and cocktail party graph.

The color Laplacian energy is defined as $L_c(G) = D(G) - A_c(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of the graph G. The eigenvalues $\{\mu_1, \mu_2, \dots, \mu_n\}$ of $L_c(G)$ are called as the color Laplacian eigenvalues of the graph G. Color Laplacian energy of G of order n and size m, denoted by $LE_c(G)$ is defined as

$$LE_c(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$$

The article is organized as follows. In section II, we recall the definitions of generalised color complements and some preliminary results. In section III, we derive color Laplacian spectrum and energy of generalized complements of some families of graph.

II. Generalised k and $k(i)$ color complements of a graph

In 2020, the authors in [14] have introduced generalised color complements of a graph.

Manuscript received November 30, 2020; revised July 25, 2021.

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Definition 3. *[14] Let $G_c = (V, E)$ be a color graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of order $k \geq 1$. The k -color complement $\{G_c\}_k^P$ of G_c is defined as follows:*

For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j and add the edges in which end vertices have different colors in G_c .

- The graph G_c is $k-$ self color complementary $(k{-}s.c.c)$ with respect to P if $\{G_c\}_k^P \cong G_c$.
- Further, G_c is $k-$co$-$self color complementary $(k - co - s.c.c)$ if $\{G_c\}_k^P \cong \overline{G}_c$.

Example 4. *$2-$color complement of path P_4 is $2-$self color complementary as $(P_4)_c \cong \{(P_4)_c\}_2^P$.*

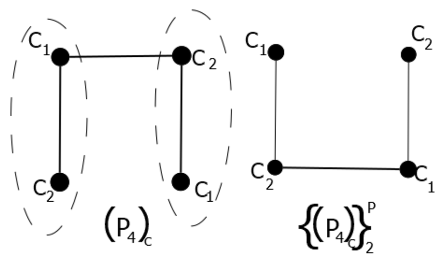

Fig. 1. Colored Path P_4 and its $2-$ complement.

Definition 5. *[14] For each set V_r in the partition P, remove the edges of G_c inside V_r, and add the edges of $\overline{G_c}$ (the complement of G_c) joining the vertices of V_r. The graph $\{G_c\}_{k(i)}^P$ thus obtained is called the $k(i)-$ color complement of G_c with respect to the partition P of V.*

- The graph G_c is $k(i)-$ self color complementary $(k(i)-s.c.c)$ if $\{G_c\}_{k(i)}^P \cong G_c$ for some partition P of order k.
- Further, G_c is $k(i)-$co$-$self color complementary $(k(i) - co - s.c.c)$ if $\{G_c\}_{k(i)}^P \cong \overline{G}_c$.

Example 6.

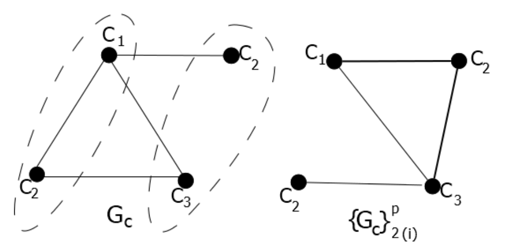

Fig. 2. Graph G_c and $\{G_c\}_{2(i)}^P$

Proposition 7. *[3] Let A= $\begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$ be a symmetric 2×2 block matrix. Then spectrum of A is the union of spectra of $A_0 + A_1$ and $A_0 - A_1$.*

Proposition 8. *[3] Let $M; N;\ P; Q$ be matrices and M be invertible. Let S= $\begin{bmatrix} M & N \\ P & Q \end{bmatrix}$.*
Then $det\ S = det\ M.\ det[Q - PM^{-1}N]$. If M and P commute, then $det\ S = det\ [MQ - PN]$.

III. Color Laplacian spectrum of generalised color complements of a graph

In this section, we consider color Laplacian characteristic polynomial, color Laplacian spectrum and color Laplacian energy of $k-$ color complement of graph G with respect to minimum number of colors and we denote these by $P_\chi(G,\mu)_k^P$, $Lspec_\chi(G_k^P)$ and $LE_\chi(G_k^P)$ respectively. Similar notations are followed for $k(i)-$ color complementary graph. Throughout this paper $0, I, J$ and B represent zero matrix, identity matrix, matrix of all 1's and adjacency matrix of complete subgraph respectively.

A. Star graph

Theorem 9. *Let $K_{1,n-1}$ be colored star graph with partition $P = \{V_1, V_2, \ldots, V_k\}$, where $V_1 = \{v_1, v_2, \ldots, v_m\}$, v_1 being central vertex. Then*

(i) $P_\chi(K_{1,n-1},\mu)_k^P = (\mu+1)^{n-m-1}\mu^{m-2}[\mu^3 + (3-n-m)\mu^2 + (mn-n-4m+4)\mu - 2m - n + mn + 2]$.

(ii) $P_\chi(K_{1,n-1},\mu)_{k(i)}^P = \mu^{n-m-1}(\mu+1)^{m-2}[\mu^3 + (m-2n+2)\mu^2 + (4m-4n-mn+n^2)\mu + 2m - mn - 2n + n^2]$.

Proof:

(i) Since $\chi(K_{1,n-1}) = 2$, color adjacency matrix of $k-$ color complement of star graph is $A_\chi(K_{1,n-1})_k^P$

$$= \left[\begin{array}{c|c|c} 0_1 & J_{1\times m-1} & 0_{1\times n-m} \\ \hline J_{m-1\times 1} & -B_{m-1} & -J_{m-1\times n-m} \\ \hline 0_{n-m\times 1} & -J_{n-m\times m-1} & -B_{n-m} \end{array}\right]_n$$

Diagonal matrix of vertex degree is

$$D(K_{1,n-1})_k^P = diag(m-1, 1, \ldots, 1, 0, \ldots, 0).$$

Color Laplacian matrix of $k-$color complement of $K_{1,n-1}$ is

$$L_\chi(K_{1,n-1})_k^P = D(K_{1,n-1})_k^P - A_\chi(K_{1,n-1})_k^P$$

$$= \left[\begin{array}{c|c|c} (m-1)_1 & -J_{1\times m-1} & 0_{1\times n-m} \\ \hline -J_{m-1\times 1} & (I+B)_{m-1} & J_{m-1\times n-m} \\ \hline 0_{n-m\times 1} & J_{n-m\times m-1} & B_{n-m} \end{array}\right]$$

$$P_\chi(K_{1,n-1},\mu)_k^P = |\mu I - L_\chi(K_{1,n-1})_k^P|. \quad (10)$$

Step 1: For rows $i = 2, 3, 4, \ldots, m-1, m+1, \ldots, n-1$, replace R_i by $R_i - R_{i+1}$ in expression (10).
Then $P_\chi(K_{1,n-1},\mu)_k^P = \mu^{m-2}(\mu+1)^{n-m-1}\det(C)$.
Step 2: In $\det(C)$, performing $C_i \to C_i + C_{i-1} + \ldots + C_2$, $i = n, n-1, \ldots, m+1, m-1, \ldots, 3$, it reduces to a determinant of order 3.

$$\det(C) = \begin{vmatrix} \mu - m + 1 & m-1 & 0 \\ 1 & \mu - m + 1 & m - n \\ 0 & 1 - m & \mu - n + m + 1 \end{vmatrix}$$

Step 3: By simplifying, we get
$P_\chi(K_{1,n-1},\mu)_k^P = \mu^{m-2}(\mu+1)^{n-m-1}[\mu^3 + (3-n-m)\mu^2 + (mn-n-4m+4)\mu - 2m - n + mn + 2]$.

