A SIAR Model of COVID-19 with Control

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***Abstract*—During 2020, the new coronavirus disease (COVID-19) was spreading rapidly, having devastating consequences and causing serious problems in 213 countries and territories. Until July 15, 2020, the total recorded cases was 12 415 672 and total death around 557 925 in the World, with an increasing trend of global new cases making COVID-19 one of the most disastrous outbreak since the Spanish flu. Prevention, testing and control are the main methods to reduce the propagation of the virus. Mathematical modeling remains one of the best ways to analyse the spread of this virus and control its prevalence. In this sense, we propose, in the present paper, a SIAR compartment model with control in order to reduce the reproduction number R_0 and slow down the epidemic outbreak.**

A stability analysis of equilibrium points is carried out and numerical simulations are performed to stress on the impact of different prevention levels and control the spread of the pandemic.

***Index Terms*—COVID-19, Mathematical model, Basic reproduction number, Stability analysis, Simulation**

I. INTRODUCTION

The year 2020 was marked by a new coronavirus which emerged first in the city of Wuhan, Hubei, China, in early December, 2019 and has spread all around the world. Until July 15, 2020, the virus has affected 213 countries and territories. The virus caused serious problems in terms of morbidity, mortality (12 415 672 total cases and total deaths 557 925 [1]), economic loss (the average index loss of industrial production across countries is 18% for high income countries, 24% for upper-middle income countries and 22% for lower-middle income countries [2]) and general social disturbances.

According to the World Health Organization (WHO)[3], coronaviruses are a large family of viruses that can be pathogenic in animals or humans. It is known that in humans, several coronaviruses can cause respiratory infections in different levels from the common cold to more serious illnesses such as Middle East Respiratory Syndrome (MERS) and Severe Acute Respiratory Syndrome (SARS). The last coronavirus is responsible for coronavirus disease 2019 (COVID-19). This new virus and disease has been labeled a "pandemic" after its outbreak and has, consequently, affected many countries around the World.

The most common symptoms of COVID-19 are fever, dry cough, and fatigue. However, some people can present other less common symptoms such as body aches and pains, nasal congestion, headache, conjunctivitis, sore throat, diarrhea, loss of taste or smell, rash, or discoloration of the fingers on the hand or foot. Although showing very mild signs, some infected people can transmit the virus (silently) to others. These individuals are called asymptomatic carriers and represent 40% to 45% of COVID-19 infections[4].

In order to help in the management and the reduction of this pandemic's risky effects, mathematical models can be a source of information helping decision makers to develop and implement efficient strategies ([5], [6]). The best example is the SIR model developed in 1924 by Kermack and McKendrick [7] and based on a system of differential equations which revolve around the reproduction number of the virus R_0 and its evolution [8].

Several mathematical models have been proposed to investigate the pandemic of COVID-19. The epidemic spreading in China has been analysed in [9], [10], [11], [12], [13], [14], [15], [16]. Furthermore, the impact of simulation modelling to reduce the spread of COVID-19 has been studied in [17], [18]. *Prem et al.* [19] examined how changes in population mixing have affected outbreak progression in Wuhan.

In this study, we use a modified SIR model that includes the compartment of asymptomatic persons a SIAR model [20] with control. The present work aims to propose different strategies to reduce the infected population and slow down the epidemic outbreak.

II. FORMULATION OF THE MODEL AND STABILITY ANALYSIS

A. Parameters of the Model

Let N denotes the population size. Death is proportional to the population size with death rate constant μ and we assume a constant Λ due to births and immigration. consequently, we write:

$$\frac{dN}{dt} = \Lambda - \mu N$$

This population of size N is formed by Susceptible S, symptomatic Infective I, Asymptomatic Infective A and Removed R.

Otherwise, the formula:

$$\frac{(\beta_1 I + \beta_2 A)S}{N}$$

gives the incidence i.e. the rate at which susceptible individuals become infectious. If the time unit is days, then the incidence is the number of new infection per day.

The contact rate β_1 is the average number of adequate contacts of susceptible with symptomatic infected person per day while the daily contact rate β_2 is the average number of adequate contacts of susceptible with asymptomatic infected person.

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$\frac{I}{N}$ is the population fraction of symptomatic infected people while $\frac{A}{N}$ represents the population fraction of asymptomatic infected people.
The parameters σ_1 and σ_2 represent respectively infection duration among symptomatic and asymptomatic infected people while α is the death rate from COVID-19.
It should be noted that the man life span $(1/\mu)$ is taken equal to 76 years and time units of weeks, months or years could also be used.

B. Equations of the Model

A schematic representation of the model is shown in figure Fig. 1.

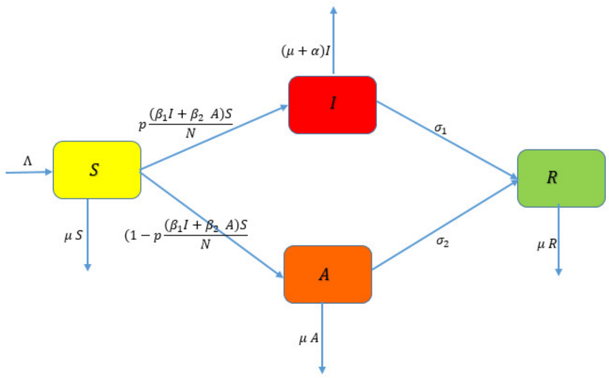

Fig. 1: Compartments population

We consider SIAR compartmental model that is to say that susceptible individuals become symptomatic infected with a $p\dfrac{(\beta_1 I+\beta_2 A)S}{N}$ rate, or asymptomatic infected with a $(1-p)\dfrac{(\beta_1 I+\beta_2 A)S}{N}$ rate, then removed with immunity after recovery from infection.

The dynamics of this disease is described by the following differential equations:

$$\begin{cases} \dfrac{dS(t)}{dt} &= \Lambda - \dfrac{S(t)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - \mu S(t) \\ \dfrac{dI(t)}{dt} &= \dfrac{pS(t)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - (\sigma_1+\mu+\alpha) I(t) \\ \dfrac{dA(t)}{dt} &= \dfrac{(1-p)S(t)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - (\sigma_2+\mu) A(t) \\ \dfrac{dR(t)}{dt} &= \sigma_1 I(t) + \sigma_2 A - \mu R(t) \end{cases}$$

and with the condition $S(t)+I(t)+A(t)+R(t)=N(t)$.
So:

$$R(t) = N(t) - S(t) - I(t) - A(t)\ .$$

The previous system becomes:

$$\begin{cases} \dfrac{dS(t)}{dt} &= \Lambda - \dfrac{S(t)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - \mu S(t) \\ \dfrac{dI(t)}{dt} &= \dfrac{pS(t)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - (\sigma_1+\mu+\alpha) I(t) \\ \dfrac{dA(t)}{dt} &= \dfrac{(1-p)S(t)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - (\sigma_2+\mu) A(t) \\ \dfrac{dN(t)}{dt} &= \Lambda - \mu N(t) - \alpha I(t) \end{cases}$$

Up to now, there is no vaccine against COVID-19 but prevention and control can avoid the catastrophic effect of this disease. Mathematically, we introduce controls u_1 and u_2 which represent:

- Prevention denoted u_1: it combines the different measures such as confining healthy individuals, isolating infected individuals and tracing every contact. Moreover, it also takes into consideration physical distancing, self-protection and testing.
- Patient care u_2: hospitalization, treatment and screening.

