Balanced Effectiveness of Toxic Mitigation Measures under Aquatic Environments: Game-theoretical Analysis for Combinatorial Mechanism among Antidotes and Catalysts

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Abstract—**Aquatic environmental pollution has become a sustainable issue that requires substantial efforts to develop and implement effective control for reducing and mitigating the pollution effects. Recently, the phenomenon of compound aquatic environmental pollution generated by multiplex contaminants is very familiar. Many studies reveal that the toxicity varies due to the catalytic effects of multiple contaminants, producing a wide quantity of complexity to toxicity in the aquatic environments. Antidotes and catalysts, naturally or artificially originated, have been always evaluated to be required elements in multitudinous toxic mitigation measures under aquatic environments. Though many antidotes can react under its specific characteristics, the effectiveness of toxic mitigation measures can be uplifted by applying harmonious combinations of various antidotes and catalysts. Another aspect, game-theoretical notions have been adopted to derive the harmonious or optimal situations for behavior simulation, efficiency adjustment, portfolio distribution and utility control among different fields. Based on game-theoretical evaluations under aquatic environments, this study is devoted to simulate and generate the most effective harmonious combinations for a collection of antidotes and catalysts with various toxic mitigation measures. Therefore, a power index for combinatorial mechanism is introduced by means of the ingredients and its reactive behavior simultaneously. Some reasonable properties and related axiomatic outcomes are also provided to evaluate the rationality and the accuracy of this power index. In conjunction with the proposed game-theoretical outcomes related to aquatic environments, this study farther analyzes and evaluate the balanced combinations among antidotes and catalysts for toxic mitigation measures.**

Index Terms—**Aquatic environments, toxic mitigation measures, game-theoretical outcome; power index, combinatorial mechanism.**

I. Introduction

Environmental pollution is driving the presence of various toxins to aquatic ecological niches, and it is very quickly changing the future prospects for current aquatic ecology and related economic activities, at a scale far quicker than expected. In this respect, the toxic mitigation measures under aquatic environments provides very crucial information in terms of risk control decisions of contaminants, chemical and biochemical management strategies, and prevention of aquatic environmental pollution concerns. Antidotes and catalysts, artificially or naturally originated, have been often considered to be essential ingredients in numerous toxic mitigation measures under aquatic environments. There are many different types of antidotes. Though each antidote can react under its possessed identities, the effectiveness of toxic mitigation measures can be promoted by a harmonious combination of various antidotes and catalysts. Apart from ameliorating reversible affects, the harm of irreversible affects can also be alleviated. To evaluate the usefulness of the combination of antidotes and catalysts, empiric prognostication, inference of posterior outcomes, or simulation of similar programs, as well as the simulation, construction and derivation of academic theories in distinct spheres, may be conducted to examine the accuracy, suitability, feasibility, validity and plausibility of such combination. The scientific community is capitalizing on many innovations to transform traditional approaches used for numerous toxic mitigation measures under aquatic environments. Related investigations have been introduced wildly, such as Habschied [2], Mouchbahani-Constance et al. [12], Peles et al. [15], Reichwaldt et al. [16], Sotnichenko et al [18], and so on.

Under recent academic literature, game-theoretical notions and related outcomes have been widely adopted to seek the harmonious or optimal situations for behavior simulation, efficiency adjustment, portfolio distribution and utility control. Under standard surroundings, a efficacy mapping is considered by analyzing whole subsets in the collection of ingredients. This presents that the options available for every ingredient are either to react thoroughly under a reaction procedure or not to react radically. In real-world situations, however, allocation, domination, regulation and imitation always vary comparatively to each other in response to the abruptly altering interplay among units, flocks, and circumstances. Hence, a *multi-choice surrounding* could be premeditated as a natural generalization of a standard surrounding in which every ingredient displays distinct reactive behavior. Another aspect, the power indexes have been utilized to examine the efficacy of each ingredient of a system. Each ingredient will possess a certain quantity of reactive behavior, and so its efficacy will be different. Many power indexes have been utilized under multi-choice surroundings. By determining overall affects for a specific ingredient under multi-choice surroundings, Cheng et al. [1], Hwang and Liao [6], [7], Liao [8], [9], Liao et al. [10] and Nouweland et al. [14] utilized different indexes and related outcomes by respectively extending the conceptions of some allocations of standard surroundings. This study considers a generalized analogue of the *pseudo equal allocation of non-separable*

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costs (PEANSC). Under standard surroundings, Hsieh and Liao [4] firstly applied the individual index to define the PEANSC, and further adopted a reduction and its consonance to conclude that the PEANSC is a suitable power index matching some useful properties.

These mentioned above derive one motivation:

- whether the proposed outcomes of the PEANSC could be generalized to simulate, construct, analyze and generate the most effective harmonious combinations for a collection of antidotes and catalysts with various toxic mitigation measures of aquatic environments under multi-choice consideration.

This study is committed to analyzing this motivation. The main outcomes of this study are presented as follows.

- Similar to Hwang and Liao [5], a generalized analogue of the PEANSC, the *multi-choice aggregate-individual index* (MAII), is considered by means of the ingredients and its reactive behavior simultaneously in Section 2.
- In order to evaluate the validity and the justifiability of the MAII, alternative extensions of the consonance and related outcomes due to Hsieh and Liao [4] are introduced to characterize the MAII in Section 3.
- By applying the proposed game-theoretical outcomes to the combinatorial processes among antidotes and catalysts for toxic mitigation measures under aquatic environments, this study farther analyzes, demonstrates and examines the accuracy, suitability, feasibility, validity and plausibility of the MAII in Section 4. Some interpretations, comparisons and applications are also presented throughout this study.

