Fuzzy Stress-Strength Reliability Subject to Exponentiated Power Generalized Weibull

Neama Salah Youssef Temraz

Abstract— **In this paper, a study of the stress-strength reliability model is introduced subject to the exponentiated power generalized Weibull distribution. The maximum likelihood estimator for the stress-strength reliability function is deduced. The asymptotic confidence interval for the stress-strength reliability function is derived. The fuzzy stress-strength reliability function is discussed using the triangular membership function. A Bayesian estimator for the stress-strength reliability function is deduced. A real data application is introduced to show the results for the stress-strength model based on real data and compare the use of exponentiated power generalized Weibull distribution with existing distributions.**

Index Terms— **Reliability, Stress-strength, exponentiated power generalized Weibull distribution, maximum likelihood estimation, Bayesian estimation, fuzzy number, triangular membership function.**

I. Introduction

Wong (2012) presented an asymptotic interval estimation for $P(Y < X)$ when X and Y are two independent variables that follow the generalized Pareto distribution based on the modified signed log-likelihood ratio statistic. Asgharzadeh et al. (2013) deduced the maximum likelihood estimator and of R and its asymptotic confidence interval in case of the stress and strength variables having a generalized logistic distribution with the same unknown scale but different shape parameters or with the same unknown shape but different scale parameters. Also, the Bayesian estimator for R was deduced. Hussian (2013) presented the estimation of $R = P(Y < X)$ when X and Y are two variables that follow the generalized inverted exponential distribution with different parameters. He discussed the maximum likelihood and the Bayes estimators for the reliability function.

Ghitany et al. (2015) developed the study of the point and interval estimation of the reliability of a stress-strength system from power Lindley distribution using different methods of the maximum likelihood, nonparametric and parametric bootstrap. Li and Hao (2016) studied the estimation of $R = P(Y < X)$ when X and Y follow generalized exponential distributions containing one outlier. Mokhlis (2017) proposed the study of the reliability of the stress-strength model subject to an exponential distribution with general form. Li and Hao (2017) introduced the estimation of $R = P(Y < X)$ when X and Y are two independent variables that follow inverse Weibull distributions with different parameters. Iranmanesh et al. (2018) presented the estimation of stress-strength reliability parameter $R = P(Y < X)$ when X and Y are independent random variables that follow inverted gamma distribution. Mohie El-Din et al. (2018) discussed the stress-strength reliability model $R = P(Y < X)$ when X and Y follow an exponentiated generalized inverse Weibull distribution with different parameters. Juvairiyya and Anilkumar (2018) introduced the likelihood and Bayesian estimation methods for the stress-strength reliability under the Pareto distribution with upper record values.

Muhammad et al. (2020) proposed the estimation of stress-strength reliability parameter $R = P(Y < X)$ based on complete samples when the two independent variables X and Y have Poisson half logistic distribution. Al-omari et al. (2020) presented the estimation of the stress-strength reliability for exponentiated Pareto distribution using median and ranked set sampling methods.

Most papers in the literature assumed the study of the deterministic reliability of the stress-strength models. This paper presents the fuzzy reliability function for the stress-strength model assuming that reliability with triangular membership function under the exponentiated power generalized Weibull distribution. To obtain the fuzzy reliability function, the maximum likelihood estimator and the asymptotic confidence interval for the stress-strength reliability function are obtained. Bayesian estimators and the credible interval for the stress-strength reliability function are discussed. A real data application is introduced to show the results for the stress-strength model and compare different distributions.

II. Exponentiated Power Generalized Weibull Distribution

Peña-Ramírez et al. (2018) proposed the exponentiated power generalized Weibull distribution with four parameters with cumulative distribution function and probability density function which given as follows

$$F(x) = \left[1 - e^{1-(1+\mu x^{\alpha})^{\gamma}}\right]^{\beta},\ x > 0,\ \mu,\alpha,\gamma,\beta > 0$$

and

$$f(x) = \beta\left[1 - e^{1-(1+\mu x^{\alpha})^{\gamma}}\right]^{\beta-1} e^{1-(1+\mu x^{\alpha})^{\gamma}}\gamma(1+\mu x^{\alpha})^{\gamma-1}\mu\alpha x^{\alpha-1}$$

where μ is the scale parameter and $\alpha, \gamma\ and\ \beta$ are the shape parameters. This distribution is flexible to model the failure rates of reliability applications.

