New Compact Finite Difference Schemes with Fourth-order Accuracy for the Extended Fisher-Kolmogorov Equation

Jinming Zuo

Abstract—**In this paper, two high-order compact finite difference schemes are proposed and analyzed for solving the extended Fisher-Kolmogorov equation. The first compact finite difference scheme is two-level and nonlinear implicit. The second scheme is three-level and linearized implicit. The existence, uniqueness of difference solutions and priori estimates are obtained. Furthermore, the present schemes are convergent, unconditionally stable, and the numerical convergence orders in l_∞-norm are of $O(\tau^2+h^4)$. Numerical experiments demonstrate that the present schemes are efficient and reliable.**

Index Terms—**Extended Fisher-Kolmogorov equation, Finite difference scheme, Unconditional stability, High-order convergence.**

I. Introduction

In this paper, we consider the following periodic initial value problem of the extended Fisher-Kolmogorov (EFK) equation

$$u_t + \gamma u_{xxxx} - u_{xx} - u + u^3 = 0, x \in R, t \in (0, T], \quad (1)$$

subject to the boundary conditions

$$u(x + L, t) = u(x, t), \quad x \in R, \quad t \in (0, T], \quad (2)$$

and the initial condition

$$u(x, 0) = u_0(x), \quad x \in R, \quad (3)$$

where $u = u(x, t)$ is a real-valued function which defined on $R \times [0, T], T > 0$, γ is a positive constant and u_0 is a given periodic-valued function regular enough. To solve the periodic initial value problem (1)-(3), we can restrict it on a bounded domain $\Omega = (0, L)$. When $\gamma = 0$, the above extended Fisher-Kolmogorov equation (1) reduces into the usual Fisher-Kolmogorov equation.

In recent years, there has been a growing interest in the computation of the extended Fisher-Kolmogorov equation, Danumjaya and Pani [1] have studied the convergence of numerical solution by using the second-order splitting combined with orthogonal cubic spline collocation method, and they also developed a finite element Galerkin method for the two-dimensional extended Fisher-Kolmogorov equation with optimal error estimates [2]. Kadri and Omrani [3] developed a second-order Crank-Nicolson scheme for the above 1D extended Fisher-Kolmogorov equation (1), where the stability and convergence of the numerical solution are proved in L_∞-norm. Moreover, Khiari and Omrani [4] developed an energy stable nonlinear scheme for the 2D extended Fisher-Kolmogorov equation, where the stability and convergence of the numerical solution are also proved in L_∞-norm. He [5] developed a second-order three-level linearly implicit finite difference method for solving the extended Fisher-Kolmogorov equation in both 1D and 2D, where the stability and convergence of the numerical solution are also proved in L_∞-norm. Xu et al. [6] utilized a reduced high-order compact finite difference scheme on POD technique for the 2D extended Fisher-Kolmogorov equation. Recently, Khiari and Omrani [7] proposed a fourth-order nonlinear finite difference scheme and their numerical analysis showed that the method can be applied to study the periodic solution in a long time. Ismail et al. [8] proposed a fourth-order linear finite difference scheme and their numerical analysis also showed that the method can be applied to study the periodic solution in a long time. However, the above finite-difference schemes in Refs. [7, 8], although they have the fourth-order numerical precision, employ a seven-point discrete method. Thus, the purpose of this paper is to establish two new high-order compact finite-difference schemes for solving the EFK equation. The coefficient matrices of these new schemes are both five-point discrete method. And we rigorously prove that the two schemes are unconditionally stable. More difference schemes can be found for the KdV equation [9-11], the Benjamin-Bona-Mahony equation or RLW equation [12-14], the Rosenau equation [15], the Rosenau-RLW equation [16-18], the Kawahara equation [19], the Rosenau-Kawahara equation [20, 21], the Rosenau-Kawahara-RLW equation [22-25], and many others [26-31].

The outline is as follows. In Sect. 2, a two-time-level nonlinear finite difference scheme for the problem (1)-(3) is described in detail, and the unique solvability, the prior error estimates, and the unconditional convergence of the difference scheme are proved. In Sect. 3, a three-time-level linearized compact finite difference scheme is constructed. The unique solvability, the prior error estimates, and the unconditional convergence of the difference scheme are also proved. In Sect. 4, we present some numerical examples to show the performance of the schemes and confirm our theoretical analysis. Finally, conclusions are drawn in the last section.

Manuscript received June 1, 2021; revised November 17, 2021. The work was supported by the Natural Science Foundation of Shandong Province of China (ZR2017MA021).

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II. Nonlinear compact difference scheme

In this section, we propose a two-time-level nonlinear fourth-order compact finite difference scheme for the problem (1)-(3).

We first describe our solution domain and its grid. The computational domain is defined as $\{(x,t)|0 \le x \le L, 0 \le t \le T\}$, which is covered by a uniform grid $\{(x_i,t_n)|x_i = ih, t_n = n\tau, i = 0,1,\cdots,J, n = 0,1,\cdots,N\}$, with spacing $h = \frac{L}{J}, \tau = \frac{T}{N}$. We denote U_i^n is the numerical approximation of $u(x_i,t_n)$ and $R_{per}^J = \{U = (U_i)|U_i = U_{i+J}, i \in Z\}$. Throughout this paper, we denote C as a general positive constant, which may have different values independent of h and τ, that varies in different occurrences in the context. For convenience, the difference operators, inner product and norms are defined as

$$\begin{aligned}
&\bar{U}_i^n = \frac{U_i^{n+1}+U_i^{n-1}}{2}, U_i^{n+\frac{1}{2}} = \frac{U_i^{n+1}+U_i^n}{2},\\
&(U_i^n)_t = \frac{U_i^{n+1}-U_i^n}{\tau}, (U_i^n)_{\hat{t}} = \frac{U_i^{n+1}-U_i^{n-1}}{2\tau},\\
&(U_i^n)_x = \frac{U_{i+1}^n-U_i^n}{h}, (U_i^n)_{\bar{x}} = \frac{U_i^n-U_{i-1}^n}{h},\\
&(U_i^n)_{\hat{x}} = \frac{U_{i+1}^n-U_{i-1}^n}{2h}, \langle U^n, V^n\rangle = h\sum_{i=1}^{J} U_i^n V_i^n,\\
&\|U^n\| = \sqrt{\langle U^n, U^n\rangle}, \|U^n\|_\infty = \max_{1\le i\le J}\|U_i^n\|,\\
&\|U^n\|_{H^1} = \sqrt{\|U^n\|^2 + \|U_x^n\|^2}.
\end{aligned}$$