II. PRELIMINARIES

A. The multi-choice aggregate-individual index

Denote that UA be the collection of ingredients, for instance, the gathering constituted by antidotes of the Earth. Arbitrary $p \in UA$ is an ingredient of UA, for instance, an antidote of the Earth. Let $p \in UA$ and $b_p \in \mathbb{N}$, $B_p = \{0, 1, \cdots, b_p\}$ could be regarded as the reactive behavior collection of ingredient p and $B_p^+ = B_p \backslash \{0\}$, where 0 indicates no reaction. Let $A \subseteq UA$ be the largest collection of total ingredients of an interactive environment under UA, for instance, all antidotes of a aquatic environment in the Earth. Denote that $B^A = \prod_{p\in A} B_p$ is the product collection of the reactive behavior collections of total ingredients of A. For all $K \subseteq A$, one could define the vector $\kappa^K \in B^A$ to be $\kappa_p^K = 1$ if $p \in K$, and $\kappa_p^K = 0$ if $p \in A \setminus K$. Assume that 0_A is the zero vector in $\mathbb{R}^A$.

A **multi-choice surrounding** is denoted by (A, b, E), where $A \neq \emptyset$ is a collection of ingredients, the vector $b = (b_p)_{p\in A}$ presents the amount of total reactive behavior for every ingredient, and $E : B^A \to \mathbb{R}$ is a efficacy map with $E(0_A) = 0$ which appoints to every $\gamma = (\gamma_p)_{p\in A} \in B^A$ the efficacy that the ingredients can arise when each ingredient p reacts with reactive behavior γ_p. As $b \in \mathbb{R}$ is fixed under this study, one could write (A, E) rather than (A, b, E).

The class of total multi-choice surroundings is denoted by Δ. Let $(A, E) \in \Delta$ and $\gamma \in B^A$, one could write that $N(\gamma) = \{p \in A |\ \gamma_p \neq 0\}$, γ_K means the restriction of γ at K for every $K \subseteq A$ and $\|\gamma\| = \sum_{p\in A} \gamma_p$.

Define that $L^A = \{(p, k_p) \mid p \in A, k_p \in B_p^+\}$ for every $(A, E) \in \Delta$. A **power index** on Δ is a mapping ζ appointing to every $(A, E) \in \Delta$ the following element

$$\zeta(A, E) = \Big(\zeta_{p,k_p}(A, E)\Big)_{(p,k_p)\in L^A} \in \mathbb{R}^{L^A}.$$

Here $\zeta_{p,k_p}(A, E)$ is the affect of the ingredient p when it reacts with behavior k_p in (A, E). For convenience, one could define that $\zeta_{p,0}(A, E) = 0$ for every $p \in A$.

A generalized analogue of the pseudo equal allocation of non-separable costs is defined on multi-choice surroundings as follows.

Definition 1: The **multi-choice aggregate-individual index (MAII)**, $\overline{\theta}$, is the map on Δ which appoints to every $(A, E) \in \Delta$, every ingredient $p \in A$ and every $k_p \in B_p^+$ the affect

$$\overline{\theta}_{p,k_p}(A, E) = \theta_{p,k_p}(A, E) + \frac{1}{\|b\|}\Big[E(b) - \sum_{t\in A}\sum_{q=1}^{b_t} \theta_{t,q}(A, E)\Big],$$

where $\theta_{p,k_p}(A, E) = E\big(b_p, 0_{A\setminus\{p\}}\big) - E\big(k_p - 1, 0_{A\setminus\{p\}}\big)$ is the **individual-level distinction** of the ingredient p from its behavior $k_p - 1$ to b_p. Under the allocation notion of $\overline{\theta}$, ingredients firstly apply its aggregate-behavior distinctions under corresponding behavior, and further allot equally the rest of efficacy under all reactive behavior.

B. Motivating and actual examples

In order to demonstrate how the notion of multi-choice surrounding and the MAII could be utilized and to guarantee its meaning more apparent and clear, this subsection would like to provide motivating and actual examples due to combinations among antidotes and catalysts for toxic mitigation measures under aquatic environments.

Example 1: Human use of pesticides is to remove pests that harm crops. Sometimes other biological agents or fungicides are substituted for agriculture. However, the use and development of synthetic pesticides is booming, and human dependence and demand for pesticides are increasing day by day. Food safety and other concerns have always been very serious and increasingly serious alternatives in a modern society with high environmental awareness. Since Taiwan has now widely used pesticides to prevent weevil, aphids, root nodule nematodes and other pests that affect the growth of crops, poisoning incidents of poultry, fish, dogs and other animals. The water-soluble pesticide is easy to enter aquatic environment after being sprayed in agriculture. Such pesticides combine with heavy metal pollutants that are common in the environment, and their toxicity may therefore exhibit enhanced toxicity. Most of the current reports on pesticide protection and metal toxicity data, however, are mostly individual acute toxic reactions. Data on the rapid toxicity and chronic toxicity of pesticide heavy metal impurities are limited. Therefore, as mentioned in introduction, it seems to be reasonable that game-theoretical notions might be adopted to investigate, simulate and generate harmonious or optimal situations of the combinatorial processes among antidotes and catalysts for toxic mitigation measures under aquatic environments.