III. Stress-Strength Reliability

Let X and Y are two independent random variables, then the stress-strength reliability function will be given by

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$$R = P(Y < X) = \int_0^\infty \int_0^x f(x)f(y)dydx = \int_0^\infty \left[\int_0^x f_y(y)dy\right] f_x(x)dx = \int_0^\infty F_y(x)f_x(x)dx$$

$$R = \int_0^\infty \left[1 - e^{1-(1+\mu x^\alpha)^\gamma}\right]^{\beta_2} \beta_1 \left[1 - e^{1-(1+\mu x^\alpha)^\gamma}\right]^{\beta_1 - 1} e^{1-(1+\mu x^\alpha)^\gamma} \gamma (1 + \mu x^\alpha)^{\gamma - 1} \mu \alpha x^{\alpha - 1} dx$$

Solving the integral yields an expression for the stress-strength reliability function

$$R = \frac{\beta_1}{\beta_1 + \beta_2}$$

A. Maximum Likelihood Estimation

Assume the two independent random samples $(X_1, X_2, \dots, X_n)$ and $(Y_1, Y_2, \dots, Y_m)$ are observed from the exponentiated power generalized Weibull distributions with the parameters $(\mu, \alpha, \gamma, \beta_1)$ and $(\mu, \alpha, \gamma, \beta_2)$, respectively. The likelihood function of $\mu, \alpha, \gamma, \beta_1$ *and* β_2 for the observed samples is given as follows

$$L = \prod_{i=1}^n f(x_i) \prod_{j=1}^m f(y_j)$$

$$L = \prod_{i=1}^n \beta_1 \gamma \mu \alpha x_i^{\alpha-1} (1 + \mu x_i^\alpha)^{\gamma-1} e^{1-(1+\mu x_i^\alpha)^\gamma} \left[1 - e^{1-(1+\mu x_i^\alpha)^\gamma}\right]^{\beta_1 - 1} \prod_{j=1}^m \beta_2 \gamma \mu \alpha y_j^{\alpha-1} (1 + \mu y_j^\alpha)^{\gamma-1} e^{1-(1+\mu y_j^\alpha)^\gamma} \left[1 - e^{1-(1+\mu y_j^\alpha)^\gamma}\right]^{\beta_2 - 1}$$

The log-likelihood function of $\mu, \alpha, \gamma, \beta_1$ *and* β_2 is obtained as follows

$$\begin{aligned} logL &= nlog(\beta_1) + mlog(\beta_2) + (n+m)log(\mu) + (n+m)\log(\alpha) \\ &+ (n+m)\log(\gamma) + (\alpha - 1)\sum_{i=1}^n log(x_i) \\ &+ (\gamma - 1)\sum_{i=1}^n log(1 + \mu x_i^\alpha) + n - \sum_{i=1}^n (1 + \mu x_i^\alpha)^\gamma \\ &+ (\beta_1 - 1)\sum_{i=1}^n log\left[1 - e^{1-(1+\mu x_i^\alpha)^\gamma}\right] \\ &+ (\alpha - 1)\sum_{j=1}^m log(y_j) + (\gamma - 1)\sum_{j=1}^m log(1 + \mu y_j^\alpha) \\ &+ m - \sum_{j=1}^m (1 + \mu y_j^\alpha)^\gamma \\ &+ (\beta_2 - 1)\sum_{j=1}^m log\left[1 - e^{1-(1+\mu y_j^\alpha)^\gamma}\right] \end{aligned}$$

The partial derivatives of the log-likelihood function with respect to $\mu, \alpha, \gamma, \beta_1$ *and* β_2 are obtained as follows

$$\begin{aligned} \frac{\partial logL}{\partial \gamma} &= \frac{n+m}{\gamma} + \sum_{i=1}^n log(1 + \mu x_i^\alpha) - \sum_{i=1}^n (1 + \mu x_i^\alpha)^\gamma log(1 + \mu x_i^\alpha) \\ &+ (\beta_1 - 1)\sum_{i=1}^n \frac{(1 + \mu x_i^\alpha)^\gamma log(1 + \mu x_i^\alpha) e^{1-(1+\mu x_i^\alpha)^\gamma}}{1 - e^{1-(1+\mu x_i^\alpha)^\gamma}} \\ &+ \sum_{j=1}^m log(1 + \mu y_j^\alpha) - \sum_{j=1}^m (1 + \mu y_j^\alpha)^\gamma log(1 + \mu y_j^\alpha) \\ &+ (\beta_2 - 1)\sum_{j=1}^m \frac{(1 + \mu y_j^\alpha)^\gamma log(1 + \mu y_j^\alpha) e^{1-(1+\mu y_j^\alpha)^\gamma}}{1 - e^{1-(1+\mu y_j^\alpha)^\gamma}} \end{aligned}$$

$$\begin{aligned} \frac{\partial logL}{\partial \mu} &= \frac{n+m}{\mu} + (\alpha - 1)\sum_{i=1}^n \frac{x_i^\alpha}{1 + \mu x_i^\alpha} - \alpha \sum_{i=1}^n (1 + \mu x_i^\alpha)^{\gamma-1} x_i^\alpha \\ &+ \alpha(\beta_1 - 1)\sum_{i=1}^n \frac{(1 + \mu x_i^\alpha)^{\gamma-1} x_i^\alpha e^{1-(1+\mu x_i^\alpha)^\gamma}}{1 - e^{1-(1+\mu x_i^\alpha)^\gamma}} \\ &+ (\alpha - 1)\sum_{j=1}^m \frac{y_j^\alpha}{1 + \mu y_j^\alpha} - \alpha \sum_{j=1}^m (1 + \mu y_j^\alpha)^{\gamma-1} y_j^\alpha \\ &+ \alpha(\beta_2 - 1)\sum_{j=1}^m \frac{(1 + \mu y_j^\alpha)^{\gamma-1} y_j^\alpha e^{1-(1+\mu y_j^\alpha)^\gamma}}{1 - e^{1-(1+\mu y_j^\alpha)^\gamma}} \end{aligned}$$