To get the high-order scheme, we use the following fourth-order compact finite-difference operator [13, 25, 30]

$$L_x U_i^n = U_i^n + \frac{h^2}{12}(U_i^n)_{x\bar{x}} = \frac{1}{12}(U_{i-1}^n + 10U_i^n + U_{i+1}^n).$$

For the discretization of the second-order derivative u_{xx} of the function $u(x,t)$, we have the following formulas [25]

$$u_{xx}(x_i,t^n) = L_x^{-1}(U_i^n)_{x\bar{x}} + O(h^4).$$

Omitting the small terms $O(h^4)$, we obtain

$$u_{xx}(x_i,t^n) \approx L_x^{-1}(U_i^n)_{x\bar{x}}.$$

We now introduce the vectors $U = (U_1,U_2,\cdots,U_J)^{\mathrm{T}}$, $\Lambda_h u_{xx} = [u_{xx}(x_1), u_{xx}(x_2), \cdots, u_{xx}(x_J)]^{\mathrm{T}}$ and the matrix

$$M = \frac{1}{12}\begin{pmatrix} 10 & 1 & 0 & \cdots & 1\\ 1 & 10 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 1 & 10 & 1\\ 1 & \cdots & 0 & 1 & 10\end{pmatrix},$$

where $[\ \cdot\ ]^{\mathrm{T}}$ is the transpose of the vector $[\ \cdot\ ]$. Thus, the corresponding matrix form is

$$\Lambda_h U_{xx} \approx M^{-1}U_{x\bar{x}}.$$

Since M is a real symmetric positive definite matrix, there exists a real symmetric positive definite matrix H, such that $H = M^{-1}$.

We first propose a nonlinear compact finite difference scheme for the EFK equation as

$$\begin{aligned}&(U_i^n)_t + \gamma(L_x^{-1})^2(U_i^{n+\frac{1}{2}})_{x\bar{x}x\bar{x}} - L_x^{-1}(U_i^{n+\frac{1}{2}})_{x\bar{x}}\\ &- U_i^{n+\frac{1}{2}} + (U_i^{n+\frac{1}{2}})^3 = 0,\end{aligned} \tag{4}$$

where $n = 0,1,\cdots,N-1, i = 1,2,\cdots,J$. Thus, the compact finite-difference scheme (4) can be rewritten in the following matrix form

$$\begin{aligned}&(U^n)_t + \gamma H^2(U^{n+\frac{1}{2}})_{x\bar{x}x\bar{x}} - H(U^{n+\frac{1}{2}})_{x\bar{x}} - U^{n+\frac{1}{2}}\\ &+ (U^{n+\frac{1}{2}})^3 = 0, \quad n = 0,1,\cdots,N-1,\end{aligned} \tag{5}$$

and the initial-boundary conditions are discretized as

$$U_{i+M}^n = U_i^n, \quad i = 1,2,\cdots,M, \quad n = 0,1,\cdots,N, \tag{6}$$

$$U_i^0 = u_0(x_i), \quad i = 1,2,\cdots,M. \tag{7}$$

The following lemmas are some properties for our compact scheme. They are essential for existence, uniqueness, convergence, and stability of the numerical solution.

Lemma 1 (See [30]). For any two mesh functions $U, V \in R_{per}^J$, we have

$$\langle U_x, V\rangle = -\langle U, V_{\bar{x}}\rangle, \quad \langle U_{x\bar{x}}, V\rangle = -\langle U_x, V_x\rangle.$$

Furthermore, $\langle U_{x\bar{x}}, U\rangle = -\|U_x\|^2, \langle U_{x\bar{x}x\bar{x}}, U\rangle = \|U_{x\bar{x}}\|^2$.

Lemma 2 For any mesh function $U \in R_{per}^J$, we have

$$\sum_{i=1}^{J} L_x U_i = \sum_{i=1}^{J} U_i.$$

Similarly, L_x^{-1} also has the above properties,

$$\sum_{i=1}^{J} L_x^{-1} U_i = \sum_{i=1}^{J} U_i.$$

Proof. Notice that $U_i = U_{i+J}, i \in Z$. We can find

$$\sum_{i=1}^{J} L_x U_i = \frac{1}{12}\sum_{i=1}^{J} U_{i-1} + \frac{10}{12}\sum_{i=1}^{J} U_i + \frac{1}{12}\sum_{i=1}^{J} U_{i+1} = \sum_{i=1}^{J} U_i.$$

Lemma 3 (See [25, 30]). For any mesh function $U \in R_{per}^J$, we have

$$\begin{aligned}&\langle HU_{x\bar{x}}, U\rangle = -\langle RU_x, RU_x\rangle = -\|RU_x\|^2,\\ &\langle H^2 U_{x\bar{x}x\bar{x}}, U\rangle = \langle HU_{x\bar{x}}, HU_{x\bar{x}}\rangle = \|U_{x\bar{x}}\|^2,\end{aligned}$$

where R is obtained by the Cholesky decomposition of H, denoted as $H = R^{\mathbf{T}}R$.

Lemma 4 (See [25, 30]). For any mesh function $U \in R_{per}^J$, we have

$$\begin{aligned}&\|U\|^2 \le \langle HU, U\rangle = \|RU\|^2 \le C\|U\|^2,\\ &\|U\|^2 \le \langle HU, HU\rangle = \|HU\|^2 \le C\|U\|^2.\end{aligned}$$

A. *Existence and prior estimates*

To show the existence of the approximations U^n $(n = 1,2,\cdots,N)$ for the scheme (5)-(7), we introduce the following *Brouwer* fixed point theorem [32].