(ii) Color Laplacian matrix of $k(i)-$ color complement of $K_{1,n-1}$ is $L_\chi(K_{1,n-1})_{k(i)}^P$

$$= \left[\begin{array}{c|c|c} n-m & 0_{1\times m-1} & -J_{1\times n-m} \\ \hline 0_{m-1\times 1} & B_{m-1} & J_{m-1\times n-m} \\ \hline -J_{n-m\times 1} & J_{n-m\times m-1} & (I+B)_{n-m} \end{array}\right]_n$$

Consider $|\mu I - L_\chi(K_{1,n-1})_k^P|$

$$= \left|\begin{array}{c|c|c} \mu - n + m & 0 & J \\ \hline 0 & \mu I - B & -J \\ \hline J & -J & (\mu - 1)I - B \end{array}\right|_n$$

Then repeating the steps 1 and 2 of Theorem $9(i)$, the result follows.

■

Theorem 11. *Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of colored star graph $K_{1,n-1}$ such that only central vertex be in V_1. Then*

(i) $LE_\chi(K_{1,n-1})_k^P = 2(n-2)$.

(ii) $LE_\chi(K_{1,n-1})_{k(i)}^P = \dfrac{4(n-1)(n-2)}{n}$.

Proof: Since $K_{1,n-1}$ is $k-$co$-$self color complementary with respect to the given partition P, $LE_\chi(K_{1,n-1})_k^P = 2(n-2)$. Also $G_{k(i)}^P \cong G$ if and only if $G_k^P \cong \overline{G}$. So $LE_\chi(K_{1,n-1})_{k(i)}^P = \dfrac{4(n-1)(n-2)}{n}$.

Proof is similar to Theorem [4.5, 4.6] of [9]. ∎

Observation 12. *For complete graph K_n,*

1) $LE(K_n)_k^P = LE_\chi(K_n)_k^P$.

2) $LE(K_n)_{k(i)}^P = LE_\chi(K_n)_{k(i)}^P$.

B. Double star

Definition 13. *A double star $S\{l,m\}$ is the graph consisting of union of two stars $K_{1,l-1}$ and $K_{1,m-1}$ together with the line joining their centers.*

Theorem 14. *Let $P = \{V_1, V_2\}$ be a partition of colored double star $S\{l,m\}$ such that $\langle V_1\rangle = K_{1,l-1}$ and $\langle V_2\rangle = K_{1,m-1}$. Then*

(i) $P_\chi(S\{l,m\},\mu)_2^P = (\mu-l+1)^{m-2}(\mu-m+1)^{l-2}[\mu^4 + (6-3m-3l)\mu^3 + (3l^2+6lm-14l+3m^2-14m+16)\mu^2 + (-l^3-4l^2m+10l^2-4lm^2+18lm-23l-m^3+10m^2-23m+18)\mu + l^3m - 2l^3 + l^2m^2 - 6l^2m + 8l^2 + lm^3 - 6lm^2 + 14lm - 12l - 2m^3 + 8m^2 - 12m + 7]$.

(ii) $P_\chi(S\{l,m\},\mu)_{2(i)}^P = (\mu+1)^{l+m-4}[\mu^4 + (2-l-m)\mu^3 + (lm-l-m-2)\mu^2 + 2(l+m-3)\mu + 2(l+m) - lm - 3]$.

Proof:

(i) Since chromatic number of double star is 2, Laplacian matrix of $2-$color complement is

$$L_\chi(S\{l,m\})_2^P = \left[\begin{array}{c|c|c} \begin{pmatrix} l-1 & 0 \\ 0 & m-1 \end{pmatrix}_2 & -C_{2\times l-1} & C_{2\times m-1} \\ \hline -C'_{l-1\times 2} & (mI+B)_{l-1} & -J_{l-1\times m-1} \\ \hline C'_{m-1\times 2} & -J_{m-1\times l-1} & (lI+B)_{m-1} \end{array}\right]_n,$$

$$C = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ -1 & -1 & \ldots & -1 \end{pmatrix}.$$

Consider $|\mu I - L_\chi(S\{l,m\})_2^P|$.

Step 1: Using row operation $R_i \to R_i - R_{i+1}, i = 3,4,\ldots,l-2,l,\ldots,m-2$ and by further simplification, we obtain $(\mu-l+1)^{m-2}(\mu-m+1)^{l-2}\det(D)$.

Step 2: On applying the column operation $C_i \to C_i + C_{i+1} + \ldots + C_n, i = 3,4,\ldots,n-1$ on $\det(D)$, the result follows.

(ii) Color Laplacian matrix of $2(i)-$ color complement of double star is $L_\chi(S\{l,m\})_{2(i)}^P$

$$= \left[\begin{array}{c|c|c} (I-B)_{2\times 2} & -C_{2\times l-1} & -D_{2\times m-1} \\ \hline -C'_{l-1\times 2} & B_{l-1\times l-1} & 0_{l-1\times m-1} \\ \hline -D'_{m-1\times 2} & 0_{m-1\times l-1} & B_{m-1\times m-1} \end{array}\right]_n,$$

$$C = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ -1 & -1 & \ldots & -1 \end{pmatrix} \text{ and}$$

$$D = \begin{pmatrix} -1 & -1 & \ldots & -1 \\ 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Consider $|\mu I - L_\chi(S\{l,m\})_{2(i)}^P|$.

Step 1: For rows $i = 3,4,\ldots,l-2,l,\ldots,m-2$, using row operation $R_i \to R_i - R_{i+1}$, we get $(\mu+1)^{l+m-4}\det(E)$.

Step 2: On applying the column operations $C_i \to C_i + C_{i-1} + \ldots + C_l, i = m-1, m-2, \ldots, l+1$ and $C_j \to C_j + C_{j-1} + \ldots + C_3, j = l-1, l-2, \ldots, 4$ on $\det(E)$, we obtain new determinant F so that $|\mu I - L_\chi(S\{l,m\})_{2(i)}^P| = (\mu+1)^{l+m-4}\det(F)$.