Let A be the collection of pesticides, toxic heavy metal impurities, antidotes and catalysts under aquatic environment.

Further, let the amount of the reactive behavior of each $p \in A$ is B_p. Different these reactive ingredients can influence each other according to the chemical structure, type characteristics and dosage of the compound itself, resulting in various interactions and mutual influence on each others reactions. Thus, a multi-choice coalition $\gamma \in B^A$ could been regarded as a combination of reactive situations for these reactive ingredients meant to assess affects or impacts, which are coincident to its ingredients. At the same time, the toxicity changes of different toxic substances after mixing can still be used. It is determined by detecting the toxic ingredients of a single metabolic pathway of the cell, but the complexity of the interaction of toxic substances in living organisms is much higher than that of the cellular level. The synergistic affects of a multi-choice behavior vector γ of these reactive ingredients after toxic mitigation measures (i.e. $E(\gamma)$) are evaluated under the mode that these reactive ingredients both unleash multiple kinds of reactions. Based on the notion of multi-choice surrounding, a toxic mitigation measure under aquatic environment with high densities of these reactive ingredients could be constituted to be (A, E). It is anticipated that the game-theoretical outcomes of the MAII could be applied to generate harmonious configurations for evaluating the toxicity of pesticides and heavy metal impurities, and the toxic mitigation measure under aquatic environment.

Example 2: Lead (Pb) and lead-derived compounds are extensively used in different industries and have been causing a widespread pollution problem in the world. People expose to the lead mostly because of water or food contamination and air pollution derived from industrial discharges and petrol containing lead or lead-derived compounds. Research evidence has revealed that lead or lead-derived compounds make many harmful impacts on the human health. Lead and lead-derived compounds can enter aquatic environments from different sources like metal water pipes, battery manufacturing, paint, gasoline and urban sewage and disperse in the whole aquatic environments. Many studies have shown that low-level lead exposure on fish for a period of time might cause embryotoxicity, behavioral disorders, and memory impairments.

Chelating agents like CaNa2EDTA and DMSA, which are lead antagonists and can form a complex preventing the binding of lead to body ligands, have been utilized on the treatment of lead poisoning for a long time. Besides, it also shown DMSA could reverse neurobehavioral dysfunction of fishes exposed to lead or lead-derived compounds. Suppose that A is the set of lead (Pb), lead-derived compounds, chelating agents and related catalysts under aquatic environment. Further, let the amount of the reactive behavior of each $p \in A$ is B_p. Different ingredients might influence each other according to the chemical structure, type characteristics and dosage of the compound itself, resulting in various interactions and mutual influence on each others reactions. Thus, a multi-choice coalition $\gamma \in B^A$ could been treated as a combination of reactive processes for these ingredients meant to assess affects or impacts, which are coincident to its ingredients. The synergistic affects of a multi-choice behavior vector γ of these ingredients after toxic mitigation measures (i.e. $E(\gamma)$) are evaluated under the mode that these ingredients both unleash multiple kinds of reactions. Based on the notion of multi-choice surrounding, a toxic mitigation measure under aquatic environment with high concentrations of these ingredients could be constituted to be (A, E). It is anticipated that the game-theoretical outcomes of the MAII could be applied to generate harmonious configurations for evaluating the toxicity of lead (Pb) and lead-derived compounds, and the toxic mitigation measure under aquatic environment.

III. Game-theoretical axiomatizations

In order to evaluate the rationality of the MAII, we demonstrate that there exist some useful properties that could be applied to characterize the MAII. Some more properties are needed. Let ζ be a power index on Δ.

- ζ contents **behavior completeness (BCOM)** if $\sum_{p\in A}\sum_{q=1}^{b_p}\zeta_{p,q}(A,E) = E(b)$ for all $(A,E)\in\Delta$.
- ζ contents **criterion for aggregate surroundings property (CACP)** if $\zeta(A,E) = \overline{\theta}(A,E)$ for all $(A,E)\in\Delta$ with $|A|\leq 2$.
- ζ contents **aggregate equal affect property (AEAP)** if for all $(A,E)\in\Delta$ with $E(\gamma,k_p,0)-E(\gamma,k_p-1,0) = E(\gamma,0,k_h)-E(\gamma,0,k_h-1)$ for some $(p,k_p),(h,k_h)\in L^A$ and for all $\gamma\in B^{A\setminus\{p,h\}}$, $\zeta_{p,k_p}(A,E) = \zeta_{h,k_h}(A,E)$.
- ζ contents **behavior synchronization (BSYN)** if for all $(A,E),(A,D)\in\Delta$ with $E(\gamma) = D(\gamma) + \sum_{p\in N(\gamma)}\sum_{q=1}^{\gamma_p}\mu_{p,q}$ for some $\mu\in\mathbb{R}^{L^A}$ and for all $\gamma\in B^A$, $\zeta(A,E) = \zeta(A,D)+\mu$.