Lemma 5. Let H be a finite-dimensional inner product space, $\|\cdot\|$ be the associated norm, and $g : H \to H$ be continuous. Assume, moreover, that $\exists\alpha > 0, \forall z \in H, \|z\| = \alpha, \langle\omega(z), z\rangle > 0$. Then, there exists a $z^* \in H$ such that $g(z^*) = 0$ and $\|z^*\| \le \alpha$.

Theorem 1. There exists $U^{n+1} \in R_{per}^J$ which satisfies the scheme (5)-(7).

Proof. In order to prove the theorem by the mathematical induction. It follows from the original problem (1)-(3) that U^0 satisfies the scheme (5)-(7). Assume there exists $U^1, U^2, \cdots, U^n \in R_{per}^J$ which satisfy the scheme (5)-(7), as $n \leq N-1$, now we try to prove that $U^{n+1} \in R^J_{per}$, satisfy the scheme (5)-(7).

We define ω on R^J_{per} as follows

$$\begin{aligned}\omega(\nu) = 2\nu - 2U^n + \tau\gamma H^2 \nu_{x\bar{x}x\bar{x}} \\ - \tau H \nu_{x\bar{x}} - \tau\nu + \tau\nu^3.\end{aligned} \tag{8}$$

Computing the inner product of (8) with ν and using Lemma 3, we obtain

$$\begin{aligned}\langle \omega(\nu), \nu\rangle &= 2\|\nu\|^2 - 2\langle U^n, \nu\rangle + \tau\gamma\|H\nu_{x\bar{x}}\|^2 \\ &+ \tau\|R\nu_x\|^2 - \tau\|\nu\|^2 + \tau\|\nu^2\|^2 \\ &\geq 2\|\nu\|^2 - \|\nu\|^2 - \|U^n\|^2 - \tau\|\nu\|^2 \\ &= (1-\tau)\|\nu\|^2 - \|U^n\|^2.\end{aligned}$$

Hence, when $\tau < 1$, for $\forall \nu \in R^J_{per}$, $\|\nu\|^2 = \frac{1}{1-\tau}\|U^n\|^2 + 1$, there exists $\langle \omega(\nu), \nu\rangle \geq 0$. It follows from Lemma 5 that exists $\nu^* \in R^J_{per}$ which satisfies $\omega(\nu^*) = 0$. Let $U^{n+1} = 2\nu - U^n$, then it can be proved that $U^{n+1} \in R^J_{per}$ is the solution of the scheme (5)-(7). This completes the proof of Theorem 1.

Next we shall give some priori estimates of difference solutions. First the following two Lemmas are introduced.

Lemma 6 (Discrete Sobolev's Inequality [5, 33]). For any mesh function $U \in R^J_{per}$, we have

$$\|U\|_\infty \leq C\|U\|_{H^1}.$$

Lemma 7 (Discrete Gronwall's Inequality). Suppose that $\omega(k)$ and $\rho(k)$ are nonnegative functions while $\rho(k)$ is a non-decreasing function. If

$$\omega(k) \leq \rho(k) + C\tau\sum_{l=0}^{k-1}\omega(l),$$

for any k, then $\omega(k) \leq \rho(k)e^{C\tau k}$.

Theorem 2. Suppose $u^0 \in C^{8,3}[0, L]$, if τ is sufficiently small, then the solution of finite difference scheme (5)-(7) satisfies $\|U^n\|_\infty \leq C$ for any $n \geq 1$.

Proof. Taking the inner product of $U^{n+\frac{1}{2}}$ for both sides of (5), we obtain

$$\begin{aligned}&\frac{\|U^{n+1}\|^2 - \|U^n\|^2}{2\tau} + \gamma\|HU^{n+\frac{1}{2}}_{x\bar{x}}\|^2 + \|RU^{n+\frac{1}{2}}_x\|^2 \\ &- \|U^{n+\frac{1}{2}}\|^2 + \left\langle (U^{n+\frac{1}{2}})^3, U^{n+\frac{1}{2}}\right\rangle = 0,\end{aligned} \tag{9}$$

where

$$\begin{aligned}&\langle U^n_t, U^{n+\frac{1}{2}}\rangle = \frac{\|U^{n+1}\|^2 - \|U^n\|^2}{2\tau}, \\ &\langle H^2 U^{n+\frac{1}{2}}_{x\bar{x}x\bar{x}}, U^{n+\frac{1}{2}}\rangle = \|HU^{n+\frac{1}{2}}_{x\bar{x}}\|^2, \\ &\langle HU^{n+\frac{1}{2}}_{x\bar{x}}, U^{n+\frac{1}{2}}\rangle = -\|RU^{n+\frac{1}{2}}_x\|^2\end{aligned}$$

are used. From (9), we get

$$\begin{aligned}\frac{\|U^{n+1}\|^2 - \|U^n\|^2}{2\tau} &= -\gamma\|HU^{n+\frac{1}{2}}_{x\bar{x}}\|^2 - \|RU^{n+\frac{1}{2}}_x\|^2 \\ &+ \|U^{n+\frac{1}{2}}\|^2 - \left\langle (U^{n+\frac{1}{2}})^3, U^{n+\frac{1}{2}}\right\rangle \\ &\leq \|U^{n+\frac{1}{2}}\|^2 \leq \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2},\end{aligned} \tag{10}$$

where $\left\langle (U^{n+\frac{1}{2}})^3, U^{n+\frac{1}{2}}\right\rangle = \|(U^{n+\frac{1}{2}})^2\|^2 \geq 0$ is used.

Thus,

$$\|U^{n+1}\|^2 - \|U^n\|^2 \leq \frac{2\tau}{1-\tau}\|U^n\|^2. \tag{11}$$

If τ, which is sufficiently small, satisfies $\tau \leq \frac{1}{2}$, then $1-\tau \geq \frac{1}{2}$ and (11) gives

$$\|U^{n+1}\|^2 - \|U^n\|^2 \leq 4\tau\|U^n\|^2. \tag{12}$$

Then, summing (12) from 0 to $n-1$, we get

$$\|U^n\|^2 \leq \|U^0\|^2 + 4\tau\sum_{k=0}^{n-1}\|U^k\|^2,$$

for any $1 \leq n \leq N$.

