Fixed-Time Stabilization for Asymmetric Constrained Nonholonomic Systems via State Feedback Control

Jiacai Huang, Sinan Lv, Fangzheng Gao and Yanling Shang

***Abstract*—In this paper, the problem of fixed-time stabilization is addressed for a category of chained nonholonomic systems with asymmetric time-varying output constraints. A novel barrier Lyapunov function (UBLF) is introduce to addressed asymmetric time-varying output constraint requirements. Based on this, a state feedback controller is developed by employing switching control technique. The given controller is able to drive the states of the closed-loop system (CLS) to zero in a fixed time without violation of the constraints. Finally, the efficacy of the proposed control scheme is confirmed by simulation results.**

***Index Terms*—fixed-time stabilization, nonholonomic systems, state feedback.**

I. Introduction

THE research on control of nonholonomic systems (NSs) has aroused wide attention in past decades due to its significant value in practice [1]. However, the distinctive feature that the number of control inputs less than that of freedom, makes the control of nonholonomic systems challenging [2]. To deal with such difficulty, several constructive control approaches including discontinuous time-invariant feedback [3–5], smooth time-varying feedback [6,7] and hybrid feedback [8] have been developed in the literature. Thanks to these effective methods, a lot important results have been made in the stabilization of NSs, see, for instance, [9–15] and the references therein.

Noticed that most of the existing references mainly centre around the asymptotic behavior of system trajectories as time verges to infinity. Nevertheless, many practical applications are more expectation that system trajectories converge to the interested equilibrium in finite time. In addition, finite-time stable systems also perform the good properties of faster convergence and better robustness against uncertainty [16]. Motivated by the facts, the research on finite-time stabilization of NSs has gained increasing attention in recent years. Specifically, a pioneering work was carried out in [17] by studying finite-time stabilization by state feedback for a kind of NSs with some weak-drift uncertainties. Subsequently, the problems of adaptive finite-time stabilization with uncertainties of linear and nonlinear parameterization were respectively addressed in [18] and [19]. By relaxing the restriction on system growth, the finite-time stabilization was studied in [20] for a category of NSs with more general nonlinearities. However, a common drawback that the convergence time seriously relies on and increases with the initial condition of studied systems, which makes the above-mentioned studies inapplicable to achieve the desired performance in an exact predefined time. Worse still, when the initial conditions are practically unknown, the proposed control methods in [17–20] fail to work because their switch strategies are established on the knowledge of the system convergence time. Recently, the fixed-time stability that requires the convergence time of a global finite-time stable system being bounded independent of initial conditions, was first introduced in [21] and further studied in [22, 23]. Such stability offers a new perspective to address the finite-time control problems and has stimulated numerous excellent results [24–30]. However, the effect of the state/output-constraints is not considered in the aforementioned papers.

In point of fact, almost all engineering systems are subjected to state/output constraints, violation of which may cause system damage, unpredictability danger or performance degradation. Drew by the practical demands, the control design of constrained systems has become a hot research topic in recent years [31–33]. Nevertheless, few result are available in the literature for state/output-constrained nonholonomic systems. Therefore, the interesting questions are put forward accordingly. *For a nonholonomic system with output constraints, is it possible for us to design a controller to fixed-time stabilization? If possible, how can it be designed?*

In this paper we give affirmative answers to above questions. Specifically, by introducing a novel universal barrier Lyapunov function (UBLF) to cope with asymmetric time-varying output constraints, a fixed-time control scheme is presented for the state feedback stabilization problem of a family of chained-form NSs. The main contributions of this paper are highlighted as follows.

(a) Under the unified framework of the considered system with symmetric/asymmetric constraints or without constraint requirements, the fixed-time stabilization problem of NSs is addressed.

(b) A novel tan-type UBLF fully taking advantage of system structure feature is introduced to ensure the requirements of the system output constraints.

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(c) By using the adding a power integrator technique and switching strategy, a state feedback control design procedure is presented to drive the states of CLS to zero in a fixed time while the asymmetric time-varying output constraints are not violated.

The notations used in this work are fairly standard. Specifically, for a vector $z = (z_1, \ldots, z_n)^T \in \mathbb{R}^n$ and two positive functions $a_1(t)$ and $a_2(t)$, denote $\bar{z}_j = (z_1, \ldots, z_j)^T \in \mathbb{R}^j$, $j = 1, \ldots, n$, $\Gamma_j^{a(t)} = \{\bar{z}_j : \bar{z}_j \in \mathbb{R}$ with $-a_1(t) < z_1| < a_2(t)\}$. For any $b > 0$ and $z \in \mathbb{R}$, the function $\lceil z \rfloor^b$ is defined as $\lceil z \rfloor^b = \text{sign}(z)|z|^b$.

II. Problem Formulation and Preliminaries

Consider the following chained-form NSs:

$$\begin{aligned} \dot{p}_0 &= u_0, \\ \dot{p}_i &= p_{i+1}u_0, \quad i = 1, \ldots, n-1, \\ \dot{p}_n &= u_1, \end{aligned} \tag{1}$$

where $(p_0, p)^T = (p_0, p_1, \ldots, p_n)^T \in \mathbb{R}^{n+1}$, $u = (u_0, u_1)^T \in \mathbb{R}^2$, $y = (p_0, p_1)^T \in \mathbb{R}^2$ are the system state, control input and system measurable output, respectively. If the p_0-subsystem is abandoned, then the system (1) degenerates to the extensively studied standard chain-form nonlinear systems. This paper suppose the output y suffering from the following time-varying constraints

$$\Omega_{p_i} = \{-k_{i1}(t) < p_i(t) < k_{i2}(t)\}, \quad i = 0, 1, \tag{2}$$

with some pre-specified functions $k_i(t) > 0$.