Introducing these two controls to the model above leads to the following controlled model:

$$\begin{cases} \dfrac{dS(t)}{dt} &= \Lambda - \dfrac{S(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - \mu S(t) \\ \dfrac{dI(t)}{dt} &= \dfrac{pS(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} \\ & \quad -(\sigma_1+\mu+\alpha+u_2) I(t) \\ \dfrac{dA(t)}{dt} &= \dfrac{(1-p)S(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} \\ & \quad -(\sigma_2+\mu) A(t) \\ \dfrac{dN(t)}{dt} &= \Lambda - \mu N(t) - \alpha I(t) \end{cases} \tag{1}$$

C. Positivity of Solutions

Theorem 1:
The set $\Omega = \{(S,I,A,N) \in \mathbb{R}^4 / 0 \le S,I,A,N \le \dfrac{\Lambda}{\mu}\}$ is positively invariant under system (1). Thus the model is epidemiologically and mathematically well posed.
Proof:
See Appendix B.1

D. Stability analysis

1) ***Equilibrium points and*** *R_0:*

For the model above, equilibrium points are defined such that there is no variations in S, I, A, N with respect to t:

$$\Lambda - \frac{S(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - \mu S(t) = 0 \tag{2}$$

$$\frac{pS(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - (\sigma_1+\mu+\alpha+u_2) I(t) = 0 \tag{3}$$

$$\frac{(1-p)S(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - (\sigma_2+\mu) A(t) = 0 \tag{4}$$

$$\Lambda - \mu N(t) - \alpha I(t) = 0 \tag{5}$$

From the equation (5), we have:

$$N^* = \frac{\Lambda}{\mu} - \frac{\alpha}{\mu} I^*$$

From the equation (2), we have:

$$\frac{S^*\left(\beta_1' I^* + \beta_2' A^*\right)}{N^*} = \Lambda - \mu S^* .$$

Where $\beta_1' = (1-u_1)\beta_1$ and $\beta_2' = (1-u_1)\beta_2$.
So, from the equation (3), we have:

$$p(\Lambda - \mu S^*) - \delta_1 I^* = 0 ,$$

where $\delta_1 = (\sigma_1 + \mu + \alpha + u_2)$

$$\Longrightarrow S^* = \frac{\Lambda}{\mu} - \frac{\delta_1}{p\mu} I^* .$$

So, from the equation (4) we have:

$$(1-p)\frac{\delta_1}{p} I^* - \delta_2 A^* = 0 ,$$

where $\delta_2 = (\sigma_2 + \mu)$

$$\Longrightarrow A^* = \frac{(1-p)\delta_1}{p\delta_2} I^* .$$

In the equation (3), we substitute N^* , S^* and A^* . Therefore:

$$p\left(\frac{\Lambda}{\mu} - \frac{\delta_1}{p\mu} I^*\right)\left(\frac{1}{\frac{\Lambda}{\mu} - \frac{\alpha}{\mu} I^*}\right)\left(\beta_1' I^* + \beta_2' \frac{(1-p)\delta_1}{p\delta_2} I^*\right) - \delta_1 I^* = 0$$

$$p\left(\frac{\Lambda}{\mu} - \frac{\delta_1}{p\mu} I^*\right)\left(\beta_1' I^* + \beta_2' \frac{(1-p)\delta_1}{p\delta_2} I^*\right) - \delta_1 \left(\frac{\Lambda}{\mu} - \frac{\alpha}{\mu} I^*\right) I^* = 0$$

$$I^*\left[p\left(\frac{\Lambda}{\mu} - \frac{\delta_1}{p\mu} I^*\right)\left(\beta_1' + \beta_2' \frac{(1-p)\delta_1}{p\delta_2}\right) - \delta_1 \left(\frac{\Lambda}{\mu} - \frac{\alpha}{\mu} I^*\right)\right] = 0$$

$$I^*\left[p\left(\frac{\Lambda}{\mu} - \frac{\delta_1}{p\mu} I^*\right) \times \frac{p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1}{p\delta_2} - \delta_1 \left(\frac{\Lambda}{\mu} - \frac{\alpha}{\mu} I^*\right)\right] = 0$$

$$I^*\left[\left(p\Lambda - p\delta_1 I^*\right) \times \frac{p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1}{p\delta_2} - \delta_1\left(\Lambda - \alpha I^*\right)\right] = 0$$

So,

$$I^* = 0$$

or

$$\left(p\Lambda - p\delta_1 I^*\right) \times \frac{p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1}{p\delta_2} - \delta_1\left(\Lambda - \alpha I^*\right) = 0$$

$$\Longrightarrow I^* = 0$$

or

$$\delta_1 \alpha I^* - \delta_1 I^* \times \frac{p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1}{p\delta_2} = \delta_1 \Lambda - p\Lambda \times \frac{p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1}{p\delta_2}$$

$$\Longrightarrow I^* = 0$$

or

$$\left(p\alpha - \frac{p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1}{\delta_2}\right) I^* = p\Lambda\left(1 - \frac{p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1}{\delta_1\delta_2}\right)$$

$$\Longrightarrow I^* = 0$$

or

$$\left(p\alpha - \frac{p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1}{\delta_2}\right) I^* = p\Lambda\left(1 - \frac{p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1}{\delta_1\delta_2}\right)$$

$$\Longrightarrow I^* = 0$$

or

$$I^* = \frac{p\Lambda\left(R_0 - 1\right)}{\left(\delta_1 R_0 - p\alpha\right)} ,$$

where,

$$\begin{aligned} R_0 &= \frac{p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1}{\delta_1\delta_2} \\ &= \frac{p\beta_1(1-u_1)}{\delta_1} + \frac{(1-p)\beta_2(1-u_1)}{\delta_2} \\ &= \frac{p\beta_1(1-u_1)}{\sigma_1 + u_2 + \mu + \alpha} + \frac{(1-p)\beta_2(1-u_1)}{\sigma_2 + \mu} \end{aligned}$$

is the **basic reproduction number** for the model (see Appendix A.1):

- If $I^* = 0$, then $A^* = 0$ and $S^* = N^* = \dfrac{\Lambda}{\mu}$.

Consequently the first equilibrium point is $E_0 = \left(\dfrac{\Lambda}{\mu}, 0, 0, \dfrac{\Lambda}{\mu}\right)$.