BCOM declares that total ingredients allot whole the efficacy entirely if total ingredients react at all behavior in a surrounding. CACP presents a self-sufficient situation if there is only one ingredient in the surrounding, but if there are two ingredients in the surrounding, then each of them first gets what they could have occurred alone, and at the remaining part of the surrounding, they partake all the rest of profits and losses. AEAP declares that the affects of two ingredients should be coincident if the aggregate-distinctions of these two ingredients are the same. BSYN could be regarded as a weakness of *additivity*. In the following sections, the interaction among above game-theoretical axioms and combinatorial procedures for antidotes and catalysts under toxic mitigation measures will be interpreted in detail. Based on Definition 1, it is trivial to demonstrate that the MAII matches CACP.

Next we propose a generalized analogue of the reduction defined by Hsieh and Liao [4]. Let $(A,E)\in\Delta$, $K\subseteq A$ and ζ be a power index. The **reduced surrounding** (K, E^{ζ}_K) is defined to be for each $\gamma\in B^K$,

$$E^{\zeta}_K(\gamma) = \begin{cases} 0 & \gamma = 0_K, \\ E(\gamma_p, 0_{A\setminus\{p\}}) & K\geq|2|,\ N(\gamma) = \{p\} \text{ for some } p, \\ E(\gamma, b_{A\setminus K}) - \sum_{p\in A\setminus K}\sum_{q=1}^{b_p}\zeta_{p,q}(A,E) & \text{o.w..} \end{cases}$$

The *consonance* property can be interpreted as follows. Given a power index ζ. For arbitrary pair of two ingredients in a surrounding, one considers a "reduced surrounding" among them by pondering the quantities remaining after the rest of the ingredients are offered the affects assigned by ζ. ζ is *consonant* if it always generates coincident affects as in initial surrounding when it is utilized to any reduced surrounding. Formally, ζ matches **consonance (CSE)** if for every $(A,E)\in\Delta$ with $|A|\geq 3$, for every $K\subseteq A$ with $|K|=2$ and for every $(p,k_p)\in L^K$, $\zeta_{p,k_p}(A,E)=\zeta_{p,k_p}(K,E^{\zeta}_K)$.

Subsequently, some results related to the consonance property are provided. The following lemma would examine the consonance property of the MAII.

Lemma 1: The MAII $\overline{\theta}$ contents CSE.

Proof: Let $(A,E)\in\Delta$, $|A|\geq 3$ and $H\subseteq A$, $|H|=2$. Assume that $H=\{p,h\}$. By the definition of $\overline{\theta}$, for all $(p,k_p)\in L^H$,

$$\begin{aligned}&\overline{\theta}_{p,k_p}(H,E^{\overline{\theta}}_H)\\ =\;&\theta_{p,k_p}(H,E^{\overline{\theta}}_H)+\tfrac{1}{\|b_H\|}\Big[E^{\overline{\theta}}_H(b_H)-\sum_{t\in H}\sum_{q=1}^{b_t}\theta_{t,q}(H,E^{\overline{\theta}}_H)\Big].\end{aligned}\tag{1}$$

Based on the definitions of θ and $E^{\overline{\theta}}_H$, for every $k_p\in B^+_p$,

$$\begin{aligned}\theta_{p,k_p}(H,E^{\overline{\theta}}_H)&=E^{\overline{\theta}}_H(b_p,0)-E^{\overline{\theta}}_H(k_p-1,0)\\&=E\big(b_p,0_{A\setminus\{p\}}\big)-E\big(k_p-1,0_{A\setminus\{p\}}\big)\\&=\theta_{p,k_p}(A,E).\end{aligned}\tag{2}$$

By applying (1), (2) and definitions of $E^{\overline{\theta}}_H$ and $\overline{\theta}$,

$$\begin{aligned}&\overline{\theta}_{p,k_p}(H,E^{\overline{\theta}}_H)\\ =\;&\theta_{p,k_p}(A,E)+\tfrac{1}{\|b_H\|}\Big[E^{\overline{\theta}}_H(b_H)-\sum_{t\in H}\sum_{q=1}^{b_t}\theta_{t,q}(A,E)\Big]\\ =\;&\theta_{p,k_p}(A,E)+\tfrac{1}{\|b_H\|}\Big[E(b)-\sum_{t\in A\setminus H}\sum_{q=1}^{b_t}\overline{\theta}_{t,q}(A,E)\\&\qquad-\sum_{t\in H}\sum_{q=1}^{b_t}\theta_{t,q}(A,E)\Big]\\ =\;&\theta_{p,k_p}(A,E)+\tfrac{1}{\|b_H\|}\Big[\sum_{t\in H}\sum_{q=1}^{b_t}\overline{\theta}_{t,q}(A,E)\\&\qquad-\sum_{t\in H}\sum_{q=1}^{b_t}\theta_{t,q}(A,E)\Big]\\ =\;&\theta_{p,k_p}(A,E)+\tfrac{1}{\|b_H\|}\Big[\tfrac{\|b_H\|}{\|b\|}\big[E(b)-\sum_{t\in A}\sum_{q=1}^{b_t}\theta_{t,q}(A,E)\big]\Big]\\ =\;&\theta_{p,k_p}(A,E)+\tfrac{1}{\|b\|}\Big[E(b)-\sum_{t\in A}\sum_{q=1}^{b_t}\theta_{t,q}(A,E)\Big]\\ =\;&\overline{\theta}_{p,k_p}(A,E).\end{aligned}$$

Similarly, $\overline{\theta}_{h,k_h}(H,E^{\overline{\theta}}_H)=\overline{\theta}_{h,k_h}(A,E)$ for all $k_h\in B^+_h$. So, the MAII matches CSE. ■

The following lemma would present alternative meaning of BCOM by applying CACP and CSE.