The following assumption is needed in this paper.

Assumption 1. The time-varying output constraints $k_{ij}(t)$ ($i = 0, 1, j = 1, 2$) are continuous differentiable and there are positive constants $\underline{k}_{i1}$, $\underline{k}_{i2}$, $\overline{k}_{i3}$ and $\overline{k}_{i4}$ such that $\underline{k}_{i1} \le k_{i1}(t)$, $\underline{k}_{i2} \le k_{i2}(t)$,$|\dot{k}_{i1}(t)| \le \overline{k}_{i3}$ and $|\dot{k}_{i2}(t)| \le \overline{k}_{i4}$.

For the sake of completeness, we review the definition and criterion of fixed-time stability of nonlinear systems.

Consider the nonlinear system

$$\dot{x} = f(t, x), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{3}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector field that can be discontinuous with respect to the state variable, and satisfies $f(t, 0) = 0$. In this case, the solutions of the system (3) are understood in the sense of Filippov [34].

Definition 1[23]**.** The origin of system (3) is said to be globally finite-time stable if it is globally (asymptotically) stable and finite-time convergent. By "finite-time convergence", it means that if, for any initial condition $x_0 \in \mathbb{R}^n$, there is a function $T : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$, called the settling time function, such that every solution $x(t, x_0)$ of (3) satisfies $x(t, x_0) = 0$ for any $t \ge T(x_0)$.

Definition 2[23]**.** The origin of system (3) is said to be globally fixed-time stable if it is globally finite-time stable and the settling time function $T(x_0)$ is bounded, that is, there exists a positive constant $T_{\max}$ such that $T(x_0) \le T_{\max}$, $\forall x_0 \in \mathbb{R}^n$.

Lemma 1[23]**.** Consider the nonlinear system (3). Suppose there exist a C^1, positive definite and radially unbounded function $V(x) : \mathbb{R}^n \to \mathbb{R}$ and real numbers $c > 0$, $d > 0$, $0 < \alpha < 1$, $\gamma > 1$, such that $\dot{V}(x) \le -cV^\alpha(x) - dV^\gamma(x)$, $\forall x \in \mathbb{R}^n$. Then, the origin of system (3) is globally fixed-time stable and the settling time $T(x_0)$ satisfies $T(x_0) \le T_{\max} := 1/(c(1-\alpha)) + 1/(d(\gamma - 1))$, $\forall x_0 \in \mathbb{R}^n$.

III. Fixed-Time Control Design

In this section, we provide a constructive procedure for the design of fixed-time controller that stabilizes system (1) for any given finite settling time $T > 0$, while preventing the violation of the time-varying output constraints (2).

A. A novel tan-type UBLF

To avoid the state p_1 violating the asymmetric time-varying constraints, a novel tan-type UBLF $V_{b_1} : \Gamma_1 \to \mathbb{R}$ is introduced as follows:

$$V_{b_1}(p_1) = \frac{2k_{b_1}^{2-\tau}}{\pi(2-\tau)} \tan\left(\frac{\pi |p_1|^{2-\tau}}{2k_{b_1}^{2-\tau}}\right), \tag{4}$$

where $\tau \in (-\frac{1}{n}, 0)$ and $k_{b_1} = k_{11}$, if $p_1 > 0$, otherwise $k_{b_1} = k_{12}$.

The properties of $V_{b_1}(p_1)$ are characterized by the following proposition:

Proposition 1. $V_{b_1}(p_1)$ is C^1, positive definite on Γ_1 and satisfies

$$\begin{aligned} \frac{\partial V_{b_1}(p_1)}{\partial p_1} &= \Phi_{b_1}(p_1)\lceil p_1 \rfloor^{1-\tau}, \\ \frac{\partial V_{b_1}(p_1)}{\partial k_{b_1}} &= \frac{2k_{b_1}^{2-\tau}}{\pi} \tan\left(\frac{\pi |p_1|^{2-\tau}}{2k_{b_1}^{2-\tau}}\right) \\ &\quad - \frac{1}{k_{b_1}} \Phi_{b_1}(p_1)|p_1|^{2-\tau}, \end{aligned} \tag{5}$$

with $\Phi_{b_1}(p_1)$ being defined as

$$\Phi_{b_1}(p_1) = \begin{cases} \Phi_{k_{12}}(p_1) = \sec^2\left(\frac{\pi |p_1|^{2-\tau}}{2k_{12}^{2-\tau}}\right), & p_1 > 0, \\ \Phi_{k_{11}}(p_1) = \sec^2\left(\frac{\pi |p_1|^{2-\tau}}{2k_{11}^{2-\tau}}\right), & p_1 \le 0. \end{cases} \tag{6}$$