Step 3: On expanding $\det(F)$ along the rows from 3^{rd} row to $(l-2)^{th}$ row and then from l^{th} row to $(m-2)^{th}$ row, we get $det(F) = (\mu+1)^{l+m-2}\det(G)$.

$$\det(G) = \begin{vmatrix} \mu-1 & 1 & 0 & 1-m \\ 1 & \mu-1 & 1-l & 0 \\ 0 & -1 & \mu-l+2 & 0 \\ -1 & 0 & 0 & \mu-m+2 \end{vmatrix}$$

Step 4: The characteristic polynomial is obtained by expanding $\det(G)$ and by back substitution. ∎

C. Complete bipartite graph

Theorem 15. *Let $P = \{V_1, V_2\}$ be a partition of colored complete bipartite graph $K_{r,s}$ such that $\langle V_1 \cup V_2\rangle$ be union of color complete bipartite subgraphs. Then* $P_\chi(K_{r,s},\mu)_2^P = P_\chi(K_{r,s},\mu)_{2(i)}^P = (\mu-a+1)^{b-1}(\mu-b+1)^{a-1}(\mu-r+a+1)^{s-b-1}(\mu-s+b+1)^{r-a-1}[\mu^4 + (4-2s-2r)\mu^3 + (-a^2-6ab+ar+3as-b^2+3br+bs+r^2+rs-6r+s^2-6s+6)\mu^2 + (a^2r-a^2s-2a^2+4abr+4abs-12ab-ar^2-ars+2ar-2as^2+6as-b^2r+b^2s-2b^2-2br^2-brs+6br-bs^2+2bs+2r^2+2rs-6r+2s^2-6s+4)\mu + a^2r + a^2s^2 - a^2s - a^2 - 2abrs + 4abr + 4abs - 6ab - ar^2 - ars + ar - 2as^2 + 3as + b^2r^2 - b^2r + b^2s - b^2 - 2br^2 - brs + 3br - bs^2 + bs + r^2 + rs - 2r + s^2 - 2s + 1]$.

Proof: As 2 and $2(i)$ color complement of complete bipartite graph are union of colored complete bipartite subgraphs i.e, $K_{a,b} \cup K_{r-a,s-b}$, we have

$$L_\chi(K_{r,s})_2^P = \left[\begin{array}{c|c|c|c} [bI+B]_a & -J_{a\times b} & J_{a\times r-a} & 0_{a\times s-b} \\ \hline -J_{b\times a} & [aI+B]_b & 0_{b\times r-a} & J_{b\times s-b} \\ \hline J_{r-a\times a} & 0_{r-a\times b} & [(s-b)I+B]_{r-a} & -J_{r-a\times s-b} \\ \hline 0_{s-b\times a} & J_{s-b\times b} & -J_{s-b\times r-a} & [(r-a)I+B]_{s-b} \end{array}\right]_n$$

Consider $|\mu I - L_\chi(K_{r,s})_2^P|$.

Step 1: Applying the row operation $R_i \to R_i - R_{i+1}$, where $i \neq a, b, r-a, s-b$, we see that $|\mu I - L_\chi(K_{r,s})_2^P| = (\mu-a+1)^{b-1}(\mu-b+1)^{a-1}(\mu-r+a+1)^{s-b-1}(\mu-s+b+1)^{r-a-1}\det(C)$.

Step 2: On applying column operation $C_i \to C_i + C_{i+1} + \ldots + C_n$ on $\det(C)$ for $i = 1,2,\ldots,n-1$, we get $\det(D)$. Hence

$$P_\chi(K_{r,s},\mu)_2^P = (\mu-a+1)^{b-1}(\mu-b+1)^{a-1}$$
$$(\mu-r+a+1)^{s-b-1}(\mu-s+b+1)^{r-a-1}\det(D). \quad (16)$$

i.e, $\det(D) =$

$$\begin{vmatrix} \mu-a-b+1 & b & a-r & 0 \\ a & \mu-a-b+1 & 0 & b-s \\ -a & 0 & \mu+a+b-r-s+1 & s-b \\ 0 & -b & r-a & \mu+a+b-r-s+1 \end{vmatrix}$$

By expanding $\det(D)$ and substituting in equation 16, we obtain

$P_\chi(K_{r,s},\mu)_2^P = (\mu-a+1)^{b-1}(\mu-b+1)^{a-1}(\mu-r+a+1)^{s-b-1}(\mu-s+b+1)^{r-a-1}[\mu^4 + (4-2s-2r)\mu^3 + (-a^2-6ab+ar+3as-b^2+3br+bs+r^2+rs-6r+s^2-6s+6)\mu^2 + (a^2r-a^2s-2a^2+4abr+4abs-12ab-ar^2-ars+2ar-2as^2+6as-b^2r+b^2s-2b^2-2br^2-brs+6br-bs^2+2bs+2r^2+2rs-6r+2s^2-6s+4)\mu + a^2r + a^2s^2 - a^2s - a^2 - 2abrs + 4abr + 4abs - 6ab - ar^2 - ars + ar - 2as^2 + 3as + b^2r^2 - b^2r +$

$b^2s-b^2-2br^2-brs+3br-bs^2+bs+r^2+rs-2r+s^2-2s+1]$. ■

Theorem 17. *For a colored complete bipartite graph $K_{r,s}$ with respect to partition of same color class is*

(i) $LE_\chi(K_{r,s})^P_2 = 2(r+s-2)$.

(ii) $LE_\chi(K_{r,s})^P_{2(i)}$

$$= \begin{cases} \frac{2}{r+s}[r^2+s^2+(r+s)\sqrt{rs}-1], & if\ r=s\ and\ s=r+1 \\ \frac{2\sqrt{rs}}{r+s}[(s-r)\sqrt{rs}+(r+s)], & if\ s>r+1 \end{cases}$$

Proof: Since $K_{r,s}$ with respect to same color class partites is 2−co−self and 2(i)−self color complementary, proof of Theorem 17 follows from Theorem [4.7, 4.8] of [9]. ■

D. Crown graph

Theorem 18. *Let $P=\{V_1,V_2\}$ be a partition of crown graph S^0_n with $|V_1|=1$. Then*

(i) $P_\chi(S^0_n,\mu)^P_2 = [\mu^2+(5-2n)\mu+n^2-5n+5]^{n-2}[\mu^4+(6-5n)\mu^3+(7n^2-15n+13)\mu^2-(3n^3-8n^2-n+8)\mu+n^3-6n^2+11n-6]$.

(ii) $P_\chi(S^0_n,\mu)^P_{2(i)} = [\mu^2-\mu-1]^{n-2}[\mu^4+(2-3n)\mu^3+(3n^2-7n+3)\mu^2+(-n^3+6n^2-9n+4)\mu-n^3+4n^2-5n+2]$.

Proof:

(i) Color Laplacian matrix of 2−color complement of S^0_n is $L_\chi(S^0_n)^P_2 =$

$$\left[\begin{array}{c|c|c|c} J_1 & J_{1\times n-1} & -J_1 & 0_{1\times n-1} \\ \hline J_{n-1\times 1} & [(n-2)I+J]_{n-1} & -J_{n-1\times 1} & [I-J]_{n-1} \\ \hline -J_1 & -J_{1\times n-1} & n & J_{n-1\times 1} \\ \hline 0_{n-1\times 1} & [I-J]_{n-1} & J_{n-1\times 1} & [(n-3)I+J]_{n-1} \end{array}\right]$$

Consider $|\mu I - L_\chi(S^0_n)^P_2|$.