$$\begin{aligned} \frac{\partial logL}{\partial \alpha} &= \frac{n+m}{\alpha} + \sum_{i=1}^n log(x_i) + (\alpha - 1)\mu \sum_{i=1}^n \frac{x_i^\alpha log(x_i)}{1 + \mu x_i^\alpha} \\ &- \gamma\mu \sum_{i=1}^n (1 + \mu x_i^\alpha)^{\gamma-1} x_i^\alpha log(x_i) \\ &+ \mu(\beta_1 - 1)\sum_{i=1}^n \frac{(1 + \mu x_i^\alpha)^{\gamma-1} x_i^\alpha log(x_i) e^{1-(1+\mu x_i^\alpha)^\gamma}}{1 - e^{1-(1+\mu x_i^\alpha)^\gamma}} \\ &+ \sum_{j=1}^m log(y_j) + (\alpha - 1)\mu \sum_{j=1}^m \frac{y_j^\alpha log(y_j)}{1 + \mu y_j^\alpha} \\ &- \gamma\mu \sum_{j=1}^m (1 + \mu y_j^\alpha)^{\gamma-1} y_j^\alpha log(y_j) \\ &+ \mu(\beta_2 - 1)\sum_{j=1}^m \frac{(1 + \mu y_j^\alpha)^{\gamma-1} y_j^\alpha log(y_j) e^{1-(1+\mu y_j^\alpha)^\gamma}}{1 - e^{1-(1+\mu y_j^\alpha)^\gamma}} \end{aligned}$$

$$\frac{\partial logL}{\partial \beta_1} = \frac{n}{\beta_1} + \sum_{i=1}^n log\left[1 - e^{1-(1+\mu x_i^\alpha)^\gamma}\right]$$

$$\frac{\partial logL}{\partial \beta_2} = \frac{m}{\beta_2} + \sum_{j=1}^m log\left[1 - e^{1-(1+\mu y_j^\alpha)^\gamma}\right]$$

Equating the partial derivatives to zero and then solving the resulting equations numerically yields the maximum likelihood estimators for the parameters μ, α, γ.

$$\frac{\partial logL}{\partial \gamma} = 0, \frac{\partial logL}{\partial \mu} = 0, \frac{\partial logL}{\partial \alpha} = 0, \frac{\partial logL}{\partial \beta_1} = 0, \frac{\partial logL}{\partial \beta_2} = 0$$

The maximum likelihood estimators for the parameters β_1 *and* β_2 can be obtained from the following relations

$$\hat{\beta}_1 = \frac{-n}{\sum_{i=1}^n log\left[1 - e^{1-(1+\hat{\mu} x_i^{\hat{\alpha}})^{\hat{\gamma}}}\right]}$$

$$\hat{\beta}_2 = \frac{-m}{\sum_{j=1}^m log\left[1 - e^{1-(1+\hat{\mu} x_i^{\hat{\alpha}})^{\hat{\gamma}}}\right]}$$

The maximum likelihood estimators for the stress-strength reliability function can be obtained as

$$\hat{R} = \frac{\hat{\beta}_1}{\hat{\beta}_1 + \hat{\beta}_2}$$

B. Asymptotic Confidence Interval

The asymptotic variance (AV) of an estimate $\hat{R}$ which is a function of the two independent statistics $\hat{\beta}_1$ *and* $\hat{\beta}_2$ is given by (see Rao (1973))

$$AV(\hat{R}) = V(\hat{\beta}_1)\left(\frac{\partial R}{\partial \beta_1}\right)^2 + V(\hat{\beta}_2)\left(\frac{\partial R}{\partial \beta_2}\right)^2$$

where

$$V(\hat{\beta}_1) = \left[E\left(\frac{\partial^2 L}{\partial \beta_1^2}\right)\right]^{-1} \quad and \quad V(\hat{\beta}_2) = \left[E\left(\frac{\partial^2 L}{\partial \beta_2^2}\right)\right]^{-1}$$

$As\ n \to \infty, m \to \infty, \frac{\hat{R}-R}{AV(\hat{R})} \xrightarrow{d} N(0,1)$ and the asymptotic $100(1-\delta)\%$ confidence interval for R is given by

$$\hat{R} \pm z_{1-\frac{\delta}{2}}\sqrt{AV(\hat{R})} = \hat{R} \pm z_{1-\frac{\delta}{2}} \frac{\hat{\beta}_1\hat{\beta}_2}{(\hat{\beta}_1+\hat{\beta}_2)^2}\sqrt{\frac{1}{n}+\frac{1}{m}}$$

C. Fuzzy Stress-Strength Reliability

Assume that the stress-strength reliability is a fuzzy function with triangular membership function. the interval for the fuzzy stress-strength reliability is given by

$$[\tilde{R}_L, \tilde{R}_U] = [L + \alpha - cut(M-L), U - \alpha - cut(U-M)]$$

where $\alpha - cut = 0, 0.1, \dots, 0.9$, [L, U] is the confidence interval for R and M is the maximum likelihood estimator for R.