- If $I^* = \dfrac{p\Lambda\left(R_0 - 1\right)}{\left(\delta_1 R_0 - p\alpha\right)}$, then

$$A^* = \frac{(1-p)\delta_1\Lambda\left(R_0 - 1\right)}{\delta_2\left(\delta_1 R_0 - p\alpha\right)}$$

$$\begin{aligned} S^* &= \frac{\Lambda}{\mu} - \frac{\delta_1}{p\mu}\frac{p\Lambda\left(R_0 - 1\right)}{\left(\delta_1 R_0 - p\alpha\right)} \\ &= \frac{\Lambda}{\mu}\left(1 - \frac{\delta_1\left(R_0 - 1\right)}{\left(\delta_1 R_0 - p\alpha\right)}\right) \\ &= \frac{\Lambda}{\mu}\left(\frac{\delta_1 - p\alpha}{\delta_1 R_0 - p\alpha}\right) \end{aligned}$$

and

$$\begin{aligned} N^* &= \frac{\Lambda}{\mu} - \frac{\alpha}{\mu}\frac{p\Lambda\left(R_0 - 1\right)}{\left(\delta_1 R_0 - p\alpha\right)} \\ &= \frac{\Lambda}{\mu}\left(1 - \frac{p\alpha\left(R_0 - 1\right)}{\left(\delta_1 R_0 - p\alpha\right)}\right) \\ &= \frac{\Lambda}{\mu}\left(\frac{R_0\left(\delta_1 - p\alpha\right)}{\delta_1 R_0 - p\alpha}\right) \end{aligned}$$

Then if $R_0 > 1$, the second equilibrium point is:

$$E_1 = \left(S^*, I^*, A^*, N^*\right) ,$$

where

$$S^* = \frac{\Lambda}{\mu}\left(\frac{\delta_1 - p\alpha}{\delta_1 R_0 - p\alpha}\right),$$
$$I^* = \frac{p\Lambda\left(R_0 - 1\right)}{\left(\delta_1 R_0 - p\alpha\right)},$$
$$A^* = \frac{(1-p)\delta_1\Lambda\left(R_0 - 1\right)}{\delta_2\left(\delta_1 R_0 - p\alpha\right)}$$

and $$N^* = \tfrac{\Lambda}{\mu}\left(\frac{R_0\left(\delta_1 - p\alpha\right)}{\delta_1 R_0 - p\alpha}\right).$$

Theorem 2: **:**
The previous system admits two equilibrium points:

1) If $R_0 < 1$, the system admits a trivial equilibrium $E_0 = (\dfrac{\Lambda}{\mu}, 0, 0, \dfrac{\Lambda}{\mu})$.
2) If $R_0 > 1$, then there exists an endemic equilibrium $E_1 = (S^*, I^*, A^*, N^*)$.

Where,

$$S^* = \frac{\Lambda}{\mu}\left(\frac{\delta_1 - p\alpha}{\delta_1 R_0 - p\alpha}\right), I^* = \frac{p\Lambda\left(R_0 - 1\right)}{\left(\delta_1 R_0 - p\alpha\right)},$$

$$A^* = \frac{(1-p)\delta_1 \Lambda\left(R_0 - 1\right)}{\delta_2\left(\delta_1 R_0 - p\alpha\right)} \text{ and } N^* = \frac{\Lambda}{\mu}\left(\frac{R_0\left(\delta_1 - p\alpha\right)}{\delta_1 R_0 - p\alpha}\right)$$

Theorem 3:
The basic reproduction number of the system (2) is given by

$$\begin{aligned} R_0 &= \frac{p\beta_1(1-u_1)}{\sigma_1 + u_2 + \mu + \alpha} + \frac{(1-p)\beta_2(1-u_1)}{\sigma_2 + \mu} \\ &= R_I + R_A \end{aligned}$$

R_I : the contribution of symptomatic infectious individuals.
R_A : the contribution of Asymptomatic infectious individuals.

*2) **Impact of control**:*
The effect of prevention and patient care is measured by considering their impact on the reproduction number R_0. Thus, this effect is determined by differentiating R_0 with respect to u_1 (respectively u_2)

$$\frac{\partial R_0}{\partial u_1} = -\left(\frac{p\beta_1}{\sigma_1 + u_2 + \mu + \alpha} + \frac{(1-p)\beta_2}{\sigma_2 + \mu}\right) \leq 0$$
$$\frac{\partial R_0}{\partial u_2} = -\frac{1}{2}\left(\frac{p\beta_1(1-u_1)}{(\sigma_1 + u_2 + \mu + \alpha)^2} + \frac{(1-p)\beta_2(1-u_1)}{(\sigma_2 + \mu)^2}\right) \leq 0$$

Since $\frac{\partial R_0}{\partial u_1}$ (resp. $\frac{\partial R_0}{\partial u_2}$) is negative, it implies that R_0 is a decreasing function of u_1 (resp. u_2). This indicates that the control u_1 (resp. u_2) has a negative impact on R_0 and will lead to reduction in this disease burden.

*3) **Local stability of** E_0:*
The local stability of the equilibrium points is based on the matrix of linearization (Jacobian matrix) given by:

$$\mathcal{J} =$$
$$\begin{pmatrix} -\frac{\beta_1' I^* + \beta_2' A^*}{N^*} - \mu & -\frac{S^*\beta_1'}{N^*} & -\frac{S^*\beta_2'}{N^*} & \tau \\ \frac{p\left(\beta_1' I^* + \beta_2' A^*\right)}{N^*} & \frac{pS^*\beta_1'}{N^*} - \delta_1 & \frac{pS^*\beta_2'}{N^*} & -p\tau \\ \frac{(1-p)\left(\beta_1' I^* + \beta_2' A^*\right)}{N^*} & \frac{(1-p)S^*\beta_1'}{N^*} & \frac{(1-p)S^*\beta_2'}{N^*} - \delta_2 & -(1-p)\tau \\ 0 & -\alpha & 0 & -\mu \end{pmatrix}$$

with $\tau = \frac{S^*\left(\beta_1' I^* + \beta_2' A^*\right)}{N^{*2}}$

For E_0, the matrix of linearization (Jacobian matrix) becomes:

$$\mathcal{J}(E_0) = \begin{pmatrix} -\mu & -\beta_1' & -\beta_2' & 0 \\ 0 & p\beta_1' - \delta_1 & p\beta_2' & 0 \\ 0 & (1-p)\,\beta_1' & (1-p)\,\beta_2' - \delta_2 & 0 \\ 0 & -\alpha & 0 & -\mu \end{pmatrix}$$

Then the characteristic polynomial of $\mathcal{J}(E_0)$ is giving by:

$$\begin{aligned} P_0(\lambda) = (\lambda + \mu)^2\,(\lambda^2 + (\delta_1 + \delta_2 - p\beta_1' - (1-p)\beta_2')\,\lambda \\ + \delta_1\delta_2(1 - R_0)) \end{aligned}$$

Thus the eigenvalues of the matrix $\mathcal{J}(E_0)$ are $-\mu$ and the roots of the polynomial:

$$Q_0(\lambda) = \lambda^2 + (\delta_1 + \delta_2 - p\beta_1' - (1-p)\beta_2')\,\lambda + \delta_1\delta_2(1 - R_0).$$