Step 1: Using row operation $R_i \to R_i - R_{i+1}$, $i = 2,3,\ldots,n-2,n-1,n+2,n+3,\ldots,2n-1$ and column operations $C_i \to C_i + C_{i-1} + \ldots + C_{n+2}$, $i = 2n, 2n-1, \ldots, n+3$, $C_j \to C_j + C_{j-1} + \ldots + C_2$, $i = n, n-1, \ldots, 3$ on $|\mu I - L_\chi(S^0_n)^P_2|$, we get $\det(A)$.

Step 2: On applying the column operation $C_{n+i} \to \mu C_{n+i} - C_i, i = 2,3,\ldots,n-1$ on $\det(A)$, we obtain the result.

(ii) Color Laplacian matrix of 2(i)− color complement of S^0_n is $L_\chi(S^0_n)^P_{2(i)}$

$$= \left[\begin{array}{c|c|c|c} (n-1)J_1 & J_{1\times n-1} & 0_1 & -J_{1\times n-1} \\ \hline J_{n-1\times 1} & J_{n-1} & 0_{n-1\times 1} & -I_{n-1} \\ \hline 0_1 & 0_{1\times n-1} & 0_1 & J_{n-1\times 1} \\ \hline -J_{n-1\times 1} & -I_{n-1} & J_{n-1\times 1} & [I+J]_{n-1} \end{array}\right]_{2n}$$

Consider $|\mu I - L_\chi(S^0_n)^P_{2(i)}|$.

Step 1: First we apply row operation $R_i \to R_i - R_{i+1}, i = 2,3,\ldots,n-2,n-1,n+2,n+3,\ldots,2n-1$ and followed by column operations $C_i \to C_i + C_{i-1} + \ldots + C_{n+2}, i = 2n, 2n-1, \ldots, n+3$, $C_j \to C_j + C_{j-1} + \ldots + C_2, i = n, n-1, \ldots, 3$ on $|\mu I - L_\chi(S^0_n)^P_2|$, we get $\det(A)$.

Step 2: On applying the column operation $C_{n+i} \to C_{n+i}(\mu - n + 2) + C_i, i = 2,3,\ldots,n-1$ on $\det(A)$, we get the result.

■

Theorem 19. *Let $P=\{V_1,V_2\}$ be a partition of crown graph S^0_n with vertices of same color class. Then* $LE_\chi(S^0_n)^P_2 = LE_\chi(S^0_n)^P_{2(i)} = 4(n-1)$.

Proof: Since S^0_n is 2−co self and 2(i)−self color complementary with respect to the partition of same color class, we obtain

$LSpec_\chi(S^0_n)^P_2 = \begin{Bmatrix} -1 & n-1 & 1 & n+1 \\ n-1 & 1 & n-1 & 1 \end{Bmatrix}$ and

$LSpec_\chi(S^0_n)^P_{2(i)} = \begin{Bmatrix} n-3 & n-1 & 3(n-1) \\ n-1 & n & 1 \end{Bmatrix}$.

Proof follows from [Theorem 4.9, 4.10] of [9]. ■

E. Friendship graph

Theorem 20. *For a colored friendship graph F_n with partition $P = \{V_1, V_2, \ldots, V_{n+1}\}$ such that central vertex is in V_1 and $\langle V_i\rangle = K_2$ for $i = 2,3,\ldots,n+1$,*

(i) $LE_\chi(F_n)^P_{n+1} = \frac{2(4n^2+n-1)}{2n+1}$.

(ii) $P_\chi(F_n,\mu)^P_{n+1(i)} = \mu^{2n-2}[\mu^3-4n\mu^2+(5n^2-2n)\mu-2n^3+2n^2]$.

Proof:

(i) Since $\chi(F_n)=3$, $L_\chi(F_n)^P_{n+1}$

$$= \left[\begin{array}{c|c|c|c|c} (nI-B)_2 & -C_2 & \ldots & -C_2 & 0_{2\times 1} \\ \hline -C_2 & (nI-B)_2 & \ldots & -C_2 & 0_{2\times 1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline -C_2 & -C_2 & \ldots & (nI-B)_2 & 0_{2\times 1} \\ \hline 0_{1\times 2} & 0_{1\times 2} & \ldots & 0_{1\times 2} & 0_1 \end{array}\right]_{2n+1},$$

where $C = J - 2I$.

Now $P_\chi(F_n,\mu)^P_{n+1} = |\mu I - L_\chi(F_n)^P_{n+1}|$.

Step 1: On applying row operation $R_i \to R_i + R_{i+1}$, $i = 1,3,...,2n-1$ and then using column operation $C_i \to C_i - C_{i-1}, i = 2,4,\ldots,2n$, we get

$$P_\chi(F_n,\mu)^P_{n+1} = \mu(\mu-n+1)^n\left|(\mu-n-1)I-2B\right|_n.$$

Step 2: Again using row operation $R_i \to R_i - R_{i+1}, i = 1,2,\ldots,n-1$ and followed by column operation $C_i \to C_i + C_{i-1} + \ldots + 1, i = n, n-1, \ldots, 2$, we obtain $|\mu I - L_\chi(F_n)^P_{n+1}| = \mu(\mu-n+1)^{2n-1}(\mu-3n+1)$.

Hence, color Laplacian spectrum of $(F_n)^P_{n+1}$ is

$Lspec_\chi(F_n)^P_{n+1} = \begin{pmatrix} 0 & n-1 & 3n-1 \\ 1 & 2n-1 & 1 \end{pmatrix}$

and average vertex degree of $(F_n)^P_{n+1}$ is $\frac{2n^2}{2n+1}$.

Thus

$$\begin{aligned} L_\chi(F_n)^P_{n+1} = & \left|\frac{2n^2}{2n+1} - 0\right| + (2n-1) \\ & \left|\frac{2n^2}{2n+1} - (n-1)\right| \\ & + \left|3n-1-\frac{2n^2}{2n+1}\right| \\ = & \frac{2(4n^2+n-1)}{2n+1}. \end{aligned}$$

(ii) Further $L_\chi(F_n)^P_{n+1(i)}$

$$= \left[\begin{array}{c|c|c|c|c} I_2 & I_2 & \ldots & I_2 & -J_{2\times 1} \\ \hline I_2 & I_2 & \ldots & I_2 & -J_{2\times 1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline I_2 & I_2 & \ldots & I_2 & -J_{2\times 1} \\ \hline -J_{1\times 2} & -J_{1\times 2} & \ldots & -J_{1\times 2} & 2nI_1 \end{array}\right]_{(2n+1)}$$

By following the step 1 and step 2 of Theorem 20(i), we get $P_\chi(F_n,\mu)^P_{n+1(i)} = \mu^{2n-2}[\mu^3-4n\mu^2+(5n^2-2n)\mu-2n^3+2n^2]$.