Remark 1. Note that if the constraint functions are symmetric and constant ones, namely, if $k_1 = k_2 = k_b = cont$, the BLF (5) becomes the one used in [31]. Compared with the conventional log-type or tan-type BLF, the constructed novel tan-type BLF $V_{b_1}(p_1)$ takes full advantage of structure feature of the system (1) and has a more attractive property that

$$\begin{aligned} \lim_{k_{11}\to\infty} V_{k_{11}}(p_1) &= \lim_{k_{11}\to\infty} \frac{2k_{11}^{2-\tau}}{\pi(2-\tau)} \tan\left(\frac{\pi |p_1|^{2-\tau}}{2k_{11}^{2-\tau}}\right) \\ &= \frac{1}{2-\tau}|p_1|^{2-\tau}. \end{aligned} \tag{7}$$

$$\begin{aligned} \lim_{k_{12}\to\infty} V_{k_{12}}(p_1) &= \lim_{k_{12}\to\infty} \frac{2k_{12}^{2-\tau}}{\pi(2-\tau)} \tan\left(\frac{\pi |p_1|^{2-\tau}}{2k_{12}^{2-\tau}}\right) \\ &= \frac{1}{2-\tau}|p_1|^{2-\tau}. \end{aligned} \tag{8}$$

That is, when there is no constraint requirements on the lower and/or upper bound of p_1, by letting $k_{11} \to \infty$ and/or $k_{12} \to \infty$, $V_{k_{b_1}}(p_1)$ in (4) will become the Lyapunov function widely employed for the control problem without constraints. As a consequence, the present tan-type UBLF $V_{b_1}(p_1)$ serves as a unified tool for handling the control problem simultaneously with asymmetric constraints, symmetric constraints or without constraint requirements.

B. Fixed-time stabilization of the p-subsystem

For the p_0-subsystem, we take the following control law

$$u_0 = (|\operatorname{sign}(p_0(0))| - \operatorname{sign}(p_0(0)) - 1)\, c_0^*, \tag{9}$$

where c_0^* is a positive constant satisfying $c_0^* < k_{01}/(\delta T)$ for a constant $\delta \in (0,1)$. As a result, the following lemma is established by some simple derivations.

Proposition 2. For any initial condition $p_0(0) \in \Omega_{p_0}$, the corresponding solution $p_0(t)$ is well defined on $[0, \delta T)$ and satisfies $p_0(t) \in \Omega_{p_0}$.

Under (9), the p-subsystem is rewritten as

$$\begin{aligned} \dot{p}_1 &= h_1 p_2, \\ \dot{p}_i &= h_i p_{i+1}, \quad i = 2, \ldots, n-1, \\ \dot{p}_n &= h_n u_1, \end{aligned} \tag{10}$$

where $h_i = (|\operatorname{sign}(p_0(0))| - \operatorname{sign}(p_0(0)) - 1)\, c_0^*$, $i = 1, \ldots, n-1$ and $h_n = 1$.

In the sequel, we will stabilize the system (10) within the settling time δT by a step-by-step manner. Before proceeding, we introduce the following coordinate transformation:

$$\begin{aligned} &\xi_i = \lceil p_i \rceil^{\frac{1}{r_i}} - \lceil p_i^* \rceil^{\frac{1}{r_i}}, \quad p_{i+1}^* = -\beta_i(\bar{p}_i)\lceil \xi_i \rceil^{r_{i+1}}, \\ &i = 1, \ldots, n, \end{aligned} \tag{11}$$

where $r_i = 1 + (i-1)\tau$, $p_1^* = 0$ and $\beta_i : \mathbb{R}^i \to \mathbb{R}$ is a C^0 function to be specified later.

Step 1: Take $V_1(p_1) = V_{k_{b_1}}(p_1)$ to be the Lyapunov function for this step. Based on (2) and (4),Assumption 2.1, we have

$$\begin{aligned} \dot{V}_1 &= \frac{\partial V_{k_{b_1}}}{\partial p_1}\dot{p}_1 + \frac{\partial V_{k_{b_1}}}{\partial k_{b_1}}\dot{k}_{b_1} \\ &= \Phi_{b_1}(p_1)\lceil p_1 \rceil^{1-\tau} p_2 - \frac{1}{k_{b_1}}\Phi_{b_1}(p_1)|p_1|^{2-\tau}\dot{k}_{b_1} \\ &\quad + \frac{2k_{b_1}^{1-\tau}}{\pi}\tan\left(\frac{\pi |x_1|^{2-\tau}}{2k_{b_1}^{2-\tau}}\right)\dot{k}_{b_1} \\ &\le \Phi_{b_1}(p_1)\lceil p_1 \rceil^{1-\tau} h_1 p_2 + \frac{2}{k_{b_1}}\Phi_{b_1}(p_1)|p_1|^{2-\tau}|\dot{k}_b| \\ &\le \Phi_{b_1}(p_1) h_1 \lceil \xi_1 \rceil^{2-\tau}(p_2 - p_2^*) + \Phi_{b_1}(p_1) h_1 \lceil \xi_1 \rceil^{1-\tau} p_2^* \\ &\quad + \Phi_{b_1}(p_1)\xi_1^2 \phi_1 \end{aligned} \tag{12}$$

where $\phi_1 \ge (2\overline{K}_1 |p_1|^{-\tau})/\underline{K}_1$ with $\underline{K}_1 = \min\{\underline{k}_{11}, \underline{k}_{12}\}$ and $\overline{K}_1 = \max\{\overline{k}_{13}, \overline{k}_{14}\}$ is a smooth function and p_2^* is the virtual controller of p_2 to be specified.