Through Lemma 7, we obtain that

$$\|U^n\|^2 \leq \|U^0\|^2 e^{4n\tau} \leq \|U^0\|^2 e^{4T}.$$

Then,

$$\|U^n\| \leq C. \tag{13}$$

Now taking the inner product of U^n_t for both sides of (5), we obtain

$$\begin{aligned}&\|U^n_t\|^2 + \gamma\frac{\|HU^{n+1}_{x\bar{x}}\|^2 - \|HU^n_{x\bar{x}}\|^2}{2\tau} \\ &+ \frac{\|RU^{n+1}_x\|^2 - \|RU^n_x\|^2}{2\tau} - \langle U^{n+\frac{1}{2}}, U^n_t\rangle \\ &+ \left\langle (U^{n+\frac{1}{2}})^3, U^n_t\right\rangle = 0,\end{aligned} \tag{14}$$

where

$$\begin{aligned}&\left\langle H^2 U^{n+\frac{1}{2}}_{x\bar{x}x\bar{x}}, U^n_t\right\rangle = \frac{\|HU^{n+1}_{x\bar{x}}\|^2 - \|HU^n_{x\bar{x}}\|^2}{2\tau}, \\ &\left\langle HU^{n+\frac{1}{2}}_{x\bar{x}}, U^n_t\right\rangle = -\frac{\|RU^{n+1}_x\|^2 - \|RU^n_x\|^2}{2\tau}\end{aligned}$$

are used. From (14), we can obtain that

$$\begin{aligned}&\gamma\frac{\|HU^{n+1}_{x\bar{x}}\|^2 - \|HU^n_{x\bar{x}}\|^2}{2\tau} + \frac{\|RU^{n+1}_x\|^2 - \|RU^n_x\|^2}{2\tau} \\ &= -\|U^n_t\|^2 + \langle U^{n+\frac{1}{2}}, U^n_t\rangle - \left\langle (U^{n+\frac{1}{2}})^3, U^n_t\right\rangle \\ &\leq -\|U^n_t\|^2 + \|U^{n+\frac{1}{2}}\| \cdot \|U^n_t\| \\ &\quad + \|(U^{n+\frac{1}{2}})^3\| \cdot \|U^n_t\| \\ &\leq -\|U^n_t\|^2 + \frac{1}{2}\|U^{n+\frac{1}{2}}\|^2 + \frac{1}{2}\|U^n_t\|^2 \\ &\quad + \frac{1}{2}\|(U^{n+\frac{1}{2}})^3\|^2 + \frac{1}{2}\|U^n_t\|^2 \\ &= \frac{1}{2}\|U^{n+\frac{1}{2}}\|^2 + \frac{1}{2}\|(U^{n+\frac{1}{2}})^3\|^2 \leq C,\end{aligned}$$

where (13) is been used.

Let $D^n = \gamma\|HU^n_{x\bar{x}}\|^2 + \|RU^n_x\|^2$, then,

$$D^{n+1} - D^n \leq 2\tau C, \quad 0 \leq n \leq N-1. \tag{15}$$

Summing (15) from 0 to $n-1$, we get

$$D^n \leq D^0 + 2\tau nC \leq D^0 + 2CT \leq C, 1 \leq n \leq N.$$

Therefore, from Lemma 4, we obtain that

$$\|U^n_x\| \leq \|RU^n_x\| \leq C, 1 \leq n \leq N. \tag{16}$$

Combining (13) and (16), we obtain that

$$\|U^n_x\|_{H^1} \leq C, 1 \leq n \leq N.$$

By Lemma 6, we can get that

$$\|U^n\|_\infty \leq C, 1 \leq n \leq N. \tag{17}$$

This completes the proof.

B. uniqueness and Convergence

In this section, the following theorems will show the uniqueness and convergence of the finite difference approximate solution (5)-(7).

Theorem 3. If τ is sufficiently small, then the finite difference scheme (5)-(7) has a unique solution.

Proof. To prove the theorem, we proceed by the mathematical induction. It is obvious that U^0 is uniquely determined by the initial condition in (7). Suppose $U^1, \cdots, U^n (1 \le n \le N-1)$ are solved uniquely, we now consider (5) for U^{n+1}. Assume that $U^{n+1,1}, U^{n+1,2}$ are two solutions of (5) and let $W^{n+1} = U^{n+1,1} - U^{n+1,2}$, then it is easy to verify that W^{n+1} satisfies the following equation

$$\begin{aligned} &\frac{1}{\tau} W_i^{n+1} + \frac{\gamma}{2} H^2 (W_i^{n+1})_{x\bar{x}x\bar{x}} - \frac{1}{2} H (W_i^{n+1})_{x\bar{x}} \\ &- \frac{W_i^{n+1}}{2} + \left(U_i^{n+\frac{1}{2},1}\right)^3 - \left(U_i^{n+\frac{1}{2},2}\right)^3 = 0, \end{aligned} \tag{18}$$

where $U_i^{n+\frac{1}{2},1} = \frac{U_i^n + U_i^{n+1,1}}{2}$, $U_i^{n+\frac{1}{2},2} = \frac{U_i^n + U_i^{n+1,2}}{2}$. Taking the inner product of (18) with W^{n+1}, we have

$$\begin{aligned} &\left(\frac{1}{\tau} - \frac{1}{2}\right) \|W^{n+1}\|^2 + \frac{\gamma}{2} \|HW_{x\bar{x}}^{n+1}\|^2 + \frac{1}{2} \|RW_x^{n+1}\|^2 \\ &+ \left\langle \left(U^{n+\frac{1}{2},1}\right)^3 - \left(U^{n+\frac{1}{2},2}\right)^3, W^{n+1} \right\rangle = 0, \end{aligned} \tag{19}$$

where

$$\langle HW_{x\bar{x}}^{n+1}, W^{n+1} \rangle = -\|RW_x^{n+1}\|^2,$$
$$\langle H^2 W_{x\bar{x}x\bar{x}}^{n+1}, W^{n+1} \rangle = \|HW_{x\bar{x}}^{n+1}\|^2$$

are used from Lemmas 3. It will be easy to see that,

$$\begin{aligned} &\left\langle \left(U^{n+\frac{1}{2},1}\right)^3 - \left(U^{n+\frac{1}{2},2}\right)^3, W^{n+1} \right\rangle = \\ &\left\langle \left[\left(U^{n+\frac{1}{2},1}\right)^2 + U^{n+\frac{1}{2},1} \cdot U^{n+\frac{1}{2},2} + \left(U^{n+\frac{1}{2},2}\right)^2\right] \right. \\ &\left. \cdot \frac{W^{n+1}}{2}, W^{n+1} \right\rangle \ge 0. \end{aligned}$$