D. Bayesian Estimation

The Bayesian estimator for the stress-strength reliability will be obtained assuming that the parameters $\beta_1\ and\ \beta_2$ are independent random variables with the following prior distributions

$$\beta_1 \sim Gamma(\vartheta_1, \eta_1)$$
$$\beta_2 \sim Gamma(\vartheta_2, \eta_2)$$

The joint prior distribution of the two parameters β_1 and β_2 is given by

$$\Pi(\beta_1,\beta_2) = \frac{{\eta_1}^{\vartheta_1}{\eta_2}^{\vartheta_2}}{\Gamma\vartheta_1\Gamma\vartheta_2}\beta_1^{\vartheta_1-1}\beta_2^{\vartheta_2-1}e^{-\eta_1\beta_1-\eta_2\beta_2}, \qquad \vartheta_1,\vartheta_2,\eta_1,\eta_2,\beta_1,\beta_2 > 0$$

The joint posterior density of the parameters β_1 and β_2 is given by

$$\Pi^*(\beta_1,\beta_2|x,y) = \frac{\Pi(\beta_1,\beta_2)L(\beta_1,\beta_2,\mu,\alpha,\gamma;x,y)}{\int_0^\infty\int_0^\infty \Pi(\beta_1,\beta_2)L(\beta_1,\beta_2,\mu,\alpha,\gamma;x,y)d\beta_1 d\beta_2}$$

The Bayesian estimator of the reliability function R under the squared error loss function using the posterior mean is given by

$$R_{SEL} = E(R|x,y) = \int_0^\infty\int_0^\infty R\ \Pi^*(\beta_1,\beta_2|x,y)d\beta_1 d\beta_2$$

Using Lindley approximation (see Lindley (1980)), the Bayesian estimator of R under the squared error loss function can be obtained using the following formula

$$\begin{aligned} R_{SEL} = E(R|x,y) = R + \frac{1}{2}\left[\left(\frac{\partial^2 R}{\partial\beta_1^2}\right)(-L_{20})^{-1} + \left(\frac{\partial^2 R}{\partial\beta_2^2}\right)(-L_{02})^{-1}\right] \\ + \left(\frac{\partial(\log\Pi(\beta_1,\beta_2))}{\partial\beta_1}\right)\left(\frac{\partial R}{\partial\beta_1}\right)(-L_{20})^{-1} \\ + \left(\frac{\partial(\log\Pi(\beta_1,\beta_2))}{\partial\beta_2}\right)\left(\frac{\partial R}{\partial\beta_2}\right)(-L_{02})^{-1} \\ + \frac{1}{2}\left\{L_{30}\left(\frac{\partial R}{\partial\beta_1}\right)[(-L_{20})^{-1}]^2 \right. \\ \left. + L_{03}\left(\frac{\partial R}{\partial\beta_2}\right)[(-L_{02})^{-1}]^2\right\} \end{aligned}$$

where

$$L_{ij} = \left[\frac{\partial^2 L}{\partial\beta_i\partial\beta_j}\right], L_{30} = \frac{\partial^3 L}{\partial\beta_1^3}, L_{03} = \frac{\partial^3 L}{\partial\beta_2^3}$$

The Bayesian estimator of R under the squared error loss function is deduced as

$$\begin{aligned} R_{SEL} = \hat{R}\left\{1 + (1-\hat{R})\left[\frac{1-\hat{R}}{n} - \frac{\hat{R}}{m}\right] \right. \\ \left. + \hat{\beta}_2\left[\frac{\hat{R}}{n}\left(\frac{\vartheta_1-1}{\hat{\beta}_1} - \eta_1\right) - \frac{(1-\hat{R})}{m}\left(\frac{\vartheta_2-1}{\hat{\beta}_2} - \eta_2\right)\right]\right\} \end{aligned}$$

The Bayesian estimator of the reliability function R under Linex loss function is given by

$$\begin{aligned} R_{LL} &= -\frac{1}{\tau}log\{E(e^{-\tau R}|x,y)\} \\ &= -\frac{1}{\tau}log\left\{\int_0^\infty\int_0^\infty e^{-\tau R}\ \Pi^*(\beta_1,\beta_2|x,y)d\beta_1 d\beta_2\right\}, \quad \tau \neq 0 \end{aligned}$$