■

Theorem 21. *For a colored friendship graph F_n of same color class partition $P = \{V_1, V_2, V_3\}$,*

(i) $LE_\chi(F_n)_3^P = \dfrac{10n^2 - 4n - 6}{2n+1}$.

(ii) $P_\chi(F_n, \mu)_{3(i)}^P = \mu^{n-1}(\mu-2)^{n-1}[\mu^3 - 2(2n+1)\mu^2 + (5n^2+4n)\mu - (2n^3 + 2n^2 - 4n)]$.

Proof:

(i) Color Laplacian matrix of 3− color complement of F_n is $L_\chi(F_n)_3^P$

$$= \left[\begin{array}{c|c|c|c|c} (n-1)I_2 & -C_2 & \dots & -C_2 & 0_{2\times 1} \\ \hline -C_2 & (n-1)I_2 & \dots & -C_2 & 0_{2\times 1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline -C_2 & -C_2 & \dots & (n-1)I_2 & 0_{2\times 1} \\ \hline 0_{1\times 2} & 0_{1\times 2} & \dots & 0_{1\times 2} & 0_1 \end{array}\right]_{2n+1},$$

where $C = J - 2I$.

Consider $P_\chi(F_n, \mu)_3^P = |\mu I - L_\chi(F_n)_3^P|$.

On applying block row operation $R_i \to R_i - R_{i+1}, i = 1, 2, \dots, n-1$ and block column operation $C_i \to C_i + C_{i-1} + \dots + C_1, i = n, n-1, \dots, 2$ on $|\mu I - L_\chi(F_n)_3^P|$, we obtain

$$\begin{aligned} P_\chi(F_n, \mu)_3^P =& \mu[(\mu - n + 2)^2 - 1]^{n-1} \\ & \begin{vmatrix} \mu - 2n + 2 & n-1 \\ n-1 & \mu - 2n + 2 \end{vmatrix} \\ =& \mu[(\mu - n + 2)^2 - 1]^{n-1} \\ & [(\mu - 2n + 2)^2 - (n-1)^2]. \end{aligned}$$

Therefore, color Laplacian spectrum of $(F_n)_3^P$ is

$$Lspec_\chi(F_n)_3^P = \begin{pmatrix} 0 & n-1 & n-3 & 3(n-1) \\ 1 & n & n-1 & 1 \end{pmatrix}.$$

Average vertex degree of $(F_n)_3^P$ is $\dfrac{2n(n-1)}{2n+1}$.

Hence

$$\begin{aligned} LE_\chi(F_n)_3^P =& \frac{2n(n-1)}{2n+1} + n\left|n - 1 - \frac{2n(n-1)}{2n+1}\right| \\ & + (n-1)\left|\frac{2n(n-1)}{2n+1} - (n-3)\right| \\ & + \left|3(n-1) - \frac{2n(n-1)}{2n+1}\right| \\ =& \frac{10n^2 - 4n - 6}{2n+1}. \end{aligned}$$

(ii) Further, $L_\chi(F_n)_{3(i)}^P$

$$= \left[\begin{array}{c|c|c|c|c} (2I-B)_2 & I_2 & \dots & I_2 & -J_{2\times 1} \\ \hline I_2 & (2I-B)_2 & \dots & I_2 & -J_{2\times 1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline I_2 & I_2 & \dots & (2I-B)_2 & -J_{2\times 1} \\ \hline -J_{1\times 2} & -J_{1\times 2} & \dots & -J_{1\times 2} & 2nI_1 \end{array}\right]_{2n+1}$$

Repeating row and column operations of Theorem $21(i)$, we get the required result.

■

Theorem 22. *Let $\{V_1, V_2\}$ be a partition of colored friendship graph such that all peripheral vertices be in V_2. Then*

(i) $LE_\chi(F_n)_2^P = \dfrac{8n^2 - 2n - 2}{2n+1}$.

(ii) $P_\chi(F_n, \mu)_{2(i)}^P = [(\mu - n + 1)^2 - 1][\mu^3 + (3-6n)\mu^2 + (11n^2 - 12n + 2)\mu - (6n^3 - 13n^2 + 6n)]$.

Proof:

(i) Color Laplacian matrix of 2−color complement of F_n is $L_\chi(F_n)_2^P$

$$= \left[\begin{array}{c|c|c|c|c} (I-B)_2 & I_2 & \dots & I_2 & 0_{2\times 1} \\ \hline I_2 & (I-B)_2 & \dots & I_2 & 0_{2\times 2} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline I_2 & I_2 & \dots & (I-B)_2 & 0_{2\times 1} \\ \hline 0_{1\times 2} & 0_{1\times 2} & \dots & 0_{1\times 2} & 0_1 \end{array}\right]_{2n+1}$$

Applying row and column operations on $|\mu I - L_\chi(F_n)_2^P|$ as in Theorem $21(i)$, we obtain $P_\chi(F_n, \mu)_2^P = \mu(\mu^2 - 1)^{n-1}[(\mu - n)^2 - 1]$.

So $Lspec_\chi(F_n)_2^P = \begin{pmatrix} 0 & 1 & -1 & n+1 & n-1 \\ 1 & n-1 & n-1 & 1 & 1 \end{pmatrix}$.

Since average vertex degree of $(F_n)_2^P$ is $\dfrac{2n}{2n+1}$,

$$\begin{aligned} LE_\chi(F_n)_2^P =& \frac{2n}{2n+1} + \left|1 - \frac{2n}{2n+1}\right|(n-1) \\ & + \left|\frac{2n}{2n+1} + 1\right|(n-1) + \left|n + 1 - \frac{2n}{2n+1}\right| \\ & + \left|n - 1 - \frac{2n}{2n+1}\right| \\ =& \frac{8n^2 - 2n - 2}{2n+1}. \end{aligned}$$

(ii) Color Laplacian matrix of $2(i)$− color complement of F_n is $L_\chi(F_n)_{2(i)}^P$

$$= \left[\begin{array}{c|c|c|c|c} nI_2 & -B_2 & \dots & -B_2 & -J_2 \\ \hline -B_2 & nI_2 & \dots & -B_2 & -J_{2\times 1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline -B_2 & -B_2 & \dots & nI_2 & -J_{2\times 1} \\ \hline -J_{1\times 2} & -J_{1\times 2} & \dots & -J_{1\times 2} & (2n-1)I_1 \end{array}\right]_{2n+1}$$

On repeating row and column operations of theorem $21(i)$, we get the required result.