When $\tau < 2$, then $\frac{1}{\tau} - \frac{1}{2} > 0$. That is, Eq. (19) only admits a zero solution, implying there exists unique U^{n+1} that satisfies the scheme in Eqs. (5)-(7). This completes the proof of the uniqueness of the new scheme.

Next, we define the grid function $u_i^n = u(x_i, t^n), u^n = (u_1^n, u_2^n, \cdots, u_J^n)^{\mathrm{T}}$. Then the truncation errors of compact difference scheme (5)-(7) $r^n = (r_1^n, r_2^n, \cdots, r_J^n)^{\mathrm{T}} \in R_{per}^J$ satisfy

$$\begin{aligned} &(u^n)_t + \gamma H^2 (u^{n+\frac{1}{2}})_{x\bar{x}x\bar{x}} - H(u^{n+\frac{1}{2}})_{x\bar{x}} - u^{n+\frac{1}{2}} \\ &+ (u^{n+\frac{1}{2}})^3 = r^n, \quad n = 0, 1, \cdots, N-1. \end{aligned} \tag{20}$$

and

$$u_{i+M}^n = u_i^n, \quad i = 1, 2, \cdots, M, \quad n = 0, 1, \cdots, N, \tag{21}$$

$$u_i^0 = u_0(x_i), \quad i = 1, 2, \cdots, M. \tag{22}$$

According to the Taylor expansion, we have

$$|r_i^n| \le C(\tau^2 + h^4), i = 1, 2, \cdots, J, n = 0, 1, \cdots, N. \tag{23}$$

Theorem 4. Assume that u_0 is sufficiently smooth and $u(x,t) \in C_{x,t}^{8,3}([0, L] \times [0, T])$, If τ is sufficiently small, then the solution U^n of the compact finite difference scheme (5)-(7) converges to the solution of the problem (1)-(3) with the convergence rate of $O(\tau^2 + h^4)$ in the sense of $\|\cdot\|_\infty$ norms.

Proof. Subtracting (5)-(7) from (20) to (22) and letting $e^n = u^n - U^n$, we obtain

$$\begin{aligned} &(e^n)_t + \gamma H^2 (e^{n+\frac{1}{2}})_{x\bar{x}x\bar{x}} - H(e^{n+\frac{1}{2}})_{x\bar{x}} - e^{n+\frac{1}{2}} \\ &+ (u^{n+\frac{1}{2}})^3 - (U^{n+\frac{1}{2}})^3 = r^n, \quad n = 0, 1, \cdots, N-1, \end{aligned} \tag{24}$$

and

$$e_{i+M}^n = e_i^n, \quad i = 1, 2, \cdots, M, \quad n = 0, 1, \cdots, N, \tag{25}$$

$$e_i^0 = 0, \quad i = 1, 2, \cdots, M. \tag{26}$$

Taking the inner product of (24) with $e^{n+\frac{1}{2}}$, we obtain

$$\begin{aligned} &\frac{\|e^{n+1}\|^2 - \|e^n\|^2}{2\tau} + \gamma \|He_{x\bar{x}}^{n+\frac{1}{2}}\|^2 + \|Re_x^{n+\frac{1}{2}}\|^2 = \\ &\langle e^{n+\frac{1}{2}} + (U^{n+\frac{1}{2}})^3 - (u^{n+\frac{1}{2}})^3, e^{n+\frac{1}{2}} \rangle + \langle r^n, e^{n+\frac{1}{2}} \rangle. \end{aligned} \tag{27}$$

where

$$\langle e_t^n, e^{n+\frac{1}{2}} \rangle = \frac{\|e^{n+1}\|^2 - \|e^n\|^2}{2\tau},$$
$$\langle H^2 e_{x\bar{x}x\bar{x}}^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \rangle = \|He_{x\bar{x}}^{n+\frac{1}{2}}\|^2,$$
$$\langle He_{x\bar{x}}^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \rangle = -\|Re_x^{n+\frac{1}{2}}\|^2$$

are used. In addition,

$$\begin{aligned} &\langle e^{n+\frac{1}{2}} + (U^{n+\frac{1}{2}})^3 - (u^{n+\frac{1}{2}})^3, e^{n+\frac{1}{2}} \rangle \\ &\le \|e^{n+\frac{1}{2}} + (U^{n+\frac{1}{2}})^3 - (u^{n+\frac{1}{2}})^3\| \cdot \|e^{n+\frac{1}{2}}\|, \end{aligned} \tag{28}$$

and

$$\begin{aligned} &\langle r^n, e^{n+\frac{1}{2}} \rangle \le \|r^n\| \cdot \|e^{n+\frac{1}{2}}\| \le \frac{1}{2}\|r^n\|^2 + \frac{1}{2}\|e^{n+\frac{1}{2}}\|^2 \\ &\le \frac{1}{2}\|r^n\|^2 + \frac{1}{4}(\|e^{n+1}\|^2 + \|e^n\|^2). \end{aligned} \tag{29}$$

For the nonlinear term $\|e^{n+\frac{1}{2}} + (U^{n+\frac{1}{2}})^3 - (u^{n+\frac{1}{2}})^3\|$, we use the boundedness of $\|U^n\|_\infty$ to find that

$$\|e^{n+\frac{1}{2}} + (U^{n+\frac{1}{2}})^3 - (u^{n+\frac{1}{2}})^3\| \le C\|e^{n+\frac{1}{2}}\|. \tag{30}$$