The Bayesian estimator of R under Linex loss function is given by

$$R_{LL} = -\frac{1}{\tau}log\{E(e^{-\tau R}|x,y)\}$$

Using Lindely approximation, the Bayesian estimator of R under Linex loss function is given by

$$\begin{aligned} R_{LL} = -\frac{1}{\tau}log\left\{e^{-\tau R} + \frac{1}{2}\left\{\tau e^{-\tau R}\left[\tau\left(\frac{\partial R}{\partial\beta_1}\right)^2 - \frac{\partial^2 R}{\partial\beta_1^2}\right](-L_{20})^{-1} \right.\right. \\ \left. + \tau e^{-\tau R}\left[\tau\left(\frac{\partial R}{\partial\beta_2}\right)^2 - \frac{\partial^2 R}{\partial\beta_2^2}\right](-L_{02})^{-1}\right\} \\ + \left(\frac{\partial(\log\Pi(\beta_1,\beta_2))}{\partial\beta_1}\right)\left(-\tau e^{-\tau R}\frac{\partial R}{\partial\beta_1}\right)(-L_{20})^{-1} \\ + \left(\frac{\partial(\log\Pi(\beta_1,\beta_2))}{\partial\beta_2}\right)\left(-\tau e^{-\tau R}\frac{\partial R}{\partial\beta_2}\right)(-L_{02})^{-1} \\ + \frac{1}{2}\left[L_{30}\left(-\tau e^{-\tau R}\frac{\partial R}{\partial\beta_1}\right)[(-L_{20})^{-1}]^2 \right. \\ \left.\left. + L_{03}\left(-\tau e^{-\tau R}\frac{\partial R}{\partial\beta_2}\right)[(-L_{02})^{-1}]^2\right]\right\} \end{aligned}$$

the Bayesian estimator of R under Linex loss function is deduced as

$$\begin{aligned} R_{LL} = \hat{R} - \frac{1}{\tau}log\left\{e^{-\tau\hat{R}}\left[1 \right.\right. \\ + \frac{\tau}{2(\hat{\beta}_1+\hat{\beta}_2)^2}\left((1-\hat{R})(\tau(1-\hat{R})-2) \right. \\ \left. + \hat{R}(\tau\hat{R}-2)\right) \\ - \tau\left(\left(\frac{\vartheta_1-1}{\hat{\beta}_1} - \eta_1\right)\frac{\hat{\beta}_2\hat{R}^2}{n} \right. \\ \left. + \left(\frac{\vartheta_2-1}{\hat{\beta}_2} - \eta_2\right)\frac{\hat{\beta}_1(1-\hat{R})^2}{m}\right) \\ \left.\left. - \tau\hat{R}(1-\hat{R})\left(\frac{1}{n}+\frac{1}{m}\right)\right]\right\} \end{aligned}$$

E. The Credible Interval

The posterior probability density function of R can be derived and the result is

$$f_R(r) = \frac{{\eta_1}^{\vartheta_1}{\eta_2}^{\vartheta_2}\Gamma(\vartheta_1+\vartheta_2)r^{\vartheta_1-1}(1-r)^{\vartheta_2-1}}{\Gamma\vartheta_1\Gamma\vartheta_2[\eta_1 r + \eta_2(1-r)]^{\vartheta_1+\vartheta_2}}, \qquad r > 0$$

It can be shown that the posterior distribution of β_1 and β_2 are given by

$$\beta_1|x \sim Gamma\left(n + \vartheta_1, \eta_1 - \sum_{i=1}^{n} log\left[1 - e^{1-(1+\mu x_i{}^\alpha)^\gamma}\right]\right)$$

and

$$\beta_2|y \sim Gamma\left(m + \vartheta_2, \eta_2 - \sum_{j=1}^{m} log\left[1 - e^{1-(1+\mu y_j{}^\alpha)^\gamma}\right]\right)$$

From the relations between the gamma distribution and chi-square distribution, it can be shown that

$$2\left(\eta_1 - \sum_{i=1}^{n} log\left[1 - e^{1-(1+\mu x_i{}^\alpha)^\gamma}\right]\right)\beta_1 \sim \chi^2_{2(n+\vartheta_1)}$$

and

$$2\left(\eta_2 - \sum_{j=1}^{m} log\left[1 - e^{1-(1+\mu y_j{}^\alpha)^\gamma}\right]\right)\beta_2 \sim \chi^2_{2(m+\vartheta_2)}$$

The posterior distribution of R can be written as

$$\left\{1+\frac{(m+\vartheta_2)\left(\eta_1-\sum_{i=1}^{n} log\left[1-e^{1-(1+\mu x_i^{\alpha})^{\gamma}}\right]\right)}{(n+\vartheta_1)\left(\eta_2-\sum_{j=1}^{m} log\left[1-e^{1-(1+\mu y_j^{\alpha})^{\gamma}}\right]\right)}F\left(2(m+\vartheta_2),2(n+\vartheta_1)\right)\right\}^{-1}$$