Lemma 2: A power index ζ matches BCOM if it matches CACP and CSE.

Proof: Let ζ be a power index matching CACP and CSE, and $(A,E)\in\Delta$. It is completed for $|A|\leq 2$ by CACP. Let $|A|\geq 3$ and $h\in A$. Based on the definition of $E^{\zeta}_{\{h\}}$,

$$E^{\zeta}_{\{h\}}(b_h)=E(b)-\sum_{p\in A\setminus\{h\}}\sum_{q=1}^{b_p}\zeta_{p,q}(A,E).$$

Since ζ contents CSE, $\zeta_{h,k_h}(A,E)=\zeta_{h,k_h}\big(\{h\},E^{\zeta}_{\{h\}}\big)$ for all $k_h\in B_h$. In particular, $\zeta_{h,b_h}(A,E)=\zeta_{h,b_h}\big(\{h\},E^{\zeta}_{\{h\}}\big)$. On the other hand, by CACP of ζ, $\sum_{q=1}^{b_h}\zeta_{h,j}(A,E)=E^{\zeta}_{\{h\}}(b_h)$. Hence, $\sum_{p\in A}\sum_{q=1}^{b_p}\zeta_{p,q}(A,E)=E(b)$, i.e., ζ matches BCOM. ■

Remark 1: Based on definition of CACP and Definition 1, it is easy to see that the MAII satisfies CACP. By applying Lemmas 1 and 2, the MAII satisfies BCOM.

Inspired by Hart and Mas-Colell [3], the following theorem would be adopted to evaluate the rationality of the MAII by means of consonance and criterion for aggregate surroundings property.

Theorem 1: A power index ζ matches CSE and CACP if and only if $\zeta=\overline{\theta}$.

Proof: Clearly, $\overline{\theta}$ matches CACP. Based on Lemma 1, $\overline{\theta}$ matches CSE.

To present uniqueness, assume that ζ matches CSE and CACP on Δ. Based on Lemma 2, ζ matches BCOM. Let $(A,E)\in\Delta$. If $|A|\leq 2$, then by CACP of ζ, $\zeta(A,E)=\overline{\theta}(A,E)$. The situation $|A|>2$: Let $p\in A$ and $S=\{p,h\}$ for some $h\in A\setminus\{p\}$. For $k_p\in B^+_p,\ k_h\in B^+_h$,

$$\begin{aligned}&\zeta_{p,k_p}(A,E)-\zeta_{h,k_h}(A,E)\\ =\;&\zeta_{p,k_p}(S,E^{\zeta}_S)-\zeta_{h,k_h}(S,E^{\zeta}_S)\\ =\;&\overline{\theta}_{p,k_p}(S,E^{\zeta}_S)-\overline{\theta}_{h,k_h}(S,E^{\zeta}_S)\\ =\;&\theta_{p,k_p}(S,E^{\zeta}_S)-\theta_{h,k_h}(S,E^{\zeta}_S)\\ =\;&\Big[E^{\zeta}_S(b_p,0)-E^{\zeta}_S(k_p-1,0)-E^{\zeta}_S(0,b_h)\\&\quad+E^{\zeta}_S(0,k_h-1)\Big]\\ =\;&\Big[E(b_p,0_{A\setminus\{p\}})-E(k_p-1,0_{A\setminus\{p\}})\\&\quad-E(0_{A\setminus\{h\}},b_h)+E(0_{A\setminus\{h\}},k_h-1)\Big].\end{aligned}\tag{3}$$

$\overline{\theta}$ instead of ζ in (3), one could have that

$$\begin{aligned}&\overline{\theta}_{p,k_p}(A,E)-\overline{\theta}_{h,k_h}(A,E)\\ =\;&\Big[E(b_p,0_{A\setminus\{p\}})-E(k_p-1,0_{A\setminus\{p\}})\\&\quad-E(0_{A\setminus\{h\}},b_h)+E(0_{A\setminus\{h\}},k_h-1)\Big].\end{aligned}\tag{4}$$

Based on (3) and (4),

$$\zeta_{p,k_p}(A,E)-\zeta_{h,k_h}(A,E)=\overline{\theta}_{p,k_p}(A,E)-\overline{\theta}_{h,k_h}(A,E).$$

This implies that $\zeta_{p,k_p}(A,E)-\overline{\theta}_{p,k_p}(A,E)=a$ for all (p,k_p). It remains to demonstrate that $a=0$. By BCOM of ζ and $\overline{\theta}$,

$$\begin{aligned}0&=\sum_{p\in A}\sum_{k_p=1}^{b_p}\Big[\zeta_{p,k_p}(A,E)-\overline{\theta}_{p,k_p}(A,E)\Big]\\&=\|b\|\cdot a.\end{aligned}$$

That is, $a=0$. ■

Inspired by Maschler and Owen [11] and Moulin [13], one would like to characterize the MAII by means of consonance, behavior completeness, aggregate equal affect property and behavior synchronization.

Lemma 3: A power index ζ matches CACP if it matches BCOM, AEAP and BSYN.