Using (27)-(30), we can obtain that

$$\|e^{n+1}\|^2 - \|e^n\|^2 \le \tau\|r^n\|^2 + C\tau(\|e^{n+1}\|^2 + \|e^n\|^2),$$

which is equivalent to

$$\|e^{n+1}\|^2 - \|e^n\|^2 \le \frac{2C\tau}{1 - C\tau}\|e^n\|^2 + \frac{\tau}{1 - C\tau}\|r^n\|^2. \tag{31}$$

Summing (31) from 0 to $n-1$, we get

$$\begin{aligned} \|e^n\|^2 \le \|e^0\|^2 + \frac{2C\tau}{1 - C\tau} \sum_{l=0}^{n-1} \|e^l\|^2 \\ + \frac{\tau}{1 - C\tau} \sum_{l=0}^{n-1} \|r^l\|^2, \end{aligned} \tag{32}$$

where $1 \le n \le N$, and

$$\tau \sum_{l=0}^{n-1} \|r^l\|^2 \le n\tau \max_{1 \le l \le n-1} \|r^l\|^2 \le T \cdot C(\tau^2 + h^4)^2.$$

Since $e_i^0 = 0$. If τ, which is sufficiently small, satisfies $\tau < \frac{1}{2C}$, then $1 - C\tau > 0$ and (32) gives

$$\|e^n\|^2 \le C(\tau^2 + h^4)^2 + 4C\tau \sum_{l=0}^{n-1} \|e^l\|^2, \ 1 \le n \le N.$$

Using Lemma 7, we obtain

$$\|e^n\|^2 \le C(\tau^2+h^4)^2, \quad 1\le n\le N.$$

Thus,

$$\|e^n\| \le C(\tau^2+h^4), \quad 1\le n\le N. \tag{33}$$

Now taking the inner product of (24) with e_t^n, we obtain

$$\begin{aligned}&\|e_t^n\|^2+\gamma\frac{\|He_{x\bar{x}}^{n+1}\|^2-\|He_{x\bar{x}}^n\|^2}{2\tau}\\&+\frac{\|Re_x^{n+1}\|^2-\|Re_x^n\|^2}{2\tau}\\&=\langle e^{n+\frac12}+(U^{n+\frac12})^3-(u^{n+\frac12})^3,e_t^n\rangle+\langle r^n,e_t^n\rangle,\end{aligned}\tag{34}$$

where

$$\langle H^2e_{x\bar{x}x\bar{x}}^{n+\frac12},e_t^n\rangle=\frac{\|He_{x\bar{x}}^{n+1}\|^2-\|He_{x\bar{x}}^n\|^2}{2\tau},$$
$$\langle He_{x\bar{x}}^{n+\frac12},e_t^n\rangle=-\frac{\|Re_x^{n+1}\|^2-\|Re_x^n\|^2}{2\tau}$$

are used. In addition,

$$\begin{aligned}&\langle e^{n+\frac12}+(U^{n+\frac12})^3-(u^{n+\frac12})^3,e_t^n\rangle\\&\le\|e^{n+\frac12}+(U^{n+\frac12})^3-(u^{n+\frac12})^3\|\cdot\|e_t^n\|\\&\le C\|e^{n+\frac12}\|^2+\frac12\|e_t^n\|^2,\end{aligned}\tag{35}$$

and

$$\langle r^n,e_t^n\rangle\le\|r^n\|\cdot\|e_t^n\|\le\frac12\|r^n\|^2+\frac12\|e_t^n\|^2. \tag{36}$$

Using (34)-(36), we can obtain that

$$\begin{aligned}&\gamma\frac{\|He_{x\bar{x}}^{n+1}\|^2-\|He_{x\bar{x}}^n\|^2}{2\tau}+\frac{\|Re_x^{n+1}\|^2-\|Re_x^n\|^2}{2\tau}\\&\le C\|e^{n+\frac12}\|^2+\frac12\|r^n\|^2\\&\le\frac{C}{2}(\|e^{n+1}\|^2+\|e^n\|^2)+\frac12\|r^n\|^2.\end{aligned}$$

From (33) and $|r_i^n|\le C(\tau^2+h^4)$, we obtain that

$$\begin{aligned}&\gamma(\|He_{x\bar{x}}^{n+1}\|^2-\|He_{x\bar{x}}^n\|^2)+(\|Re_x^{n+1}\|^2-\|Re_x^n\|^2)\\&\le C(\tau^2+h^4)^2.\end{aligned}\tag{37}$$

Let $E^n=\gamma\|He_{x\bar{x}}^n\|^2+\|Re_x^n\|^2$, then,

$$E^{n+1}-E^n\le C(\tau^2+h^4)^2,\quad 0\le n\le N-1. \tag{38}$$

Summing (38) from 0 to $n-1$, we get

$$E^n\le E^0+C(\tau^2+h^4)^2.$$

For $e_i^0=0$, we have

$$E^n\le C(\tau^2+h^4)^2, 1\le n\le N.$$

Using Lemma 4, we obtain

$$\|e_x^n\|\le\|Re_x^n\|\le C(\tau^2+h^4), 1\le n\le N. \tag{39}$$

Combining (33) and (39), we obtain that

$$\|e^n\|_{H^1}\le C(\tau^2+h^4), 1\le n\le N.$$

By Lemma 6, we obtain

$$\|e^n\|_\infty\le C(\tau^2+h^4), 1\le n\le N. \tag{40}$$

This completes the proof.