■

F. Cocktail party graph

Theorem 23. *Let $K_{n\times 2}$ be a colored cocktail party graph with partition $P = \{V_1, V_2, \dots, V_k\}$ of same color class, then*

- $LE_\chi(K_{n\times 2})_k^P = 2n.$
- $LE_\chi(K_{n\times 2})_{k(i)}^P = 6(n-1).$

Proof: Colored Cocktail party graph is k−co−self and $k(i)$−self color complementary with respect to same color class partition. Hence the result follows. For proof refer Theorem [4.11, 4.12] of [9]. ■

G. Triangular book graph

Definition 24. *Triangular book graph $B(3, n)$ is a planar undirected graph with $n + 2$ vertices and $2n + 1$ edges constructed by n triangles sharing a common edge.*

Example 25.

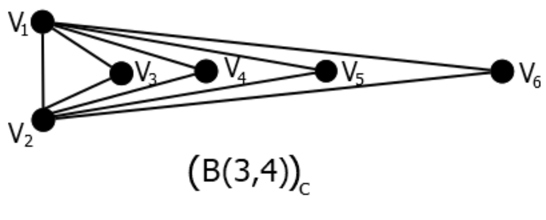

Fig. 3. Color triangular book graph with color class $C_1 = \{v_1\}, C_2 = \{v_2\}, C_3 = \{v_3, v_4, v_5, v_6\}$.

Theorem 26. *Let $\{V_1, V_2\}$ be a partition of colored triangular book graph such that $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4, \ldots, v_n\}$. Then*

(i) $LE_\chi(B(3,n))_2^P = \dfrac{2(n^2+3n-2)}{n+2}.$

(ii) $LE_\chi(B(3,n))_{2(i)}^P = \dfrac{(4n^2-7n+2)}{n+2} + \sqrt{8n+1}.$

Proof:

(i) $L_\chi(B(3,n))_2^P = \begin{bmatrix}(2I-J)_2 & 0_{2\times n}\\ 0_{n\times 2} & (J-I)_n\end{bmatrix}_{n+2}$ is color Laplacian matrix of 2−color complement of $B(3,n)$. The result is proved by showing $LZ = \mu Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue μ is same, as $L_\chi(B(3,n))_2^P$ is real and symmetric.
Let $Z = \begin{bmatrix}X_2\\ Y_n\end{bmatrix}$ be an eigenvector of order $n+2$ partitioned conformally with $L_\chi(B(3,n))_2^P$.
Consider

$$[L_\chi(B(3,n))_2^P - \mu I]\begin{bmatrix}X_2\\ Y_n\end{bmatrix} = \begin{bmatrix}[(2-\mu)I-J]X + 0Y\\ 0X + [J-(1+\mu)I]Y\end{bmatrix}. \tag{27}$$

Case 1: Let $X = 0_2$ and $Y = 1_n$.
From equation (27),

$$[J-(1+\mu)I]1_n = (-\mu+n-1)1_n.$$

Therefore, $\mu = n-1$ is color Laplacian eigenvalue with multiplicity of at least one.
Case 2: Let $X = 0_2$ and $Y = Y_j$. From equation (27),

$$[J-(1+\mu)I]Y_j = -(\mu+1)Y_j.$$

Hence $\mu = -1$ is color Laplacian eigenvalue with multiplicity of at least $n-1$ since there are $n-1$ independent vectors of the form Y_j.
Case 3: Let $X = 1_2$ and $Y = 0_n$.
From equation (27),

$$[(2-\mu)I - J]1_2 = -\mu 1_2.$$

So $\mu = 0$ is color Laplacian eigenvalue with multiplicity of at least one.
Case 4: Let $X = \begin{pmatrix}-1\\ 1\end{pmatrix}$ and $Y = 0_n$.
From equation (27),

$$[(2-\mu)I - J]\begin{pmatrix}-1\\ 1\end{pmatrix} = (2-\mu)\begin{pmatrix}-1\\ 1\end{pmatrix}.$$

Thus $\mu = 2$ is color Laplacian eigenvalue with multiplicity of at least one.
So $Lspec_\chi(B(3,n))_2^P = \begin{pmatrix}-1 & n-1 & 0 & 2\\ n-1 & 1 & 1 & 1\end{pmatrix}.$
Since average vertex degree of $(B(3,n))_2^P$ is $\dfrac{2}{n+2}$,

$$\begin{aligned}LE_\chi(B(3,n))_2^P =& \left|-1-\frac{2}{n+2}\right|(n-1)\\ &+\left|n-1-\frac{2}{n+2}\right|\\ &+\left|0-\frac{2}{n+2}\right| + \left|2-\frac{2}{n+2}\right|\\ =&\frac{2(n^2+3n-2)}{n+2}.\end{aligned}$$

(ii) $L_\chi(B(3,n))_{2(i)}^P = \begin{bmatrix}nI_2 & -J_{2\times n}\\ -J_{n\times 2} & (I+J)_n\end{bmatrix}_{n+2}$ is a color Laplacian matrix of $2(i)$−color complement of $B(3,n)$. The result is proved by showing $LZ = \mu Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue μ is same, as $L_\chi(B(3,n))_{2(i)}^P$ is real and symmetric.
Let $Z = \begin{bmatrix}X_2\\ Y_n\end{bmatrix}$ be an eigenvector of order $n+2$ partitioned conformally with $L_\chi(B(3,n))_{2(i)}^P$.
Consider

$$[L_\chi(B(3,n))_{2(i)}^P - \mu I]\begin{bmatrix}X_2\\ Y_n\end{bmatrix} = \begin{bmatrix}[(n-\mu)I]X - JY\\ -JX + [(1-\mu)I+J]Y\end{bmatrix}. \tag{28}$$

Case 1: Let $X = \dfrac{n}{n-\mu}1_2$ and $Y = 1_n$.
From equation (28),

$$\begin{aligned}-J\frac{n}{n-\mu}1_2 + [(1-\mu)I+J]1_n =& \left(\frac{-2n}{n-\mu} + 1 - \mu + n\right)1_n\\ =&\frac{\mu^2-(2n+1)\mu+n^2-n}{n-\mu}1_n.\end{aligned}$$

Therefore, $\mu = \dfrac{2n+1+\sqrt{8n+1}}{2}$ and $\mu = \dfrac{2n+1-\sqrt{8n+1}}{2}$ are the color Laplacian eigenvalues with multiplicity of at least one.
Case 2: Let $X = 0_2$ and $Y = Y_j$.
From equation (28),

$$[(1-\mu)I+J]Y_j = (1-\mu)Y_j.$$

Hence $\mu = 1$ is color Laplacian eigenvalue with multiplicity of at least $n-1$ since there are $n-1$ independent vectors of the form Y_j.
Case 3: Let $X = \begin{pmatrix}1\\ -1\end{pmatrix}$ and $Y = 0_n$.
From equation (28),

$$[(n-\mu)I]\begin{pmatrix}1\\ -1\end{pmatrix} = (n-\mu)\begin{pmatrix}1\\ -1\end{pmatrix}.$$