Proof: Let ζ be a power index matching BCOM, AEAP and BSYN. Given $(A,E)\in\Delta$ with $A=\{p,h\}$ for some $p\neq h$. One define a surrounding (A,D) to be for every $\gamma\in B^A$,

$$D(\gamma)=E(\gamma)-\sum_{t\in N(\gamma)}\sum_{q=1}^{\gamma_t}\theta_{t,q}(A,E).$$

Thus, $D(b_p,0) - D(k_p-1,0) = 0$ for every $k_p \in B_p^+$. Similarly, $D(b_h,0) - D(k_h-1,0) = 0$ for all $k_h \in B_h^+$. Since $D(b_p,0)-D(k_p-1,0) = 0 = D(b_h,0)-D(k_h-1,0)$, $\zeta_{p,k_p}(A,D) = \zeta_{h,k_h}(A,D)$ by AEAP of ζ. By BCOM of ζ,

$$\begin{aligned} D(b) &= \sum_{q=1}^{b_p} \zeta_{p,q}(A,D) + \sum_{q=1}^{b_h} \zeta_{h,q}(A,D) \\ &= \|b\| \cdot \zeta_{p,q}(A,D) \end{aligned} \quad (5)$$

for every $q \in B_p^+$. Based on (5) and definition of D,

$$\begin{aligned} \zeta_{p,q}(A,D) &= \tfrac{D(b)}{\|b\|} \\ &= \tfrac{1}{\|b\|} \cdot \big[E(b) - \sum_{t\in A}\sum_{q=1}^{b_t} \theta_{p,q}(A,E)\big]. \end{aligned}$$

By BSYN of ζ,

$$\begin{aligned} & \zeta_{p,k_p}(A,E) \\ = \; & \zeta_{p,k_p}(A,D) + \theta_{p,k_p}(A,E) \\ = \; & \tfrac{1}{\|b\|} \cdot \big[E(b) - \sum_{t\in A}\sum_{q=1}^{b_t} \theta_{t,q}(A,E)\big] + \theta_{p,k_p}(A,E) \\ = \; & \overline{\theta}_{p,k_p}(A,E). \end{aligned}$$

Similarly, $\zeta_{h,k_h}(A,E) = \overline{\theta}_{h,k_h}(A,E)$ for every $k_h \in B_h^+$. That is, ζ matches CACP. ■

Lemma 4: The MAII satisfies AEAP.

Proof: Let $(A,E) \in \Delta$. Assume that $E(\gamma,k_p,0) - E(\gamma,k_p-1,0) = E(\gamma,0,k_q) - E(\gamma,0,k_q-1)$ for some $(p,k_p),(q,k_q) \in L^A$ and for every $\gamma \in B^{A\setminus\{p,q\}}$. By taking $\gamma = 0_{A\setminus\{p,q\}}$,

$$\begin{aligned} & E(k_p,0_{A\setminus\{p\}}) - E(k_p-1,0_{A\setminus\{p\}}) \\ = \; & E(\gamma,k_p,0) - E(\gamma,k_p-1,0) \\ = \; & E(\gamma,0,k_q) - E(\gamma,0,k_q-1) \\ = \; & E(k_q,0_{A\setminus\{q\}}) - E(k_q-1,0_{A\setminus\{q\}}), \end{aligned}$$

i,e,.

$$\begin{aligned} \theta_{p,k_p}(A,E) &= E(k_p,0_{A\setminus\{p\}}) - E(k_p-1,0_{A\setminus\{p\}}) \\ &= E(k_q,0_{A\setminus\{q\}}) - E(k_q-1,0_{A\setminus\{q\}}) \\ &= \theta_{q,k_q}(A,E). \end{aligned}$$

So,

$$\begin{aligned} & \overline{\theta}_{p,k_p}(A,E) \\ = \; & \theta_{p,k_p}(A,E) + \tfrac{1}{\|b\|} \cdot \Big[E(d) - \sum_{t\in A}\sum_{k_t=1}^{b_t} \theta_{t,k_t}(A,E)\Big] \\ = \; & \theta_{q,k_q}(A,E) + \tfrac{1}{\|b\|} \cdot \Big[E(d) - \sum_{t\in A}\sum_{k_t=1}^{b_t} \theta_{t,k_t}(A,E)\Big] \\ = \; & \overline{\theta}_{q,k_q}(A,E). \end{aligned}$$

Thus, the MAII $\overline{\theta}$ matches AEAP. ■

Lemma 5: The MAII satisfies BSYN.

Proof: Let $(A,E),(A,D) \in \Delta$ with

$$E(\gamma) = D(\gamma) + \sum_{t\in N(\gamma)} \sum_{k_t=1}^{\gamma_t} \mu t, k_t$$

for some $\mu \in \mathbb{R}^{L^A}$ and for every $\gamma \in B^A$. For every $(p,k_p) \in L^A$,

$$\begin{aligned} & \theta_{p,k_p}(A,E) \\ = \; & E(k_p,0_{A\setminus\{p\}}) - E(k_p-1,0_{A\setminus\{p\}}) \\ = \; & D(k_p,0_{A\setminus\{p\}}) - D(k_p-1,0_{A\setminus\{p\}}) + \mu_{p,k_p} \\ = \; & \theta_{p,k_p}(A,D) + \mu_{p,k_p}. \end{aligned}$$