Select

$$\begin{aligned} p_2^* &= -\frac{n-1+l+l|\xi_1|^q + \phi_1}{h_1}\lceil \xi_1 \rceil^{r_1+\tau} \\ &:= -\beta_1(p_1)[\xi_1]^{r_2}, \end{aligned} \tag{13}$$

with design parameters $l > 0$ and $q > -\tau$, which results in

$$\begin{aligned} \dot{V}_1 \ \le\ & -l\Phi_{b_1}(p_1)(|\xi_1|^2 + |\xi_1|^{2+q}) - (n-1)|\xi_1|^2 \\ & + \Phi_{b_1}(p_1) h_1 \lceil \xi_1 \rceil^{1-\tau}(p_2 - p_2^*). \end{aligned} \tag{14}$$

Step 2: Choose $V(\bar{p}_2) = V_1(p_1) + W_2(\bar{p}_2)$ with

$$W_2(\bar{p}_2) = \int_{p_2^*}^{p_2} \left\lceil \lceil s \rceil^{\frac{1}{r_2}} - \lceil p_2^* \rceil^{\frac{1}{r_2}} \right\rceil^{2-\tau-r_2} \mathrm{d}s. \tag{15}$$

Since

$$\begin{cases} \dfrac{\partial W_2}{\partial p_2} = \lceil \xi_2 \rceil^{2-\tau-r_2}, \\ \dfrac{\partial W_2}{\partial p_1} = -(2-\tau-r_2)\dfrac{\partial\left(\lceil p_2^* \rceil^{\frac{1}{r_2}}\right)}{\partial p_1} \\ \qquad\qquad \times \displaystyle\int_{p_2^*}^{p_2} \left\lceil \lceil s \rceil^{\frac{1}{r_2}} - \lceil p_2^* \rceil^{\frac{1}{r_2}} \right\rceil^{1-\tau-r_2} \mathrm{d}s \end{cases} \tag{16}$$

a direct calculation yields

$$\begin{aligned} \dot{V}_2 \le & -l\Phi_{b_1}(p_1)(|\xi_1|^2 + |\xi_1|^{2+q}) - (n-1)|\xi_1|^2 \\ & + h_2\lceil \xi_2 \rceil^{2-\tau-r_2} p_3^* + h_2 \lceil \xi_2 \rceil^{2-\tau-r_2}(p_3 - p_3^*) \\ & + \Phi_{b_1}(p_1) h_1 \lceil \xi_1 \rceil^{1-\tau}(p_2 - p_2^*) + \frac{\partial W_2}{\partial p_1} h_1 p_2 \end{aligned} \tag{17}$$

where p_3^* is the virtual controller being designed later. To continue the design, we gives the estimates for last two terms in the right hand of (17).

First, by the definition of ξ_j and p_j^*, $j = 1, 2$, we have

$$\begin{aligned} |p_2 - p_2^*| &= \left| \left(\lceil p_2 \rceil^{\frac{1}{r_2}}\right)^{r_2} - \left(\lceil p_2^* \rceil^{\frac{1}{r_2}}\right)^{r_2} \right| \\ &\le 2^{1-r_2}\left| \lceil p_2 \rceil^{\frac{1}{r_2}} - \lceil p_2^* \rceil^{\frac{1}{r_2}} \right|^{r_2} \\ &= 2^{1-r_2}|\xi_2|^{r_2}. \end{aligned} \tag{18}$$

Thus, from (18), we can obtain that

$$\begin{aligned} & \Phi_{b_1}(p_1) h_1 \lceil \xi_1 \rceil^{1-\tau}(p_2 - p_2^*) \\ \le\ & 2^{1-r_2} h_1 \Phi_{max}(p_1)|\xi_1|^{1-\tau}|\xi_2|^{r_2} \\ \le\ & \frac{1}{3}|\xi_1|^2 + |\xi_2|^2 \chi_{21}. \end{aligned} \tag{19}$$

where $\Phi_{max}(p_1) = \max\{\Phi_{k_{11}}(p_1), \Phi_{k_{12}}(p_1)\}$ and $\chi_{21} > 0$ is a smooth function.

Secondly, note that

$$\begin{aligned} & (2-\tau-r_2)\int_{p_2^*}^{p_2} \left| \lceil s \rceil^{\frac{1}{r_2}} - \lceil p_2^* \rceil^{\frac{1}{r_2}} \right|^{1-\tau-r_2} \mathrm{d}s \\ \le\ & (2-\tau-r_2)\,|\xi_2|^{1-\tau-r_2}|p_2 - p_2^*| \\ \le\ & (2-\tau-r_2)\, 2^{1-r_2}|\xi_2|^{1-\tau}. \end{aligned} \tag{20}$$

Therefore, according to (2) and (20), we have

$$\begin{aligned} \frac{\partial W_2}{\partial p_1} h_2 p_2 \le\ & (2-\tau-r_2)\int_{p_2^*}^{p_2} \left| \lceil s \rceil^{\frac{1}{r_2}} - \lceil p_2^* \rceil^{\frac{1}{r_2}} \right|^{1-\tau-r_2} \mathrm{d}s \\ & \times \left| \frac{\partial\left(\lceil p_2^* \rceil^{\frac{1}{r_2}}\right)}{\partial p_1} \right| h_2 |p_2| \\ \le\ & \frac{1}{3}|\xi_1|^2 + |\xi_2|^2 \chi_{22}, \end{aligned} \tag{21}$$

where $\chi_{22} \ge 0$ is a smooth function.