III. Linearized compact difference scheme

Compared with the nonlinear scheme, the linearized scheme is the most effective in terms of accuracy and computational cost. Then in this section, we propose a three-level linearized compact finite difference scheme for the EFK problem (1)-(3)

$$\begin{aligned}&(U_i^n)_{\hat{t}}+\gamma(L_x^{-1})^2(\bar{U}_i^n)_{x\bar{x}x\bar{x}}-L_x^{-1}(\bar{U}_i^n)_{x\bar{x}}\\&-\bar{U}_i^n+(U_i^n)^2\bar{U}_i^n=0,\end{aligned}\tag{41}$$

where $n=1,2,\cdots,N-1, j=1,2,\cdots,J$. Since the scheme (41) is a three-time-level method, to start the computation, we may get U^1 by the following two levels in time method (4) as

$$\begin{aligned}&(U_i^0)_t+\gamma(L_x^{-1})^2(U_i^{\frac12})_{x\bar{x}x\bar{x}}-L_x^{-1}(U_i^{\frac12})_{x\bar{x}}\\&-U_i^{\frac12}+(U_i^{\frac12})^3=0, i=1,2,\cdots,J.\end{aligned}\tag{42}$$

Thus, the compact linearized finite-difference scheme (41) and (42) can be rewritten in the following matrix form

$$\begin{aligned}&(U^n)_{\hat{t}}+\gamma H^2(\bar{U}^n)_{x\bar{x}x\bar{x}}-H(\bar{U}^n)_{x\bar{x}}\\&-\bar{U}^n+(U^n)^2\bar{U}^n=0, n=1,2,\cdots,N-1,\end{aligned}\tag{43}$$

$$\begin{aligned}&(U^0)_t+\gamma H^2(U^{\frac12})_{x\bar{x}x\bar{x}}-H(U^{\frac12})_{x\bar{x}}\\&-U^{\frac12}+(U^{\frac12})^3=0,\end{aligned}\tag{44}$$

and the initial-boundary conditions are discretized as

$$U_{i+M}^n=U_i^n,\quad i=1,2,\cdots,M,\quad n=0,1,\cdots,N, \tag{45}$$

$$U_i^0=u_0(x_i),\quad i=1,2,\cdots,M. \tag{46}$$

A. *Existence and prior estimates*

Theorem 5. There exists $U^{n+1}\in R_{per}^J$ which satisfies the scheme (43)-(46).

Proof. In order to prove the theorem by the mathematical induction. It is obvious that U^0 and U^1 are uniquely solvable by (46) and (44), respectively. Assume there exist $U^1,U^2,\cdots,U^n\in R_{per}^J$ which satisfy the scheme (43)-(46), as $n\le N-1$, now we try to prove that $U^{n+1}\in R_{per}^J$, satisfies the scheme (43)-(46).

We define ω on R_{per}^J as follows

$$\begin{aligned}\omega(\nu)=&\nu-U^{n-1}+\tau\gamma H^2\nu_{x\bar{x}x\bar{x}}\\&-\tau H\nu_{x\bar{x}}-\tau\nu+\tau(U^n)^2\nu.\end{aligned}\tag{47}$$

Computing the inner product of (47) with ν and using Lemma 3, we obtain

$$\begin{aligned}\langle\omega(\nu),\nu\rangle&=\|\nu\|^2-\langle U^{n-1},\nu\rangle+\tau\gamma\|H\nu_{x\bar{x}}\|^2\\&+\tau\|R\nu_x\|^2-\tau\|\nu\|^2+\tau\langle(U^n)^2\nu,\nu\rangle\\&\ge\|\nu\|^2-\frac12\|\nu\|^2-\frac12\|U^{n-1}\|^2-\tau\|\nu\|^2\\&=(\frac12-\tau)\|\nu\|^2-\frac12\|U^{n-1}\|^2.\end{aligned}$$

Hence, when $\tau<\frac12$, for $\forall\nu\in R_{per}^J, \|\nu\|^2=\frac{1}{1-2\tau}\|U^{n-1}\|^2+1$, there exists $\langle\omega(\nu),\nu\rangle\ge0$. It follows from Lemma 5 that exists $\nu^*\in R_{per}^J$ which satisfies $\omega(\nu^*)=0$. Let $U^{n+1}=2\nu-U^{n-1}$, then it can be proved that $U^{n+1}\in R_{per}^J$ is the solution of the scheme (43)-(46). This completes the proof of Theorem 5.

Theorem 6. Suppose $u^0\in C^{8,3}[0,L]$, if τ is sufficiently

small, then the solution of finite difference scheme (43)-(46) satisfies $\|U^n\|_\infty \le C$ for any $n \ge 2$.
Proof. Taking the inner product of $\bar{U}^n$ for both sides of (43), we obtain

$$\begin{aligned}&\frac{\|U^{n+1}\|^2 - \|U^{n-1}\|^2}{4\tau} + \gamma\|H\bar{U}^n_{x\bar{x}}\|^2 + \|R\bar{U}^n_x\|^2 \\ &- \|\bar{U}^n\|^2 + \langle (U^n)^2\bar{U}, \bar{U}^n\rangle = 0,\end{aligned} \tag{48}$$

where

$$\begin{aligned}&\langle U^n_{\hat{t}}, \bar{U}^n\rangle = \frac{\|U^{n+1}\|^2 - \|U^{n-1}\|^2}{4\tau},\\ &\langle H^2\bar{U}^n_{x\bar{x}x\bar{x}}, \bar{U}^n\rangle = \|H\bar{U}^n_{x\bar{x}}\|^2,\\ &\langle H\bar{U}^n_{x\bar{x}}, \bar{U}^n\rangle = -\|R\bar{U}^n_x\|^2\end{aligned}$$

are used. From (48), we get

$$\begin{aligned}&\frac{\|U^{n+1}\|^2 - \|U^{n-1}\|^2}{4\tau} = -\gamma\|H\bar{U}^n_{x\bar{x}}\|^2 - \|R\bar{U}^n_x\|^2 \\ &+ \|\bar{U}^n\|^2 - \langle (U^n)^2\bar{U}, \bar{U}^n\rangle \le \|\bar{U}^n\|^2 \\ &\le \frac{1}{2}(\|U^{n+1}\|^2 + \|U^{n-1}\|^2),\end{aligned} \tag{49}$$

where $\langle (U^n)^2\bar{U}, \bar{U}^n\rangle \ge 0$ is used.