For $R_0 < 1$, we have:

$$\delta_1\delta_2(1 - R_0) > 1$$

and

$$R_0 = \frac{p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1}{\delta_1\delta_2} < 1$$

$$\Longrightarrow \delta_1\delta_2(1 - R_0) > 1 \text{ and } R_0 = \frac{p\beta_1'}{\delta_1} + \frac{(1-p)\beta_2'}{\delta_2} < 1$$
$$\Longrightarrow \delta_1\delta_2(1 - R_0) > 1,\ \frac{p\beta_1'}{\delta_1} < 1 \text{ and } \frac{(1-p)\beta_2'}{\delta_2} < 1$$
$$\Longrightarrow \delta_1\delta_2(1 - R_0) > 1,\ \delta_1 - p\beta_1' > 0 \text{ and } \delta_2 - (1-p)\beta_2' > 0$$
$$\Longrightarrow \delta_1\delta_2(1 - R_0) > 1 \text{ and } \delta_1 + \delta_2 - p\beta_1' - (1-p)\beta_2' > 0\ .$$

So for $R_0 < 1$ the coefficients of polynomial Q_0 are positive.
Then, following Routh-Hurwitz conditions for the polynomial Q_0, the state E_0 is locally asymptotically stable for $R_0 < 1$.

*4) **Global stability of** E_0:*
Theorem 4:
E_0 is globally asymptotically stable.

Proof:
Since we have an asymptotic study, we replace N by:

$$N^* = \frac{\Lambda}{\mu} \text{ (because } \lim_{t\to+\infty} N(t) = N^* = \frac{\Lambda}{\mu}).$$

So we consider the following Liapunov function:

$$V = \beta_1'\delta_2 I + \beta_2'\delta_1 A$$

So,

$$\begin{aligned} \dot{V} &= \beta_1'\delta_2\frac{dI}{dt} + \beta_2'\delta_1\frac{dA}{dt} \\ &= \beta_1'\delta_2\left(\frac{pS\,(t)\,(1-u_1)\,(\beta_1' I(t) + \beta_2' A\,(t))}{N^*} - \delta_1 I\,(t)\right) \\ &+ \beta_2'\delta_1\left(\frac{(1-p)\,S\,(t)\,(1-u_1)\,(\beta_1' I\,(t) + \beta_2' A\,(t))}{N^*}\right. \\ &\left. - \delta_2 A\,(t)\right) \\ &= (p\beta_1'\delta_2 + (1-p)\beta_2'\delta_1)\left(\frac{S\,(t)}{N^*}\,(\beta_1' I(t) + \beta_2' A\,(t))\right) \\ &- \delta_1\delta_2\,(\beta_1' I(t) + \beta_2' A\,(t)) \\ &= \left(R_0\delta_1\delta_2)\frac{S\,(t)}{N^*} - \delta_1\delta_2\right)(\beta_1' I(t) + \beta_2' A\,(t)) \\ &= \delta_1\delta_2\left(R_0\frac{S\,(t)}{N^*} - 1\right)(\beta_1' I(t) + \beta_2' A\,(t)) \\ &\leq \delta_1\delta_2\,(R_0 - 1)\,(\beta_1' I(t) + \beta_2' A\,(t)) \end{aligned}$$

So, for $R_0 \leq 1$, we obtain $\dot{V} \leq 0$.
On the other hand,

$$\dot{V} = 0 \Rightarrow \beta_1' I(t) + \beta_2' A\,(t)$$

$$\Longrightarrow I = A = 0 \text{ (because } I \geq 0 \text{ and } A \geq 0)$$

Thus the set $\{E_0\}$ is the largest invariant set within the set

$$\{(x_1, x_2, x_3, x_4)/\dot{V}(x_1, x_2, x_3, x_4) = 0\}$$

So according to the invariant set theorem, every trajectory in Ω tends to E_0 as time t increases and as E_0 is locally stable then it is globally asymptotically stable. Therefore E_0 is globally asymptotically stable.

*5) **Local stability of** E_1:*

Theorem 5:
If $R_0 > 1$ and $\sigma_2 \geq \sigma_1 + u_2 + \alpha$, then E_1 is locally asymptotically stable.

Remark 1:
The condition $\sigma_2 \geq \sigma_1 + u_2 + \alpha$ is equivalent to the condition:

$$\frac{1}{\sigma_2 + \mu} \leq \frac{1}{\sigma_1 + u_2 + \mu + \alpha} \, ,$$

where :

- $\dfrac{1}{\sigma_1 + u_2 + \mu + \alpha}$ is the average duration of the infectious period of symptomatic infectious persons.
- $\dfrac{1}{\sigma_2 + \mu}$ is the average duration of the infectious period of asymptomatic infectious persons.

(i.e we suppose that the average duration of the infectious period of symptomatic infectious persons is greater than or equal to the average duration of the infectious period of asymptomatic infectious persons.)

Proof: For E_1 the matrix of linearization (Jacobian matrix) is giving by:

$\mathcal{J}(E_1) =$

$$\begin{pmatrix} -\frac{\beta_1' I^* + \beta_2' A^*}{N^*} - \mu & -\frac{S^*\beta_1'}{N^*} & -\frac{S^*\beta_2'}{N^*} & \tau \\ \frac{p\left(\beta_1' I^* + \beta_2' A^*\right)}{N^*} & \frac{pS^*\beta_1'}{N^*} - \delta_1 & \frac{pS^*\beta_2'}{N^*} & -p*\tau \\ \frac{(1-p)\left(\beta_1' I^* + \beta_2' A^*\right)}{N^*} & \frac{(1-p)S^*\beta_1'}{N^*} & & -(1-p)*\tau \\ 0 & -\alpha & 0 & -\mu \end{pmatrix}$$

with $\tau = \dfrac{S^*\left(\beta_1' I^* + \beta_2' A^*\right)}{{N^*}^2}$.

We have,

$$S^* = \frac{\Lambda}{\mu}\left(\frac{\delta_1 - p\alpha}{\delta_1 R_0 - p\alpha}\right)$$

and

$$N^* = \tfrac{\Lambda}{\mu}\left(\frac{R_0\left(\delta_1 - p\alpha\right)}{\delta_1 R_0 - p\alpha}\right).$$

Thus,

$$\begin{aligned} \frac{S^*}{N^*} &= \frac{\Lambda}{\mu}\left(\frac{\delta_1 - p\alpha}{\delta_1 R_0 - p\alpha}\right) \times \frac{\mu}{\Lambda}\left(\frac{\delta_1 R_0 - p\alpha}{R_0\left(\delta_1 - p\alpha\right)}\right) \\ &= \frac{1}{R_0} \end{aligned}$$

So,
$\mathcal{J}(E_1) =$

$$\begin{pmatrix} -R_0 Y - \mu & -\frac{\beta_1'}{R_0} & -\frac{\beta_2'}{R_0} & Y \\ pR_0 Y & \frac{p\beta_1'}{R_0} - \delta_1 & \frac{p\beta_2'}{R_0} & -pY \\ (1-p)\,R_0 Y & \frac{(1-p)\,\beta_1'}{R_0} & \frac{(1-p)\,\beta_2'}{R_0} - \delta_2 & -(1-p)\,Y \\ 0 & -\alpha & 0 & -\mu \end{pmatrix}$$

where,

$$Y = \frac{S^*\left(\beta_1' I^* + \beta_2' A^*\right)}{{N^*}^2}.$$

Then the characteristic polynomial of $\mathcal{J}(E_1)$ is given by:

$P_1(\lambda) =$

$$\begin{vmatrix} -R_0 Y - \mu - \lambda & -\frac{\beta_1'}{R_0} & -\frac{\beta_2'}{R_0} & Y \\ pR_0 Y & \frac{p\beta_1'}{R_0} - \delta_1 - \lambda & \frac{p\beta_2'}{R_0} & -pY \\ (1-p)\,R_0 Y & \frac{(1-p)\,\beta_1'}{R_0} & \frac{(1-p)\,\beta_2'}{R_0} - \delta_2 - \lambda & -(1-p)\,Y \\ 0 & -\alpha & 0 & -\mu - \lambda \end{vmatrix}$$

$= -(\lambda + \mu) \times$

$$\begin{vmatrix} 1 & -\frac{\beta_1'}{R_0} & -\frac{\beta_2'}{R_0} & Y \\ 0 & \frac{p\beta_1'}{R_0} - \delta_1 - \lambda & \frac{p\beta_2'}{R_0} & -pY \\ 0 & \frac{(1-p)\,\beta_1'}{R_0} & \frac{(1-p)\,\beta_2'}{R_0} - \delta_2 - \lambda & -(1-p)\,Y \\ R_0 & -\alpha & 0 & -\mu - \lambda \end{vmatrix}$$

$= -(\lambda + \mu) \times$

$$\begin{vmatrix} 1 & -\frac{\beta_1'}{R_0} & -\frac{\beta_2'}{R_0} & Y \\ 0 & \frac{p\beta_1'}{R_0} - \delta_1 - \lambda & \frac{p\beta_2'}{R_0} & -pY \\ 0 & \frac{(1-p)\,\beta_1'}{R_0} & \frac{(1-p)\,\beta_2'}{R_0} - \delta_2 - \lambda & -(1-p)\,Y \\ 0 & \beta_1' - \alpha & \beta_2' & -R_0 Y - \mu - \lambda \end{vmatrix}$$

$$\begin{aligned} &= (\lambda + \mu)\,[\lambda^3 + \left(R_0 Y + \mu + \delta_1 + \delta_2 - \tfrac{(1-p)\beta_2' + p\beta_1'}{R_0}\right)\lambda^2 \\ &+ \left(R_0 Y\delta_2 + R_0 Y\delta_1 - p\alpha\,Y + \mu\left(\delta_1 + \delta_1 - \tfrac{(1-p)\beta_2' + p\beta_1'}{R_0}\right)\right)\lambda \\ &+ R_0 Y\delta_1\delta_2 - \alpha\,pY\delta_2] \end{aligned}$$

Therefore, the eigenvalues of the matrix $\mathcal{J}(E_1)$ are $-\mu$ and the roots of the polynomial:

$$Q_1(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C\,,$$

where,

$$\begin{aligned} A &= R_0 Y + \mu + \delta_1 + \delta_2 - \frac{(1-p)\,\beta_2' + p\beta_1'}{R_0} \\ B &= R_0 Y\delta_2 + R_0 Y\delta_1 - p\alpha\,Y \\ &\quad + \mu\left(\delta_1 + \delta_2 - \frac{(1-p)\,\beta_2' + p\beta_1'}{R_0}\right) \\ C &= R_0 Y\delta_1\delta_2 - \alpha\,pY\delta_2 \end{aligned}$$

- We have,
$$\delta_1 + \delta_2 - \frac{(1-p)\,\beta_2' + p\beta_1'}{R_0} = \frac{(1-p)\beta_2'\delta_1^2 + p\beta_1'\delta_2^2}{(1-p)\beta_2'\delta_1 + p\beta_1'\delta_2} \geq 0$$
So, $A > 0$.

- We have,
  $\delta_1 = \sigma_1 + u_2 + \mu + \alpha > \alpha \geq p\alpha$ and $R_0 > 1$
  $\Longrightarrow R_0\delta_1 - p\alpha > 0 \Longrightarrow B > 0$
- We have $\delta_1 = \sigma_1 + u_2 + \mu + \alpha > \alpha \geq p\alpha$ and $R_0 > 1$. then, $R_0\delta_1 - p\alpha > 0$
  $\Longrightarrow Y\delta_2(R_0\delta_1 - p\alpha) > 0 \Longrightarrow C > 0.$

On the other hand,

$$\begin{aligned} AB &= \left(R_0 Y + \mu + \delta_1 + \delta_2 - \frac{(1-p)\,\beta_2' + p\beta_1'}{R_0}\right) \times \\ &\quad (R_0 Y\delta_2 + R_0 Y\delta_1 - p\alpha Y) \\ &\quad +\mu \left(R_0 Y + \mu + \delta_1 + \delta_2 - \frac{(1-p)\,\beta_2' + p\beta_1'}{R_0}\right) \times \\ &\quad \left(\delta_1 + \delta_2 - \frac{(1-p)\,\beta_2' + p\beta_1'}{R_0}\right) \\ &= (R_0 Y + \mu)\,(R_0 Y\delta_2 + R_0 Y\delta_1 - p\alpha Y) + \delta_1 R_0 Y\delta_2 \\ &\quad + \left(\delta_2 - \frac{(1-p)\,\beta_2' + p\beta_1'}{R_0}\right) \times \\ &\quad (R_0 Y\delta_2 + R_0 Y\delta_1 - p\alpha Y) + \delta_1\,(R_0 Y\delta_1 - p\alpha Y) \\ &\quad +\mu\, A \left(\delta_1 + \delta_2 - \frac{(1-p)\,\beta_2' + p\beta_1'}{R_0}\right) \\ &= R_0 Y\delta_1\delta_2 + (R_0 Y + \mu) \times \\ &\quad \left(R_0 Y\delta_2 + R_0 Y\left(\delta_1 - \frac{p\alpha}{R_0}\right)\right) \\ &\quad + \left(\delta_2 - \frac{(1-p)\,\beta_2' + p\beta_1'}{R_0}\right) \times \\ &\quad \left(R_0 Y\delta_2 + R_0 Y\left(\delta_1 - \frac{p\alpha}{R_0}\right)\right) \\ &\quad + R_0 Y\delta_1 \left(\delta_1 - \frac{p\alpha}{R_0}\right) \\ &\quad +\mu\, A \left(\delta_1 + \delta_2 - \frac{(1-p)\,\beta_2' + p\beta_1'}{R_0}\right) \end{aligned}$$

Since

- $\delta_1 - \dfrac{p\alpha}{R_0} \geq 0$ for $R_0 > 1$
- $\delta_2 - \dfrac{(1-p)\,\beta_2' + p\beta_1'}{R_0} = \dfrac{p\beta_1'(\delta_2 - \delta_1)}{(1-p)\beta_2'\delta_1 + p\beta_1'\delta_2} \geq 0$ for $\delta_2 > \delta_1$
- $\delta_1 + \delta_2 - \dfrac{(1-p)\,\beta_2' + p\beta_1'}{R_0} = \dfrac{(1-p)\beta_2'\delta_1^2 + p\beta_1'\delta_2^2}{(1-p)\beta_2'\delta_1 + p\beta_1'\delta_2} \geq 0$

Then,

$$AB > R_0 Y\delta_1\delta_2 > C \Longrightarrow AB - C > 0.$$

We have,

$$A > 0,\ B > 0\ ,\ C > 0 \text{ and } AB - C > 0.$$

Thus, using Routh Hurwitz stability criterion, it can be concluded that E_1 is locally asymptotically stable.