Substituting (19) and (21) into (18) yields

$$\begin{aligned} \dot{V}_2 \le & -l\Phi_{b_1}(p_1)(|\xi_1|^2 + |\xi_1|^{2+q}) - (n-2)|\xi_1|^2 \\ & + \lceil \xi_2 \rceil^{2-\tau-r_2} h_2(p_3 - p_3^*) + \lceil \xi_2 \rceil^{2-\tau-r_2} h_2 p_3^* \\ & + (\chi_{21} + \chi_{22})|\xi_2|^2. \end{aligned} \tag{22}$$

Design the virtual controller

$$\begin{aligned} p_3^* &= -\frac{1}{h_2}\left(n-2+l+l|\xi_2|^q + \chi_{21} + \chi_{22}\right)\lceil \xi_1 \rceil^{r_3} \\ &:= -\beta_2(\bar{p}_2)\lceil \xi_2 \rceil^{r_3}. \end{aligned} \tag{23}$$

Then, the time derivative of V_2 becomes

$$\begin{aligned}\dot{V}_2 \leq & -l\Phi_{b_1}(p_1)(|\xi_1|^2 + |\xi_1|^{2+q}) - (n-2)(|\xi_1|^2 + |\xi_2|^2) \\ & -l(|\xi_2|^2 + |\xi_2|^{2+q}) + \lceil \xi_2 \rceil^{2-\tau-r_2} h_2(p_3 - p_3^*).\end{aligned} \tag{24}$$

Following the same arguments of Step 2, for Step i (i=2 , . . . , n), we can find a C^1 and positive definite Lyapunov function $V_i(\bar{p}_i) = V_{b_1}(p_1) + \sum_{j=2}^{i} W_j(\bar{p}_j)$ with

$$W_j(\bar{p}_j) = \int_{p_i^*}^{p_i} \left\lceil \lceil s \rceil^{\frac{1}{r_i}} - \lceil p_i^* \rceil^{\frac{1}{r_i}} \right\rceil^{2-\tau-r_i} \mathrm{d}s, \tag{25}$$

and a set of continuous virtual controllers $p_{j+1}^* = -\beta_j(\bar{p}_j)\lceil \xi_j \rceil^{r_j}$, $j = 1, \ldots, n$, such that

$$\begin{aligned}\dot{V}_i \leq & -l\Phi_{b_1}(p_1)(|\xi_1|^2 + |\xi_1|^{2+q}) - (n-i)\sum_{j=1}^{i} |\xi_j|^2 \\ & -l\sum_{j=2}^{i}(|\xi_j|^2 + |\xi_j|^{2+q}) + \lceil \xi_i \rceil^{2-\tau-r_i} h_i(p_{i+1} - p_{i+1}^*).\end{aligned} \tag{26}$$

Consequently, the following result is obtained.

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Proposition 3. For system (10), if the controller $u_1 = p_{n+1}^*(p)$ is specified by (11) with design parameters $l > 0$ and $-\tau < q < 2 - 2\tau$ satisfying

$$\frac{2(\tau-2)}{\delta l \tau} + \frac{(2-\tau)2^{\frac{2+q}{2-\tau}} n^{\frac{q+\tau}{2-\tau}}}{\delta l(q+\tau)} < T, \tag{27}$$

then, for all $p(0) \in \Theta_n^{p_1} = \{p(t) \in \mathbb{R}^n | -k_{11}(t) < p_1(t) < k_{12}(t)\}$, the following properties establish.

(i) The state p_1 stays in the set $\Omega_{p_1} = \{-k_{11}(t) < p_1(t) < k_{12}(t)\}$, $\forall t \geq 0$.

(ii) All the CLS states are regulated to zero in a fixed settling time δT.

C. *Fixed-time stabilization of the p_0-subsystem*

From Proposition 3, we know that $p(t) = 0$ when $t \geq \delta T$. Therefore, we here just need to stabilize the p_0-subsystem in a prescribed time $(1-\delta)T$. To guard against the state p_0 violating the constraint, similar to subsection III-A, we introduce a tan-type UBLF $V_0 : \Omega_0^{p_0} = \{-k_1(t) < p_0(t) < k_2(t)\} \to \mathbb{R}$ for the p_0-subsystem as

$$V_0(p_0) = \frac{2k_{b_0}^2}{\pi} \tan\left(\frac{\pi |p_0|^2}{2k_{b_0}^2}\right), \tag{28}$$

where $k_{b_0} = k_{01}$, if $p_0 > 0$, otherwise $k_{b_0} = k_{02}$. Then, the derivative of V_0 satisfies

$$\begin{aligned}\dot{V}_0(p_0) = & \sec^2\left(\frac{\pi|p_0|^2}{2k_{b_0}^2}\right) p_0 u_0 \\ & + \frac{4k_{b_0}}{\pi} \tan\left(\frac{\pi|p_0|^2}{2k_{b_0}^2}\right) \dot{k}_{b_0} - \frac{2}{k_0}\Phi_{b_0}(p_0)|p_0|^2 \dot{k}_{b_0} \\ \leq & \Phi_{b_0}(p_0)p_0 u_0 + \frac{4}{k_{b_0}}\Phi_{b_0}(p_0)|p_0|^2|\dot{k}_{b_0}|\end{aligned} \tag{29}$$

with

$$\Phi_{b_0}(p_0) = \begin{cases} \Phi_{k_{02}}(p_0) = \sec^2\left(\dfrac{\pi|p_0|^2}{2k_{02}^2}\right), & p_0 > 0, \\ \Phi_{k_{01}}(p_0) = \sec^2\left(\dfrac{\pi|p_0|^2}{2k_{01}^2}\right), & p_0 \leq 0. \end{cases} \tag{30}$$