So,

$$\begin{aligned} & \overline{\theta}_{p,k_p}(A,E) \\ = \; & \theta_{p,k_p}(A,E) + \tfrac{1}{\|b\|}\Big[E(b) - \sum_{q\in A}\sum_{k_q=1}^{b_q} \theta_{q,k_q}(A,E)\Big] \\ = \; & \theta_{p,k_p}(A,D) + \mu_{p,k_p} + \tfrac{1}{\|b\|}\Big[D(b) + \sum_{t\in A}\sum_{k_t=1}^{b_t} \mu_{t,k_t} \\ & - \sum_{q\in A}\sum_{k_q=1}^{b_q} \theta_{q,k_q}(A,D) - \sum_{q\in A}\sum_{k_q=1}^{b_q} \mu_{q,k_q}\Big] \\ = \; & \theta_{p,k_p}(A,D) + \mu_{p,k_p} + \tfrac{1}{\|b\|}\Big[D(b) \\ & - \sum_{q\in A}\sum_{k_q=1}^{b_q} \theta_{q,k_q}(A,D)\Big] \\ = \; & \overline{\theta}_{p,k_p}(A,D) + \mu_{p,k_p}. \end{aligned}$$

Thus, the MAII $\overline{\theta}$ matches BSYN. ■

The following theorem would be adopted to evaluate the rationality of the MAII by means of consonance, behavior completeness, aggregate equal affect property and behavior synchronization.

Theorem 2: A power index ζ on Δ matches BCOM, AEAP, BSYN and CSE if and only if $\zeta = \overline{\theta}$.

Proof: Based on definition of $\overline{\theta}$, Remark 1 and Lemmas 1, 4, 5, it matches BCOM, AEAP, BSYN and CSE. The rest of proof could be completed by Lemma 3 and Theorem 1. ■

IV. Application on Combinatorial Procedures for Antidotes and Catalysts

Due to the rapidly changing interactions, this study adopts game-theoretical outcomes to analyze numerous combinatorial procedures for antidotes and catalysts by inquiring ”how is the combination formed”, ”is such combination precise”, ”why do one adopt this combination”, and ”how effectual is such combination”. Throughout the proposed outcomes, it is illustrated that the main merit of the MAII and related axiomatization is that the MAII of a multi-choice surrounding exists absolutely and to generate an accurate affect for a specific ingredient reacting with a specific reactive behavior that distinct from the general claim with multi-choice surroundings, which determining a type of entire affect for a specific ingredient by gathering the distinctions of this ingredient among its total reactive behavior. It is expected that the MAII could exactly generate “harmonious outcome” of combinatorial procedures. In order to clarify how the MAII could be utilized and to rise its implication more transparent, one should further quest the interaction between game-theoretical outcomes and combinatorial procedures among antidotes and catalysts for toxic mitigation measures under aquatic environments.

1) **Behavior completeness**: Harmonious toxic mitigation measures should make full use of aquatic environmental resources. That is, a harmonious toxic mitigation measures should match the property of behavior completeness.
2) **Criterion for aggregate surroundings property**: Antidotes and catalysts possess its distinctive characteristics of reactions. Interactions among antidotes and catalysts are familiarly generated from two-ingredient

(antidote or catalyst) interactions followed by coalition interactions. Thereupon, a harmonious antidote-relating procedure should match the property of criterion for aggregate surroundings.

3) **Aggregate equal affect property**: If arbitrary two ingredients (antidotes or catalysts) are equally influential to the entire aquatic environment after the reaction of ingredient grouping, the efficacy of these two ingredients (antidotes or catalysts) on the overall aquatic environment should be coincident also. Hence, a harmonious toxic mitigation measures should match the aggregate equal affect property.
4) **Behavior synchronization**: Harmonious toxic mitigation measures, in which each ingredient (antidote or catalyst) is utilized with specific density to attain the objective reaction, rather than the quantity (small or large) basing on the objective reaction, should attain the most harmonious affect in accordance with the proportionality principle. Therefore, a harmonious toxic mitigation measures should match the property of behavior synchronization.
5) **Consonance**: Harmonious toxic mitigation measures are examined via a continuous iterative synergistic process, and should yield consistent efficacy. A harmonious toxic mitigation measures should therefore match the property of consonance.

By employing related definitions, examples and interpretations of Section 2, one could conclude that the notion of combinatorial procedures among antidotes and catalysts for toxic mitigation measures under aquatic environments could be formulated as a multi-choice surrounding. Depended on Theorems 1 and 2, it is shown that the MAII is the unique mechanism matching simultaneously behavior completeness, criterion for aggregate surroundings property, aggregate equal affect property, behavior synchronization and consonance. As mentioned in interpretations 1–5, it is clear to have that these axiomatic properties should be indispensable requirements in the framework of combinatorial procedures among antidotes and catalysts for toxic mitigation measures under aquatic environments. Thus, the MAII could be utilized to be an useful assignation concept for combinatorial procedures among antidotes and catalysts.