Thus $\mu = n$ is color Laplacian eigenvalue with multiplicity of at least one.
So $Lspec_\chi(B(3,n))_{2(i)}^P$

$$= \begin{pmatrix}1 & n & \frac{2n+1+\sqrt{8n+1}}{2} & \frac{2n+1-\sqrt{8n+1}}{2}\\ n-1 & 1 & 1 & 1\end{pmatrix}.$$

Since average vertex degree of $(B(3,n))_{2(i)}^P$ is $\dfrac{4n}{n+2}$, $LE_\chi(B(3,n))_{2(i)}^P$

$$\begin{aligned}=&\left|1-\frac{4n}{n+2}\right|(n-1) + \left|n-\frac{4n}{n+2}\right|\\ &+\left|\frac{2n+1+\sqrt{8n+1}}{2} - \frac{4n}{n+2}\right|\\ &+\left|\frac{2n+1-\sqrt{8n+1}}{2} - \frac{4n}{n+2}\right|\\ =&\frac{(4n^2-7n+2)}{n+2} + \sqrt{8n+1}.\end{aligned}$$

■

H. Book graph

Definition 29. *The $n-$book graph (B_n) is defined as the graph Cartesian product $B_n = K_{1,n} \times P_2$, where $K_{1,n}$ is a star graph and P_2 is the path graph on two vertices. Order of Book graph is $2n+2$.*

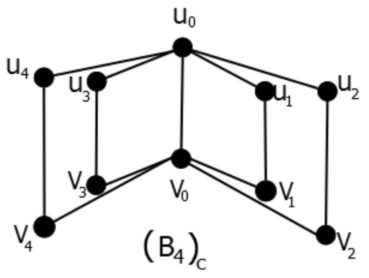

Fig. 4. Color book graph with color class $C_1 = \{u_0, v_1, v_2, v_3, v_4\}$ and $C_2 = \{v_0, u_1, u_2, u_3, u_4\}$.

Theorem 30. *Let $\{V_1, V_2\}$ be a partition of colored book graph such that $V_1 = \{u_0, u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_0, v_1, v_2, \ldots, v_n\}$. Then*

(i) $LE_\chi(B_n)_2^P = 4n.$

(ii) $LE_\chi(B_n)_{2(i)}^P = 4n.$

Proof:

(i) Color Laplacian matrix of $2-$ color complement of B_n is $L_\chi(B_n)_2^P =$

$$\begin{bmatrix} nJ_1 & -J_{1\times n} & 0_1 & J_{1\times n} \\ -J_{n\times 1} & (nI+B)_n & J_{n\times 1} & -B_n \\ 0_1 & J_{1\times n} & nJ_1 & -J_{1\times n} \\ J_{n\times 1} & -B_n & -J_{n\times 1} & (nI+B)_n \end{bmatrix}_{2n+2}$$

Consider $P_\chi(B_n,\mu)_2^P = |\mu I - L_\chi(B_n)_2^P|$, where $|\mu I - L_\chi(B_n)_2^P|$ is of the form $\begin{vmatrix} X & Y \\ Y & X \end{vmatrix}$.

Hence $P_\chi(B_n,\mu)_2^P = |X+Y||X-Y|$.

Where,

$$|X+Y| = \begin{vmatrix} \mu - n & 0_{1\times n} \\ 0_{n\times 1} & (\mu-n)I_n \end{vmatrix}_{n+1} = (\mu-n)^{n+1}.$$

and

$$|X-Y| = \begin{vmatrix} \mu - n & 2J_{1\times n} \\ 2J_{n\times 1} & [(\mu-n)I - 2B]_n \end{vmatrix}_{n+1} = (\mu-n+2)^{n-1} \{\mu^2 + (2-4n)\mu + 3n^2 - 6n\}.$$

Hence $Lspec_\chi(B_n)_2^P = \begin{pmatrix} n & n-2 & 3n \\ n+1 & n & 1 \end{pmatrix}$.

Since average vertex degree of $(B_n)_2^P$ is n,

$$\begin{aligned} LE_\chi(B_n)_2^P =& |n-n|(n+1) + |n-2-n|\,(n-1) \\ &+ |3n-n| + |n-2-n| \\ =& 4n. \end{aligned}$$

(ii) Color Laplacian matrix of $2(i)-$color complement of B_n is

$$L_\chi(B_n)_{2(i)}^P = \begin{bmatrix} J_1 & 0_{1\times n} & -J_1 & J_{1\times n} \\ 0_{n\times 1} & J_n & J_{n\times 1} & -I_n \\ -J_1 & J_{1\times n} & J_1 & 0_{1\times n} \\ J_{n\times 1} & -I_n & 0_{n\times 1} & J_n \end{bmatrix}_{2n+2}$$

Consider $P_\chi(B_n,\mu)_{2(i)}^P = |\mu I - L_\chi(B_n)_{2(i)}^P|$, where $|\mu I - L_\chi(B_n)_{2(i)}^P|$ is of the form $\begin{vmatrix} X & Y \\ Y & X \end{vmatrix}$.

Hence $P_\chi(B_n,\mu)_{2(i)}^P = |X+Y||X-Y|$.

Where,

$$|X+Y| = \begin{vmatrix} \mu & -J_{1\times n} \\ -J_{n\times 1} & (\mu I - B)_n \end{vmatrix}_{n+1} = (\mu+1)^{n-1}\{\mu^2 - (n-1)\mu - n\}.$$

and

$$|X-Y| = \begin{vmatrix} \mu - 2 & J_{1\times n} \\ J_{n\times 1} & [(\mu-2)I - B]_n \end{vmatrix}_{n+1} = (\mu-1)^{n-1}\{\mu^2 - (n+3)\mu + n + 2\}.$$

Hence $Lspec_\chi(B_n)_{2(i)}^P = \begin{pmatrix} 1 & -1 & n & n+2 \\ n & n & 1 & 1 \end{pmatrix}$.

Since average vertex degree of $(B_n)_{2(i)}^P$ is 1,

$$\begin{aligned} LE_\chi(B_n)_{2(i)}^P =& |1-1|n + |-1-1|\,n \\ &+ |n-1| + |n+2-1| \\ =& 4n. \end{aligned}$$

■

I. Amalgamation of m copies of K_n

In graph theory, graph amalgamation is a relationship between two graphs (one graph is an amalgamation of another). Amalgamations can provide a way to reduce a graph to a simpler graph while keeping certain structure intact.