Therefore, for the p_0-subsystem, the control u_0 can be adopted as

$$u_0 = -l_2\left(1 + |p_0|^d + \phi_0\right)\lceil p_0 \rceil^{\sigma}, \tag{31}$$

where $\phi_0 \geq (4\overline{K}_0 |p_0|^{1-\sigma})/\underline{K}_0$ with $\underline{K}_0 = \min\{\underline{k}_{01}, \underline{k}_{02}\}$ and $\overline{K}_0 = \max\{\overline{k}_{03}, \overline{k}_{04}\}$ is a smooth function and σ, l_2, d are design parameters to be determined later. Following the same line as in subsection B, the following result is gained.

Proposition 4. If design parameters $0 < \sigma < 1$, $l_2 > 0$ and $1 - \sigma < d < 3 - \sigma$ in (31) satisfy

$$\frac{2}{l_2(1-\sigma)(1-\delta)} + \frac{2}{l_2(\sigma + d - 1)(1-\delta)} < T, \tag{32}$$

then, for any initial condition $p_0(0) \in \Omega_{p_0}$, the following properties hold.

(i) The state p_0 keeps in the set $\Omega_{p_0} = \{-k_{01}(t) < p_0(t) < k_{02}(t)\}$, $\forall t \geq 0$ without violating the constraint.

(ii) The state p_0 is regulated to zero within a fixed settling time $(1-\delta)T$.

Consequently, the following theorem is given to summarize the main result of the paper.

Theorem 1. If the following switching control strategy with an appropriate choice of the design parameters is applied to system (1) subject to constraints (2),

$$u_0 = \begin{cases} c_0^*, & t < \delta T, \\ -l_2\left(1 + |p_0|^d + \phi_0\right)\lceil p_0 \rceil^{\sigma}, & t \geq \delta T, \end{cases} \tag{33}$$

$$u_1 = p_{n+1}^*(p), \tag{34}$$

then the states of the CLS are regulated to zero in any prescribed finite time T while, at the same time the constraints (2) are satisfied.

IV. An Application Example

Consider a unicycle-type mobile robot working in a limited area. The kinematic equations of this robot are represented by

$$\dot{x}_c = v\cos\theta, \quad \dot{y}_c = v\sin\theta, \quad \dot{\theta} = w, \tag{35}$$

where (x_c, y_c) denotes the position of the center of mass of the robot, θ is the heading angle of the robot, v is the forward velocity, w is the angular velocity of the robot and the origin is the parking position of the robot. Due to environmental limitation, we design the control laws under the constraints $-k_{01}(t) < x_c < k_{02}(t)$ and $-k_{11}(t) < y_c < k_{12}(t)$.

Introducing the following input and state transformations:

$$\begin{aligned} & p_0 = x_c, \quad p_1 = y_c, \quad p_2 = \tan\theta, \\ & u_0 = v\cos\theta, \quad u_1 = w\sec^2\theta, \end{aligned} \tag{36}$$

system (35) is transformed into

$$\dot{p}_0 = u_0, \quad \dot{p}_1 = u_0 p_2, \quad \dot{p}_2 = u_1. \tag{37}$$

Clearly, system (37) is in the form of system (1) with $n = 2$, and how to park this robot within a given time becomes