And therefore a $100(1-\varepsilon)\%$ credible interval for R will be given by

$$\left\{1+\frac{A}{B}F_{\frac{\varepsilon}{2},2(m+\vartheta_2),2(n+\vartheta_1)}\right\}^{-1},\left\{1+\frac{A}{B}F_{1-\frac{\varepsilon}{2},2(m+\vartheta_2),2(n+\vartheta_1)}\right\}^{-1}$$

where

$A=(m+\vartheta_2)\left(\eta_1-\sum_{i=1}^{n} log\left[1-e^{1-(1+\mu x_i^{\alpha})^{\gamma}}\right]\right)$ and $B=(n+\vartheta_1)\left(\eta_2-\sum_{j=1}^{m} log\left[1-e^{1-(1+\mu y_j^{\alpha})^{\gamma}}\right]\right)$

IV. REAL DATA APPLICATION

The following data from Crowder (2000) will be used which represents the breaking strengths of single carbon fibers of length 1 in data set1 and of length 10 in data set 2.

TABLE I
DATA SET 1 (THE BREAKING STRENGTHS OF SINGLE CARBON FIBERS OF LENGTH 1)

2.247	2.640	2.842	2.908	3.099	3.126	3.245	3.328	3.355	3.383
3.572	3.581	3.681	3.726	3.727	3.728	3.783	3.785	3.786	3.896
3.912	3.964	4.050	4.063	4.082	4.111	4.118	4.141	4.216	4.251
4.262	4.326	4.402	4.457	4.466	4.519	4.542	4.555	4.614	4.632
4.634	4.636	4.678	4.698	4.738	4.832	4.924	5.054	5.099	5.134
5.359	5.473	5.571	5.684	5.721	5.998	6.060			

TABLE II
DATA SET 2 (THE BREAKING STRENGTHS OF SINGLE CARBON FIBERS OF LENGTH 10)

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445
2.454	2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616
2.618	2.624	2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.937
2.937	2.977	2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235
3.243	3.264	3.272	3.294	3.332	3.346	3.377	3.408	4.435	3.493
3.501	3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024
4.027	4.225	4.395	5.020						

The maximum likelihood estimators for the parameters and the reliability of the stress-strength model according to data set 1 and data set 2 are obtained as $\hat{\mu}=0.298, \hat{\alpha}=1.202, \hat{\gamma}=1.708, \hat{\beta}_1=40.669, \hat{\beta}_2=10.711$ and the $\hat{R}=0.791$. The asymptotic 95% confidence interval for R is [0.733, 0.849]. The results of the intervals for the fuzzy stress-strength reliability function are obtained in Table III.

TABLE III
THE INTERVALS FOR THE FUZZY STRESS-STRENGTH RELIABILITY FUNCTION

$\boldsymbol{\alpha-cut}$	$[\widetilde{\boldsymbol{R}}_L,\widetilde{\boldsymbol{R}}_U]$
0.1	[0.7388, 0.8432]
0.2	[0.7446, 0.8374]
0.3	[0.7504, 0.8316]
0.4	[0.7562, 0.8258]
0.5	[0.7620, 0.8200]
0.6	[0.7678, 0.8142]
0.7	[0.7736, 0.8084]
0.8	[0.7794, 0.8026]
0.9	[0.7852, 0.7968]

To fit the two data sets with the exponentiated power generalized Weibull distribution, the Kolmogorov-Simrnov and Anderson-Darling goodness of fit tests are used and the results are shown in Table IV which indicated that the exponentiated power generalized Weibull distribution fits well to data set 1 and data set 2.

TABLE IV
KOLMOGOROV-SIMRNOV AND ANDERSON-DARLING GOODNESS OF FIT TESTS FOR DATA SETS 1 AND 2

Data Set	Kolmogorov-Simrnov Test	Anderson-Darling Test
Data Set 1	0.121 < 0.180 (critical value)	1.771 < 2.492 (critical value)
Data Set 2	0.131 < 0.170 (critical value)	1.839 < 2.492 (critical value)

The goodness of fit of the two data sets is tested for different distributions using the log-likelihood function (Log L), Akaike information criteria (AIC), Akaike information criteria corrected (AICC) and Bayesian information criteria (BIC). The goodness of fit for the exponentiated power generalized Weibull distribution (EPGW) is compared with the exponentiated Weibull distribution (EW), power generalized Weibull distribution (PGW), Nadarajah-Haghighi distribution (N-H), exponentiated Nadarajah-Haghighi distribution (EN-H) and exponentiated exponential distribution (EE). The maximum likelihood estimators (MLE) for the parameters, Log L, AIC, AICC and BIC for the data set 1 and data set 2 are shown in Table V and Table VI, respectively.