III. Simulations

Simulations are performed by taking parameters in (Table 1) and using data from Morocco [21]

To solve the system (1) numerically we will use the Gauss-Seidel-like implicit finite-difference. The time interval $[t_0, T]$ is discretized with a step h (time step size) such that $t_i = t_0 + ih$
$i = 0, 1, \cdots, n$ and $t_n = T$
The derivatives are approached by the following finite differences:

TABLE I: Input parameters of the SIAR COVID-19 model

Symbol	Meaning	Value
β_1	Effective contact rate with symptomatic infected person	0.6
β_2	Effective contact rate with asymptomatic infected person	0.4
$\dfrac{1}{\mu+\sigma_1}$	Infection duration among symptomatic infected person	7.6
$\dfrac{1}{\mu+\sigma_2}$	Infection duration among asymptomatic infected person	7.22
α	Death rate from COVID-19	0.006

$$\frac{S_{i+1} - S_i}{h} = \Lambda - \frac{S_{i+1}\,(1-u_1)\,(\beta_1\, I_i + \beta_2\, A_i)}{N_i} - \mu\, S_{i+1}$$

$$\frac{I_{i+1} - I_i}{h} = \frac{pS_{i+1}\,(1-u_1)\,(\beta_1\, I_{i+1} + \beta_2\, A_i)}{N_i} - (\sigma_1 + \mu + \alpha + u_2)\, I_{i+1}$$

$$\frac{A_{i+1} - A_i}{h} = \frac{(1-p)\, S_{i+1}\,(1-u_1)\,(\beta_1\, I_{i+1} + \beta_2\, A_{i+1})}{N_i} - (\sigma_2 + \mu)\, A_{i+1}$$

$$\frac{N_{i+1} - N_i}{h} = -\alpha\, I_{i+1} - \mu\, N_{i+1} + \Lambda$$

Then,

$$S_{i+1} = \frac{\Lambda\, hN_i + N_i S_i}{h\mu\, N_i + (1-u_1)\,(\beta_1\, I_i + \beta_2\, A_i)\, h + N_i}$$

$$I_{i+1} = \frac{p\,(1-u_1)\,\beta_2\, hA_i S_{i+1} + N_i I_i}{-p\,(1-u_1)\,\beta_1\, hS_{i+1} + \left(\frac{1}{h} + \sigma_1 + \mu + \alpha + u_2\right) hN_i}$$

$$A_{i+1} = \frac{(1-p)\,(1-u_1)\,\beta 1\, hI_{i+1}S_{i+1} + N_i A_i}{-(1-p)\,(1-u_1)\,\beta_2\, hS_{i+1} + (\sigma_2 + \mu)\, hN_i + N_i}$$

$$N_{i+1} = \frac{(-\alpha\, I_{i+1} + \Lambda)\, h + N_i}{h\mu + 1}$$

The simulation's results are given by Fig. 2 – Fig.7 below, corresponding to different scenarios developed to illustrate the effect of control on the number of reproductions.

IV. Results and discussion

The numerical simulations below concern the pandemic state evolution described by the proposed model, applied to the Moroccan case. In these simulations, we took the daily pandemic data [1] as a reference, then we tested different scenarios corresponding to different prevention and patient care strategies.

The scenario illustrated by Fig. 2 is related to the pandemic evolution without control. This figure presents a peak of the pandemic reached in an early stage with a very high number of infected cases inducing a high pressure on the health system and a high case fatality rate.
After this simulation without control, we proposed to test numerically scenarios with different control levels with an initial reproductive number $R_0 = 3.3$.

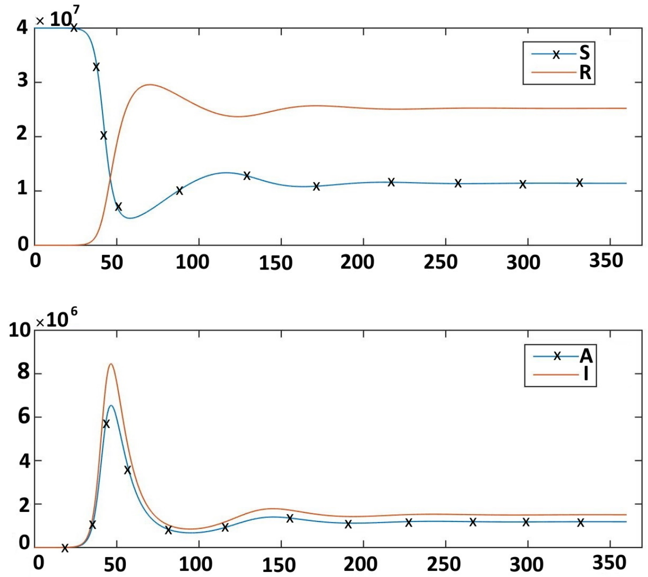

Fig. 2: Evolution of the pandemic without control $u_1 = 0, u_2 = 0,\ R_0 = 3.3$

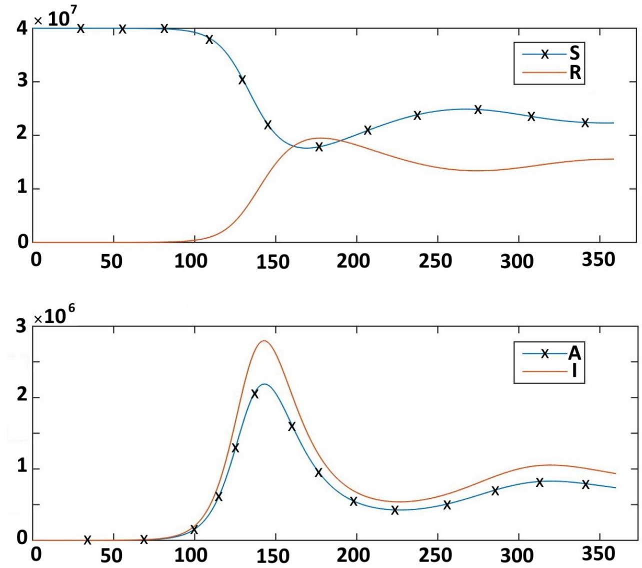

Fig. 3: Evolution of the pandemic with a moderate control of u_1 ($u_1 = 0.5, u_2 = 0\ R_0 = 1.7$)