Next, an applied process would be provided to demonstrate how the results of multi-choice surrounding and the MAII could be used to produce combinations among antidotes and catalysts for toxic mitigation measures under aquatic environments. As mentioned in previous sections, the types of catalysts for toxic mitigation measures under aquatic environments are diverse, and they take different forms, such as, enzymes, acid-base catalysts, and heterogeneous (or surface) catalysts, etc. On the other hand, there are many catalytic reactions for toxic mitigation measures under aquatic environments, for example, Tetra-Amido Macrocyclic Ligands (TAMLs) are useful catalysts with a host of applications for reducing and cleaning up pollutants; The oxidation catalysts are the first highly effective mimics of peroxidase enzymes. When partnered with hydrogen peroxide, they are able to convert harmful pollutants into less toxic substances. Thus, one could assume that A is the collection of all operational ingredients. The reactive behavior of each ingredient is not set in stone, and there are different reactive behavior in response to different situations. That is, each ingredient $p \in A$ will have different reactive behavior k_p. Moreover, the reactive behavior between all operational ingredients also affect one another as a result of different situations. For example, generated from the common units of biochemistry, carbon, hydrogen, nitrogen and oxygen round a reactive iron core, Fe-TAMLs are less toxic and useful at mighty low concentrations. However, their composition also causes in very special chemical bonds that might be affected down by the highly reactive oxygen intermediaries formed during the reaction with hydrogen peroxide. In other words, each ingredient will interact within the context of the situation, inspire and react different implementation behavior; as a result, there will be different association of behavior and corresponding advantages. That is, each ingredient will intersect with other ingredients for different situations, and adopt different reactive behavior $\gamma_p \in B_p$ for different situation nature and other different ingredients. Thus, a map E can be used to evaluate the efficacy of reactive behavior $\gamma \in B^A$ taken by all operational ingredients (i.e. $E(\gamma)$). Therefore, the toxic mitigation measures under aquatic environments can be regarded as a multi-choice surrounding (A, E). To evaluate the affect of each ingredient for toxic mitigation measures under aquatic environments, using the power index this article proposed, one could first assess the individual-level affect each operational ingredient has accumulated over reacting processes based on various and distinct behavior, which is the the individual-level distinction θ mentioned in Definition 1. The remaining generated efficacy distribution should also be allocated entirely and equally derived for each ingredient and its reactive behavior, which is the MAII $\overline{\theta}$ mentioned in Definition 1.

Finally, one would introduce a numerical application. Let $(A, E) \in \Delta$ be a aquatic environment with antidotes and catalysts collection $A = \{p, q, k\}$ and reactive behavior vector $b = (2, 1, 1)$. Define $E(2, 1, 1) = 10$, $E(2, 1, 0) = -5$, $E(2, 0, 1) = 2$, $E(2, 0, 0) = 8$, $E(1, 1, 1) = 3$, $E(1, 1, 0) = -6$, $E(1, 0, 1) = 11$, $E(1, 0, 0) = -2$, $E(0, 1, 1) = -2$, $E(0, 1, 0) = 6$, $E(0, 0, 1) = -2$ and $E(0, 0, 0) = 0$ to be the efficacy that the ingredients can produce under entire reaction environment. Depended on Definition 1,

$$
\begin{aligned}
\theta_{p,2}(A,E) &= 10, & \theta_{p,1}(A,E) &= 8,\\
\theta_{q,1}(A,E) &= 6, & \theta_{k,1}(A,E) &= -2,\\
\overline{\theta_{p,2}}(A,E) &= 7, & \overline{\theta_{p,1}}(A,E) &= 5,\\
\overline{\theta_{q,1}}(A,E) &= 3, & \overline{\theta_{k,1}}(A,E) &= -5.
\end{aligned}
$$

Clearly, the affect or the efficacy of each ingredient when it reacts at a specific behavior in (A, E). For instance, the affect of ingredient p is $\overline{\theta_{p,2}}(A, E) = 6.75$ if p reacts at the behavior 2 in (A, E).

V. Concluding Remarks

1) The aim of this study is to introduce a mechanism and related applications to the existing combinatorial procedures among antidotes and catalysts for toxic mitigation measures under aquatic environments.
   - A generalized analogue of the PEANSC, the multi-choice aggregate-individual index, is defined by

means of the ingredients and its reactive behavior simultaneously.
- In order to evaluate the validity and the justifiability of the multi-choice aggregate-individual index, two characterizations are proposed.
- By extending the proposed game-theoretical outcomes to the combinatorial processes among antidotes and catalysts for toxic mitigation measures under aquatic environments, this study farther analyzes and examines the rationality of the multi-choice aggregate-individual index by using actual examples and related interpretations.

2) By simultaneously evaluating the ingredients and its reactive behavior (operational levels), Hwang and Liao [5] considered an extended Shapley value [17] and related axiomatizations on fuzzy situations. One might compare our outcomes with the outcomes of Hwang and Liao [5]. There are some major differentiations:
   - The multi-choice aggregate-individual index and related outcomes are presented initially.
   - Power indexes on standard surroundings have only investigated participation or non-participation of all ingredients. This study proposes the power index to resolve assignation mechanism under multi-choice behavior.
   - Under multi-choice surroundings, most of existing power indexes have been defined to compute a type of entire affect for a specific ingredient by collecting the distinctions of this ingredient among its total reactive behavior. By premeditating real-world conditions, this study proposes the MAII to determine related affects by simultaneously considering a specific ingredient reacting with a specific reactive behavior.
   - The game-theoretical outcomes of this study are used to evaluate combinatorial processes among antidotes and catalysts for toxic mitigation measures. This application does not appear among related existing outcomes.

3) The outcomes introduced in this study arouse one motivation.
   - whether more game-theoretical outcomes could be expanded to analyze and generate the most effective harmonious combinations among antidotes and catalysts for toxic mitigation measures under aquatic environments.

   This is left to the readers.

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