TABLE V
MLE ESTIMATES, LOG L, AIC, AICC AND BIC FOR THE DATA SET 1

Distribution	MLE estimates	Log L	AIC	AICC	BIC
EPGW$(\mu,\alpha,\gamma,\beta)$	$\hat{\mu}=0.298$ $\hat{\alpha}=1.202$ $\hat{\gamma}=1.708$ $\hat{\beta}_1=40.669$	**-73.827**	**155.654**	**156.423**	**154.677**
Weibull(μ,α)	$\hat{\mu}=0.529$ $\hat{\alpha}=0.939$	-161.641	327.282	327.504	326.793
EW(μ,α,β)	$\hat{\mu}=0.428$ $\hat{\alpha}=1.076$ $\hat{\beta}_1=6.313$	-96.449	198.898	199.350	198.165
PGW(μ,α,γ)	$\hat{\mu}=0.664$ $\hat{\alpha}=0.916$ $\gamma=0.902$	-166.711	339.422	339.874	338.689
N-H(μ,γ)	$\hat{\mu}=0.184$ $\gamma=1.083$	-138.658	281.316	281.538	280.827
EN-H(μ,γ,β)	$\hat{\mu}=0.570$ $\hat{\gamma}=0.103$ $\hat{\beta}_1=0.479$	-211.535	429.070	429.522	428.337
Exponential(μ)	$\hat{\mu}=0.213$	-139.567	281.134	281.206	280.889
EE(μ,β)	$\hat{\mu}=0.625$ $\hat{\beta}_1=11.700$	-89.594	183.188	183.410	182.699

The results obtained in Tables V and VI, indicated that the exponentiated power generalized Weibull distribution can be a better distribution to model the data sets than the distributions power generalized Weibull, Nadarajah-Haghighi, exponentiated Nadarajah-Haghighi and exponentiated exponential.

TABLE VI
MLE ESTIMATES, LOG L, AIC, AICC AND BIC FOR THE DATA SET 2

Distribution	Estimates	Log L	AIC	AICC	BIC
EPGW$(\mu,\alpha,\gamma,\beta)$	$\hat{\mu}=0.298$ $\hat{\alpha}=1.202$ $\hat{\gamma}=1.708$ $\hat{\beta}_2=10.711$	**-191.205**	**390.410**	**391.087**	**389.634**
Weibull(μ,α)	$\hat{\mu}=0.529$ $\hat{\alpha}=0.939$	-273.413	550.826	551.022	550.438
EW(μ,α,β)	$\hat{\mu}=0.428$ $\hat{\alpha}=1.076$ $\hat{\beta}_2=3.390$	-230.357	466.714	467.114	466.132
PGW(μ,α,γ)	$\hat{\mu}=0.664$ $\hat{\alpha}=0.916$ $\gamma=0.902$	-278.864	563.728	564.128	563.146
N-H(μ,γ)	$\hat{\mu}=0.184$ $\gamma=1.083$	-268.587	541.174	541.370	540.786
EN-H(μ,γ,β)	$\hat{\mu}=0.570$ $\hat{\gamma}=0.103$ $\hat{\beta}_2=0.437$	-344.481	694.962	695.362	694.380
Exponential(μ)	$\hat{\mu}=0.213$	-268.549	539.098	539.162	538.904
EE(μ,β)	$\hat{\mu}=0.625$ $\hat{\beta}_2=5.697$	-221.483	446.966	447.162	446.578

In Table VII, the Bayesian estimate under the squared error loss and the credible interval for R are obtained for different values of the parameters of the prior distributions of β_1 and β_2. In Table VIII, The Bayesian estimate of R under Linex loss function is obtained for different values of the parameters of the prior distributions of (β_1, β_2) and different values of τ.

TABLE VII
THE BAYESIAN ESTIMATE OF R_{SEL} AND THE CREDIBLE INTERVAL FOR R

$(\vartheta_1,\vartheta_2,\eta_1,\eta_2)$	R_{SEL}	Credible Interval
(1, 1, 0.5, 0.5)	0.744	[0.680, 0.812]
(2, 2, 0.5, 0.5)	0.744	[0.681, 0.812]
(2, 3, 0.5, 0.5)	0.742	[0.678, 0.809]
(3, 3, 0.5, 0.5)	0.745	[0.682, 0.812]
(1, 1, 1.5, 1.5)	0.654	[0.616, 0.766]
(2, 2, 1.5, 1.5)	0.655	[0.617, 0.766]
(2, 3, 1.5, 1.5)	0.652	[0.614, 0.762]
(0.5, 0.5, 1, 1)	0.699	[0.644, 0.787]
(0.5, 0.5, 2, 3)	0.637	[0.621, 0.770]
(1.5, 1.5, 1, 1)	0.699	[0.645, 0.787]
(2, 3, 2, 3)	0.635	[0.620, 0.705]