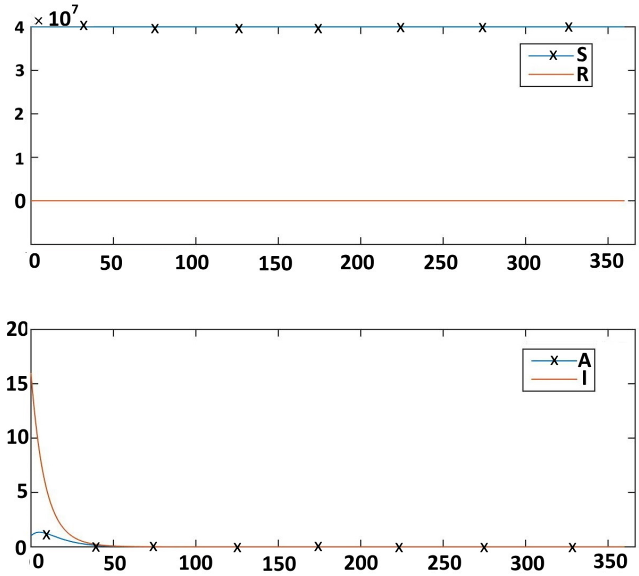

Fig. 4: Evolution of the pandemic with a high control of u_1 ($u_1 = 0.9, u_2 = 0\ R_0 = 0.3$)

First, the carried out simulations take into account the effect of the control u_1 on R_0. Figure 3 shows that with a moderate prevention, the reproductive number R_0 can be reduced to 1.7. Furthermore, while pushing the level of prevention to the maximum level, with very strict containment and isolation measures, R_0 can be reduced to 0.3. Consequently, the pandemic will be eradicated (see Fig. 4). However, this scenario is unrealistic due to its consequences on individual freedom and economic activity.

Second, the analysis of the control effect u_2 on R_0 shows only a slight decrease of R_0 even under a maximum increase of u_2 (see Fig .5 and Fig. 6). This can be explained by the fact that, susceptible people are silently infected by uncontrolled asymptomatic individuals. Hence, a combination of these two controls is necessary to reduce the R_0 drastically ($R_0 = 0.99 < 1$) without affecting the economy and individual freedom (see Fig. 7).

These results show that combining the different prevention's measures and high patient care quality improve the prevention against COVID-19 pandemic.

V. Conclusion

The spread of pandemics has always been a difficult phenomenon to grasp due to the multitude of included factors. In the case of the COVID-19 pandemic, the situation is even more complex because it is a new virus whose behavior is difficult to predict. Nevertheless, it is mandatory to figure out its mode of propagation by different tools in order to control its prevalence.
In this purpose we proposed a mathematical model that describes the pandemic evolution and allows a better understanding of the impact of prevention and control on the basic reproduction number R_0, using different scenarios according to a level of control and prevention. According to our simulations results coronavirus disease COVID-19 will continue to spread as long as there is no vaccine or acquired community immunity.
Until now the only way to fight COVID-19 is to take the essential protective measures against the new coronavirus as recommended by WHO [3].

Appendix A

1. Reproduction number

The basic reproduction number R_0, is defined as the average number of secondary infections produced when one infected individual is introduced into a host population where everyone is susceptible [22].

Using notations in [23], the matrices F and V and their Jacobian matrices for the new infection terms and the remaining transfer term evaluated at the disease free equilibrium are respectively given by ;

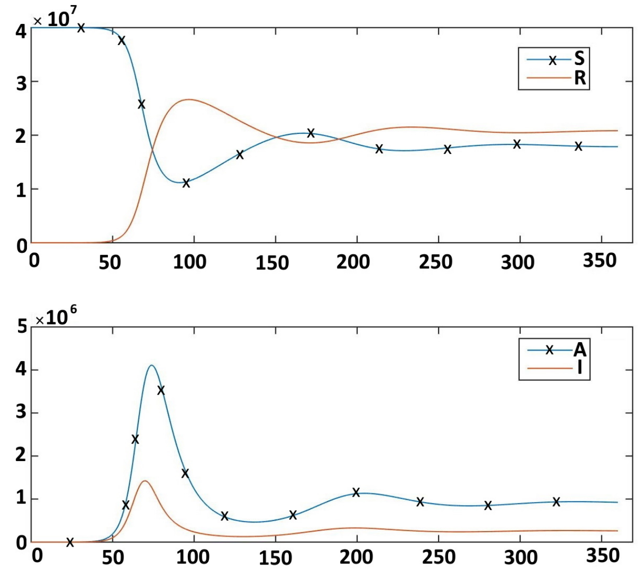

Fig. 5: Evolution of the pandemic with a moderate control of u_2 ($u_1 = 0, u_2 = 0.5$ $R_0 = 2.2$)

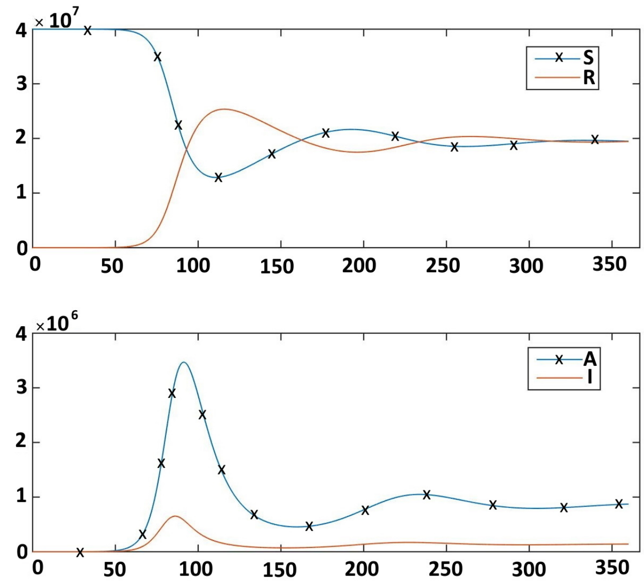

Fig. 6: Evolution of the pandemic with a high control of u_2 ($u_1 = 0, u_2 = 1$ $R_0 = 2$)

$$F = \begin{pmatrix} \dfrac{pS\left(\beta_1' I + \beta_2' A\right)}{N} \\ \dfrac{(1-p)S\left(\beta_1' I + \beta_2' A\right)}{N} \end{pmatrix} ; V = \begin{pmatrix} \delta_1 I \\ \delta_2 A \end{pmatrix}$$

$$J_F = \begin{pmatrix} \dfrac{pS\beta_1'}{N} & \dfrac{pS\beta_2'}{N} \\ \dfrac{(1-p)S\beta_1'}{N} & \dfrac{(1-p)S\beta_2'}{N} \end{pmatrix}.$$

For trivial equilibrium E_0, we have $S^* = N^*$. So,

$$J_F = \begin{pmatrix} p\beta_1' & p\beta_2' \\ (1-p)\beta_1' & (1-p)\beta_2' \end{pmatrix}$$

$$J_V = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \Rightarrow J_v^{-1} = \begin{pmatrix} \frac{1}{\delta_1} & 0 \\ 0 & \frac{1}{\delta_2} \end{pmatrix}.$$