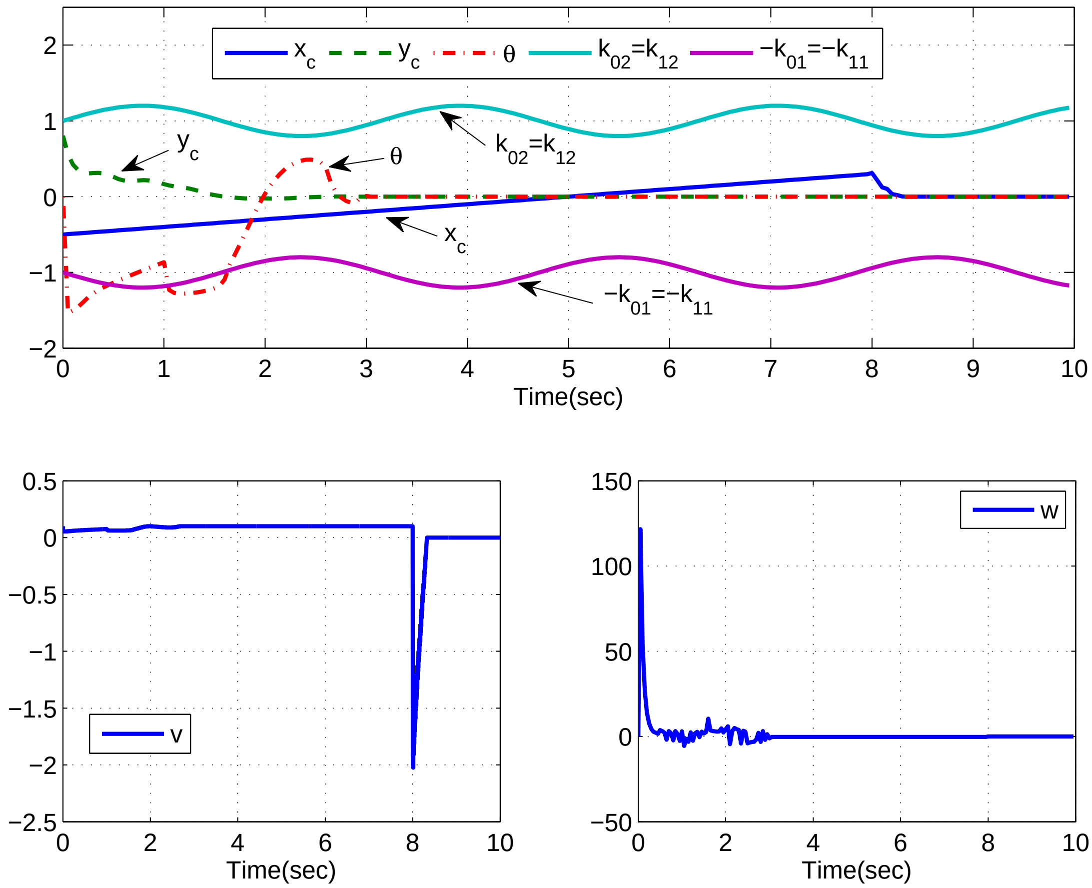

Fig. 1. The responses of the CLS. The top graph shows trajectories of x_c, y_c and θ under time-varying constraints, and the bottom graphs demonstrate trajectories of inputs v, w.

the problem of fixed-time stabilization of system (37) with output constraints (2).

To verify our proposed controller, we take $k_{01} = k_{11} = 1 + 0.12 \sin 2t$ and $k_{02} = k_{12} = 1 + 0.1 \sin 2t$. which satisfy the assumption made in this paper with $\underline{k}_{01} = \underline{k}_{11} = 0.88$, $\underline{k}_{02} = \underline{k}_{12} = 0.9$, $\bar{k}_{03} = \bar{k}_{13} = 0.12$, $\bar{k}_{04} = \bar{k}_{14} = 0.1$, For simplicity, we suppose $x_0(0) < 0$. In this case, for the p_0-subsystem, we can choose the control law $u_0 = u_0^*$, where u_0^* is a positive constant satisfying $u_0^* < k_{01}/(\delta T)$ with $\delta \in (0, 1)$.

By choosing the prescribed time $T = 10$ and the gains for the control laws as $c_0^* = 0.1$, $\delta = 0.8$, $l = 4$, $l_2 = 3$, $d = q = 2$, $\tau = 1/3$ and $\sigma = 0.5$, Fig.1 is obtained to exhibit the responses of the CLS with $(x_c(0), y_c(0), \theta(0))= (-0.5, 0.8, 0)$. We can see that the mobile robot moves to the desired location in a given prescribed time and the output constraints are never violated.

V. Conclusions

This paper has studied the problem of fixed-time stabilization for a class of asymmetric time-varying output-constrained NSs. Based on the novel barrier Lyapunov function (UBLF) to deal with the constraints, and by using adding a power integrator technique, a constructive design procedure for state feedback control is established. Together with a novel switching control strategy, the designed controller ensures that the states of the CLS are regulated to zero in any given prescribed time, while the output constraints are not violated.

Appendix

Proof of Proposition 3. The proof is divided into two parts.

Part I: Verification of the constraints $-k_{11} < p_1 < k_{12}$.

From the definitions of V_{b_1} and W_j's, we can easily verify that $V_n = V_{b_1} + \sum_{j=2}^{n} W_j$ is positive definite on Γ_n. This together with (26) renders that the CLS is asymptotically stable for all $p(0) \in \Theta_n^{p_1} = \{p(t) \in \mathbb{R}^n | -k_{11}(t) < p_1(t) < k_{12}(t)\}$. Therefore, one has for all $t \geq 0$,

$$V_{b_1}(p_1) = \frac{2k_{b_1}^{2-\tau}}{\pi(2-\tau)} \tan\left(\frac{\pi|p_1|^{2-\tau}}{2k_{b_1}^{2-\tau}}\right) \leq V_n(p) \leq V_n(p(0)). \tag{38}$$

That is,

$$\frac{\pi|p_1|^{2-\tau}}{2k_{b_1}^{2-\tau}} \leq \tan^{-1}\left(\frac{\pi(2-\tau)}{2k_{b_1}^{2-\tau}} V_n(p(0))\right) < \frac{\pi}{2}, \tag{39}$$

for all $t \geq 0$. As a result, the state p_1 will remains in the set $|p_1| < k_{b_1}$ (i.e., $-k_{11} < p_1 < k_{12}$) and never violates the constraints.

Part II: Fixed-time stable analysis

Since the CLS is asymptotically stable at the origin is showed in Part I. From Definitions 1 and 2, to achieve the fixed-time stability, we just need to prove that the settling-time function exists and is bounded by δT here. First of all, it is easily see that

$$\begin{aligned} W_j &= \int_{p_j^*}^{p_j} \left\lceil \lceil s \rceil^{\frac{1}{r_j}} - \lceil p_j^* \rceil^{\frac{1}{r_j}} \right\rceil^{2-\tau-r_j} \mathrm{d}s \\ &\leq |\xi_j|^{2-\tau-r_j} |p_j - p_j^*| \\ &\leq 2|\xi_j|^{2-\tau}. \end{aligned} \tag{40}$$