TABLE VIII
THE BAYESIAN ESTIMATE OF R_{LL}

$(\vartheta_1,\vartheta_2,\eta_1,\eta_2)$	R_{LL} $(\tau=0.5)$	R_{LL} $(\tau=-0.5)$	R_{LL} $(\tau=1)$	R_{LL} $(\tau=-1)$
(1, 1, 0.5, 0.5)	0.725	0.723	0.726	0.721
(2, 2, 0.5, 0.5)	0.730	0.728	0.731	0.727
(2, 3, 0.5, 0.5)	0.733	0.731	0.733	0.730
(3, 3, 0.5, 0.5)	0.735	0.734	0.736	0.733
(1, 1, 1.5, 1.5)	0.589	0.566	0.598	0.552
(2, 2, 1.5, 1.5)	0.594	0.572	0.603	0.559
(2, 3, 1.5, 1.5)	0.596	0.575	0.605	0.562
(0.5, 0.5, 1, 1)	0.653	0.643	0.657	0.637
(0.5, 0.5, 2, 3)	0.498	0.447	0.516	0.412
(1.5, 1.5, 1, 1)	0.658	0.649	0.662	0.644
(2, 3, 2, 3)	0.507	0.460	0.525	0.428

IV. CONCLUSION

The study of the fuzzy stress-strength reliability model subject to the exponentiated power generalized Weibull distribution is introduced. The maximum likelihood estimator and the asymptotic confidence interval for the stress-strength reliability function are obtained. Bayesian estimators and the credible interval for the stress-strength reliability function are derived. An application based on real data is introduced to show the results for the stress-strength model and compare the exponentiated power generalized Weibull distribution with other different distributions. This comparison shows that the exponentiated power generalized Weibull model can be considered a better model to fit the data sets.

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REFERENCES

[1] A. Wong, Interval estimation of P(Y<X) for generalized Pareto distribution, Journal of Statistical Planning and Inference (2012), Volume 142, Issue 2, Pages 601-607.

[2] A. Asgharzadeh, R. Valiollahi and Mohammad Z. Raqab, Estimation of the stress–strength reliability for the generalized logistic distribution, Statistical Methodology (2013), Volume 15, Pages 73-94, https://doi.org/10.1016/j.stamet.2013.05.002.

[3] M. A. Hussian, On estimation of stress strength model for generalized inverted exponential distribution, Journal of Reliability and Statistical Studies (2013), Volume 6, Issue 2, Pages 55-63.

[4] M. E. Ghitany, D. K. Al-Mutairi and S. M. Aboukhamseen, Estimation of the reliability of a stress-strength system from power Lindley distributions, (2015), Volume 44, Issue 1, Pages 118-136, https://doi.org/10.1080/03610918.2013.767910.

[5] C. Li and H. Hao, Likelihood and bayesian estimation in stress strength model from generalized exponential distribution containing outliers, IAENG International Journal of Applied Mathematics (2016), Volume 46, Issue 2, Pages 155-159.

[6] N. A. Mokhlis, E. J. Ibrahim and D. M. Gharieb, Interval estimation of stress-strength reliability for a general exponential form distribution with different unknown parameters, International Journal of Statistics and Probability (2017), Volume 6, Number 6, Pages 60-70.

[7] C. Li and H. Hao, Reliability of a stress-strength model with inverse Weibull distribution, IAENG International Journal of Applied Mathematics (2017), Volume 47, Issue 3, Pages 302-306.

[8] A. Iranmanesh, K. F. Vajargah and M. Hasanzadeh, On the estimation of stress strength reliability parameter of inverted gamma distribution, Mathematical Sciences (2018), Volume 12, Pages 71–77, https://doi.org/10.1007/s40096-018-0246-4.

[9] M. M. Mohie El-Din, A. Sadek and S. H. Elmeghawry, Stress-strength reliability estimation for exponentiated generalized inverse Weibull distribution, Journal of Statistics Applications & Probability (2018), Volume 7, Number 1, pages 127-136, http://dx.doi.org/10.18576/jsap/070112.

[10] R. M. Juvairiyya and P. Anilkumar, Estimation of stress-strength reliability for the Pareto distribution based on upper record values, STATISTICA (2018), Volume 78, Number 4, Pages 397-409.

[11] I. Muhammad, X. Wang, C. Li, M. Yan and M. Chang, Estimation of the reliability of a stress–strength system from Poisson half logistic distribution, Entropy (2020), Volume 22, Issue 11, Pages 1-28, https://doi.org/10.3390/e22111307.

[12] A. I. Al-omari, I. M. Almanjahie, A. S. Hassan and H. F. Nagy, Estimation of the stress-strength reliability for exponentiated Pareto distribution using median and ranked set sampling methods, Computers, Materials & Continua (2020), Volume 64, Number 2, Pages 835-857.

[13] F. A. PEÑA-Ramírez, R. R. Guerra, G. M. Cordeiro and P. R. D. Marinho, The exponentiated power generalized Weibull: Properties and applications, Annals of the Brazilian Academy of Sciences (2018), Volume 90, Number 3, Pages 2553-2577, http://dx.doi.org/10.1590/0001-3765201820170423.

[14] C. R. Rao, Linear Statistical Inference and its Applications, Wiley Eastern Limited, India (1973).

[15] D. V. Lindley, Approximate Bayesian methods, Trabajos de Estadistica Y de Investigacion Operativa (1980), Volume 31, Pages 223–245, https://doi.org/10.1007/BF02888353.