Definition 31. *Let $\{G_1, G_2, G_3, \ldots, G_m\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{0i} called a terminal. The amalgamation $Amal(v_{0i}, G_i)$ is formed by taking all the $G_i's$ and identifying their terminals. In particular, if we take $G_i = K_n$ for $i = 1, 2, \ldots, m$ we get amalgamation of m copies of K_n denoted by $Amal(m, K_n), m \geq 2$. For convenience we denote v_0 as the vertex of amalgamation and $v_{j2}, v_{j3}, \ldots, v_{jn}$ are the remaining vertices of the j^{th} copy of K_n, where $1 \leq j \leq m$.*

Example 32. *The amalgamation of 3 copies of K_4 is shown in Figure 5.*

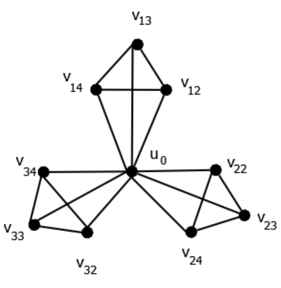

Fig. 5. Amal(3,K_4)

Theorem 33. *Let $v_0, v_{12}, v_{13}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ be the vertices of $Amal(m, K_n)$ with*

$P = \{V_1, V_2, \ldots, V_m\}$ *such that* $\langle V_1 \rangle = K_1$ *and* $\langle V_i \rangle = K_{n-1}, i = 2, 3, \ldots, m+1$. *Then*

(i) $Lspec_{\chi}(Amal(m, K_n))^P_{m+1}$
$$= \begin{pmatrix} 0 & mn-1 & m-1 & m(n-2)-1 \\ 1 & n-2 & 1 & (n-1)(m-1) \end{pmatrix}.$$

(ii) $Lspec_{\chi}(Amal(m, K_n))^P_{(m+1)(i)} =$
$$\begin{pmatrix} 0 & m & \dfrac{P+Q}{2} & \dfrac{P-Q}{2} \\ (n-1)(m-1) & n-2 & 1 & 1 \end{pmatrix},$$ *where* $P = mn$ *and* $Q = \sqrt{(m^2n^2 - 4m(m-1)(n-1)}$.

Proof:

(i) The adjacency matrix of $(m+1)-$color complement of $(Amal(m, K_n))$ is $A_{\chi}(Amal(m, K_n))^P_{m+1} =$
$$\begin{bmatrix} 0_1 & 0_{1\times n-1} & 0_{1\times n-1} & \ldots & 0_{1\times n-1} \\ 0_{n-1\times 1} & B_{n-1} & B - I_{n-1} & \ldots & B - I_{n-1} \\ 0_{n-1\times 1} & B - I_{n-1} & B_{n-1} & \ldots & B - I_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n-1\times 1} & B - I_{n-1} & B - I_{n-1} & \ldots & B_{n-1} \end{bmatrix}_{m(n-1)+1}$$
The degree matrix of the graph is
$$D = \begin{bmatrix} 0_1 & 0_{1\times m(n-1)} \\ 0_{m(n-1)\times 1} & m(n-2)I_{m(n-1)} \end{bmatrix}_{m(n-1)+1}$$
Color Laplacian matrix of $(m+1)-$ color complement of $(Amal(m, K_n))^P_{m+1}$ is $L_{\chi}(Amal(m, K_n))^P_{m+1} = D - A_{\chi}(Amal(m, K_n))^P_{m+1}$.
Step 1: Consider $|\mu I - L_{\chi}(Amal(m, K_n))^P_{m+1}|$.
By applying block row operation $R_i \to R_i - R_{i+1}, i = 2, 3, \ldots, m+1$ and block column operation $C_i \to C_i + C_{i-1} + \ldots + C_2, i = m+1, m, \ldots, 3$ on $|\mu I - L_{\chi}(Amal(m, K_n))^P_{m+1}|$, we get $\mu(\mu - m(n-2)+1)^{(n-1)(m-1)} \det(C)$, where $\det(C)$ is of the order $n-1$.
Step 2: On performing row operation $R_i \to R_i - R_{i+1}, i = 1, 2, \ldots, n-2$ and column operation $C_i \to C_i + C_{i-1} + C_{i-2} + \ldots + C_1, i = n-1, n-2, \ldots, 2$ on $\det(C)$, we obtain $(\mu - mn + 1)^{n-2}(\mu - m + 1)$.
Hence $Lspec_{\chi}(Amal(m, K_n))^P_{m+1}$
$$= \begin{pmatrix} 0 & mn-1 & m-1 & m(n-2)-1 \\ 1 & n-2 & 1 & (n-1)(m-1) \end{pmatrix}.$$

(ii) The adjacency matrix of $(m+1)(i)-$color complement of $(Amal(m, K_n))$ is $A_{\chi}(Amal(m, K_n))^P_{(m+1)(i)} =$
$$\begin{bmatrix} 0_1 & J_{1\times n-1} & J_{1\times n-1} & \ldots & J_{1\times n-1} \\ J_{n-1\times 1} & 0_{n-1} & -I_{n-1} & \ldots & -I_{n-1} \\ J_{n-1\times 1} & -I_{n-1} & 0_{n-1} & \ldots & -I_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{n-1\times 1} & -I_{n-1} & -I_{n-1} & \ldots & 0_{n-1} \end{bmatrix}_{m(n-1)+1}$$
The degree matrix of the graph is
$$D = \begin{bmatrix} m(n-1)I_1 & 0_{1\times m(n-1)} \\ 0_{m(n-1)\times 1} & I_{m(n-1)} \end{bmatrix}_{m(n-1)+1}$$
Color Laplacian matrix of $(m+1)(i)-$color complement of $(Amal(m, K_n))^P_{(m+1)(i)}$ is $L_{\chi}(Amal(m, K_n))^P_{(m+1)(i)} = D - A_{\chi}(Amal(m, K_n))^P_{(m+1)(i)}$.
Consider $P_{\chi}((Amal(m, K_n)), \mu)^P_{(m+1)(i)} = |\mu I - L_{\chi}(Amal(m, K_n))^P_{(m+1)(i)}|$.
On applying block row operation $R_i \to R_i - R_{i+1}, i = 2, 3, \ldots, m+1$ and block column operation $C_i \to C_i + C_{i-1} + \ldots + C_2, i = m+1, m, \ldots, 3$ on $|\mu I - L_{\chi}(Amal(m, K_n))^P_{(m+1)(i)}|$,
we obtain $\mu^{(n-1)(m-1)} \begin{vmatrix} (\mu - m(n-1))I & mJ \\ J & (\mu - m)I \end{vmatrix}_n$
On simplifying further, we get
$\mu^{(n-1)(m-1)}(\mu - m)^{n-2}[\mu^2 - mn\mu + n(m-1)^2]$.
Hence $Lspec_{\chi}(Amal(m, K_n))^P_{m+1(i)} =$
$$\begin{pmatrix} 0 & m & \dfrac{P+Q}{2} & \dfrac{P-Q}{2} \\ (n-1)(m-1) & n-2 & 1 & 1 \end{pmatrix},$$ where $P = mn$ and $Q = \sqrt{(m^2n^2 - 4m(m-1)(n-1)}$.

■

Conclusion: Generalised color complement of a graph not only depends on the partition of vertex set but also depends on the assigned colors to the vertices. In this paper, we have defined color Laplacian energy of generalised color complement of graph. The color Laplacian spectrum and color Laplacian energy of generalised color complements of families of graphs are derived.

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