Thus,

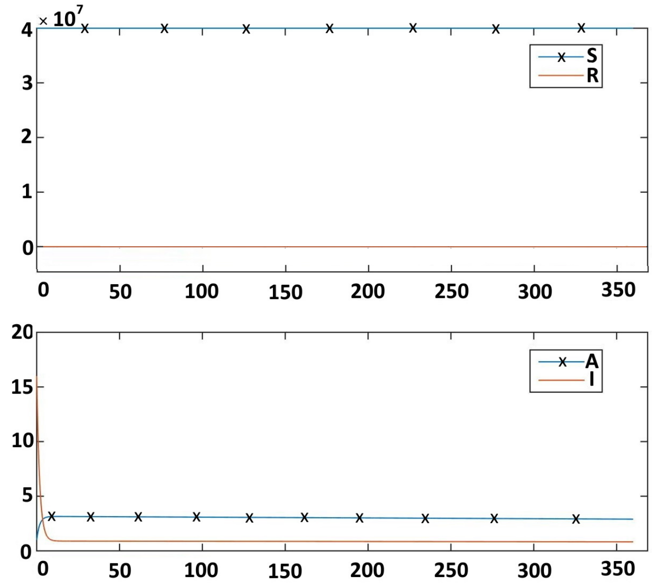

Fig. 7: Evolution of the pandemic with a moderate control of u_1 and u_2 ($u_1 = 0.55, u_2 = 0.5$, $R_0 = 0.99$)

$$J_F J_V^{-1} = \begin{pmatrix} \dfrac{p\beta_1'}{\delta_1} & \dfrac{p\beta_2'}{\delta_2} \\ \dfrac{(1-p)\beta_1'}{\delta_1} & \dfrac{(1-p)\beta_2'}{\delta_2} \end{pmatrix}$$

It follows that the basic reproduction, denoted by R_0, is given by:

$$\begin{aligned} R_0 &= \rho(J_F J_V^{-1}) \\ &= \frac{p\beta_1'}{\delta_1} + \frac{(1-p)\beta_2'}{\delta_2} \\ &= \frac{p\beta_1(1-u_1)}{\sigma_1 + u_2 + \mu + \alpha} + \frac{(1-p)\beta_2(1-u_1)}{\sigma_2 + \mu} \\ &= R_I + R_A \end{aligned}$$

Where, $\qquad R_I = \dfrac{p\beta_1(1-u_1)}{\sigma_1 + u_2 + \mu + \alpha}$

and $\qquad R_A = \dfrac{(1-p)\beta_2(1-u_1)}{\sigma_2 + \mu}.$

Appendix B

1. Proof of theorem 1

From

$$\frac{dN}{dt} = \Lambda - \mu N - \alpha I$$

$$\begin{aligned} \frac{dN}{dt} &= \Lambda - \mu N - \alpha I \\ &\geq -(\mu + \alpha)N \quad (\text{ because } I \leq N \text{ and } \alpha \geq 0). \end{aligned}$$

Then using Gronwall's inequality:

$$N(t) \geq N(0)e^{\left(-\int_0^T (\mu+\alpha)dt\right)} \implies N(t) > 0.$$

On the other hand, we have:

$$\frac{dI}{dt} = \frac{pS(t)(1-u_1)(\beta_1 I(t) + \beta_2 A(t))}{N(t)} - \delta_1 I(t)$$

Assume that there exists some time $t^* > 0$ such that $I(t^*) = 0$, other variables (S, N, A) are positive and

$I(t) > 0$ for $t \in [0, t^*[$.

So, we have:

$$\begin{aligned}\frac{dI(t)e^{\delta_1 t}}{dt} &= \delta_1 e^{\delta_1 t} I(t) + e^{\delta_1 t}\Big[\frac{pS(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} \\ &\quad -\delta_1 I(t)\Big] \\ &= e^{\delta_1 t}\left[\frac{pS(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)}\right].\end{aligned}$$

Integrating this Equation from 0 to t^* we have:

$$\int_0^{t^*} \frac{dI(t)e^{\delta_1 t}}{dt} dt = \int_0^{t^*} \left[\frac{pS(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)}\right] e^{\delta_1 t} dt,$$

then, $I(t^*) = e^{-\delta_1 t^*} I(0) + e^{-\delta_1 t^*}$
$\int_0^{t^*} \left[\frac{pS(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)}\right] e^{\delta_1 t} dt > 0$
which contradicts $I(t^*) = 0$.

Consequently, $I(t) > 0 \quad \forall t \in [0, T]$.

In the same way:

$$\frac{dA}{dt} = \frac{(1-p)S(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - \delta_2 A(t)$$

Assume that there exists some time $t^* > 0$ such that $A(t^*) = 0$, other variables (S, N, I) are positive and $A(t) > 0$ for $t \in [0, t^*[$.

So, we have:

$$\begin{aligned}\frac{dA(t)e^{\delta_1 t}}{dt} &= \delta_1 e^{\delta_2 t} A(t) + e^{\delta_2 t} \times \\ &\quad \left[\frac{(1-p)S(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} - \delta_2 A(t)\right] \\ &= e^{\delta_2 t}\left[\frac{(1-p)S(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)}\right]\end{aligned}$$

Integrating this Equation from 0 to t^* we have

$$\int_0^{t^*} \frac{dA(t)e^{\delta_2 t}}{dt} dt = \int_0^{t^*} \left[\frac{(1-p)S(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)}\right] \times e^{\delta_1 t} dt$$

then $A(t^*) = e^{-\delta_2 t^*} I(0) +$
$e^{-\delta_2 t^*} \int_0^{t^*} \left[\frac{(1-p)S(t)(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)}\right] e^{\delta_2 t} dt > 0$
which contradicts $A(t^*) = 0$.

Consequently, $A(t) > 0 \quad \forall t \in [0, T]$.

From:

$$\begin{aligned}\frac{dS}{dt} &= \Lambda - \frac{(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} S(t) - \mu S(t) \\ &\geq -\left(\frac{(1-u_1)(\beta_1 I(t)+\beta_2 A(t))}{N(t)} + \mu\right) S(t)\end{aligned}$$

(because $I(t) \geq 0$, $A(t) \geq 0$ and $N(t) \geq 0$).

Then using Gronwall's inequality
$S(t) \geq S(0) \exp\left(-\int_0^t \left(\frac{(1-u_1)(\beta_1 I(s)+\beta_2 A(s))}{N(s)} + \mu\right) ds\right)$
$\Longrightarrow S(t) > 0$.

On the other hand,

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t) - \alpha I(t) \leq \Lambda - \mu N_h(t).$$

So, $N(t) \leq \frac{\Lambda}{\mu} - \left(\frac{\Lambda}{\mu} - N(0)\right) e^{-\mu t} \Longrightarrow N(t) \leq \frac{\Lambda}{\mu}$
for initial value $N(0) \leq \frac{\Lambda}{\mu}$

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