So one has

$$\begin{aligned} V_n &= V_{b_1} + \sum_{j=2}^{n} W_j \\ &\leq \frac{2k_{b_1}^{2-\tau}}{\pi(2-\tau)} \tan\left(\frac{\pi|p_1|^{2-\tau}}{2k_{b_1}^{2-\tau}}\right) + 2\sum_{j=2}^{n} |\xi_j|^{2-\tau}. \end{aligned} \tag{41}$$

Since $2-\tau > 1$ and thus for all $p_1 \in \Gamma_1$, $0 \leq \frac{\pi|p_1|^{2-\tau}}{2k_{b_1}^{2-\tau}} \leq \frac{\pi}{2}$, according to the characteristics of tangent functions, it is obtained that

$$\begin{aligned} \tan\left(\frac{\pi|p_1|^{2-\tau}}{2k_{b_1}^{2-\tau}}\right) &\leq \frac{\pi}{2k_{b_1}^{2-\tau}} \Phi_{b_1}^{\frac{1}{2}}(p_1)|p_1|^{2-\tau} \\ &\leq \frac{\pi(2-\tau)}{2k_{b_1}^{2-\tau}} \Phi_{b_1}^{\frac{1}{2}}(p_1)|\xi_1|^{2-\tau}. \end{aligned} \tag{42}$$

$$\begin{aligned} \tan\left(\frac{\pi|p_1|^{2-\tau}}{2k_{b_1}^{2-\tau}}\right) &\leq \frac{\pi}{2k_{b_1}^{2-\tau}} \Phi_{b_1}(p_1)|p_1|^{2-\tau} \\ &\leq \frac{\pi(2-\tau)}{2k_{b_1}^{2-\tau}} \Phi_{b_1}(p_1)|\xi_1|^{2-\tau}. \end{aligned} \tag{43}$$

Noting the fact that $\Phi_{b_1}(p_1) \geq 1$ for all $p_1 \in \Gamma_1$ and $0 < 2/(2-\tau) < 1$, by (41), (43), one deduces that

$$\begin{aligned} & V_n^{\frac{2}{2-\tau}} \\ \leq & \left(\frac{2k_{b_1}^{2-\tau}}{\pi(2-\tau)} \tan\left(\frac{\pi|p_1|^{2-\tau}}{2k_{b_1}^{2-\tau}}\right) + 2\sum_{j=2}^{n} |\xi_j|^{2-\tau}\right)^{\frac{2}{2-\tau}} \\ \leq & \left(\Phi_{b_1}(p_1)|\xi_1|^{2-\tau} + 2\sum_{j=2}^{n} |\xi_j|^{2-\tau}\right)^{\frac{2}{2-\tau}} \\ \leq & \Phi_{b_1}^{\frac{2}{2-\tau}}(p_1)|\xi_1|^2 + 2^{\frac{2}{2-\tau}} \sum_{j=2}^{n} |\xi_j|^2 \\ \leq & 2\left(\Phi_{b_1}(p_1)|\xi_1|^2 + \sum_{j=2}^{n} |\xi_j|^2\right). \end{aligned} \tag{44}$$

On the other side, observing $1 < (2+q)/(2-\tau) < 2$, by taking (41) and (42) into account, one arrives

$$\begin{aligned} & V_n^{\frac{2+q}{2-\tau}} \\ \leq & \left(\frac{2k_{b_1}^{2-\tau}}{\pi(2-\tau)} \tan\left(\frac{\pi|p_1|^{2-\tau}}{2k_{b_1}^{2-\tau}}\right) + 2\sum_{j=2}^{n} |\xi_j|^{2-\tau}\right)^{\frac{2+q}{2-\tau}} \\ \leq & \left(\Phi_{b_1}^{\frac{1}{2}}(p_1)|\xi_1|^{2-\tau} + 2\sum_{j=2}^{n} |\xi_j|^{2-\tau}\right)^{\frac{2+q}{2-\tau}} \\ \leq & n^{\frac{q+\tau}{2-\tau}} \left(\Phi_{b_1}^{\frac{2+q}{4-2\tau}}(p_1)|\xi_1|^{2+q} + 2^{\frac{2+q}{2-\tau}} \sum_{j=2}^{n} |\xi_j|^{2+q}\right) \\ \leq & n^{\frac{2+q}{2-\tau}-1} 2^{\frac{2+q}{2-\tau}} \left(\Phi_{b_1}(p_1)|\xi_1|^{2+q} + \sum_{j=2}^{n} |\xi_j|^{2+q}\right). \end{aligned} \tag{45}$$

Therefore, by considering (26), (44) and (45), it follows that

$$\dot{V}_n \leq -l2V_n^{\alpha} - l2^{-\gamma}n^{1-\gamma}V_n^{\gamma}, \tag{46}$$

where $\alpha = 2/(2-\tau)$ and $\gamma = (2+q)/(2-\tau)$.

Thus, according to Lemma 1, we conclude that the equilibrium $p = 0$ of the closed-loop system is fixed-time stable and the settling time function T_1 satisfies

$$\begin{aligned} T_1 &\leq \frac{2}{l(1-\alpha)} + \frac{2^{\gamma}n^{\gamma-1}}{l(\gamma-1)} \\ &= \frac{2(\tau-2)}{l\tau} + \frac{(2-\tau)2^{\frac{2+q}{2-\tau}} n^{\frac{q+\tau}{2-\tau}}}{l(q+\tau)} \\ &< \delta T. \end{aligned} \tag{47}$$

Thus, the proof is completed.

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