Thus,

$$\begin{aligned}&\|U^{n+1}\|^2 - \|U^{n-1}\|^2 \le 2\tau(\|U^{n+1}\|^2 + \|U^{n-1}\|^2)\\ &\le 2\tau(\|U^{n+1}\|^2 + 2\|U^n\|^2 + \|U^{n-1}\|^2).\end{aligned} \tag{50}$$

Let $F^n = \|U^n\|^2 + \|U^{n-1}\|^2$, then

$$F^{n+1} - F^n \le 2\tau(F^{n+1} + F^n),$$

which is equivalent to

$$(1-2\tau)(F^{n+1} - F^n) \le 4\tau F^n. \tag{51}$$

If τ, which is sufficiently small, satisfies $\tau \le \frac{1}{3}$ then $1-2\tau \ge \frac{1}{3}$ and (51) gives

$$F^{n+1} - F^n \le \frac{4}{1-2\tau}\tau F^n \le 12\tau F^n. \tag{52}$$

Then, summing (52) from 1 to $n-1$, we get

$$F^n \le F^1 + 12\tau\sum_{l=1}^{n-1} D^l,$$

for any $2 \le n \le N$.

Through Lemma 7, we obtain that

$$F^n \le F^1 e^{12n\tau} \le F^1 e^{12T}.$$

Then,

$$\|U^n\| \le C. \tag{53}$$

Now taking the inner product of $U^n_{\hat{t}}$ for both sides of (43), we obtain the following identity

$$\begin{aligned}&\|U^n_{\hat{t}}\|^2 + \gamma\frac{\|HU^{n+1}_{x\bar{x}}\|^2 - \|HU^{n-1}_{x\bar{x}}\|^2}{4\tau}\\ &+ \frac{\|RU^{n+1}_x\|^2 - \|RU^{n-1}_x\|^2}{4\tau} - \langle \bar{U}^n, U^n_{\hat{t}}\rangle\\ &+ \langle (U^n)^2\bar{U}^n, U^n_{\hat{t}}\rangle = 0,\end{aligned} \tag{54}$$

where

$$\begin{aligned}&\langle H^2\bar{U}^n_{x\bar{x}x\bar{x}}, U^n_{\hat{t}}\rangle = \frac{\|HU^{n+1}_{x\bar{x}}\|^2 - \|HU^{n-1}_{x\bar{x}}\|^2}{4\tau},\\ &\langle H\bar{U}^n_{x\bar{x}}, U^n_{\hat{t}}\rangle = -\frac{\|RU^{n+1}_x\|^2 - \|RU^{n-1}_x\|^2}{4\tau}\end{aligned}$$

are used. From (54), we can obtain that

$$\begin{aligned}&\gamma\frac{\|HU^{n+1}_{x\bar{x}}\|^2 - \|HU^{n-1}_{x\bar{x}}\|^2}{4\tau} + \frac{\|RU^{n+1}_x\|^2 - \|RU^{n-1}_x\|^2}{4\tau}\\ &= -\|U^n_{\hat{t}}\|^2 + \langle \bar{U}^n, U^n_{\hat{t}}\rangle - \langle (U^n)^2\bar{U}^n, U^n_{\hat{t}}\rangle\\ &\le -\|U^n_{\hat{t}}\|^2 + \|\bar{U}^n\|\cdot\|U^n_{\hat{t}}\| + \|(U^n)^2\bar{U}^n\|\cdot\|U^n_{\hat{t}}\|\\ &\le -\|U^n_{\hat{t}}\|^2 + \frac{1}{2}\|\bar{U}^n\|^2 + \frac{1}{2}\|U^n_{\hat{t}}\|^2\\ &\quad + \frac{1}{2}\|(U^n)^2\bar{U}^n\|^2 + \frac{1}{2}\|U^n_{\hat{t}}\|^2\\ &\le \frac{1}{2}\|\bar{U}^n\|^2 + \frac{1}{2}\|(U^n)^2\|^2\cdot\|\bar{U}^n\|^2 \le C,\end{aligned}$$

where (53) is been used.

Let $G^n = \gamma(\|HU^n_{x\bar{x}}\|^2 + \|HU^{n-1}_{x\bar{x}}\|^2) + (\|RU^n_x\|^2 + \|RU^{n-1}_x\|^2)$, then,

$$G^{n+1} - G^n \le 4\tau C, \quad 1 \le n \le N-1. \tag{55}$$

Summing (55) from 1 to $n-1$, we get

$$\begin{aligned}&G^n \le G^1 + 4\tau(n-1)C \le G^1 + 4CT \le C,\\ &2 \le n \le N-1.\end{aligned}$$

Thus, from Lemma 4, we obtain that

$$\|U^n_x\| \le \|RU^n_x\| \le C, 2 \le n \le N. \tag{56}$$

Combining (53) and (56), we obtain that

$$\|U^n\|_{H^1} \le C, 2 \le n \le N.$$

By Lemma 6, we can get that

$$\|U^n\|_\infty \le C, 2 \le n \le N. \tag{57}$$

This completes the proof.

B. uniqueness and Convergence

Theorem 7. If τ is sufficiently small, then the finite difference scheme (43)-(46) has a unique solution.
Proof. To prove the theorem, we proceed by the mathematical induction. It is obvious that U^0 and U^1 are uniquely solvable by (46) and (44), respectively. Suppose $U^0, U^1, \cdots, U^n (1 \le n \le N-1)$ are solved uniquely, we now consider (43) for U^{n+1}. Assume that $U^{n+1,1}, U^{n+1,2}$ are two solutions of (43) and let $\widetilde{W}^{n+1} = U^{n+1,1} - U^{n+1,2}$, then it is easy to verify that $\widetilde{W}^{n+1}$ satisfies the following equation

$$\begin{aligned}&\frac{1}{2\tau}\widetilde{W}^{n+1}_i + \frac{\gamma}{2}H^2(\widetilde{W}^{n+1}_i)_{x\bar{x}x\bar{x}} - \frac{1}{2}H(\widetilde{W}^{n+1}_i)_{x\bar{x}}\\ &- \frac{1}{2}\widetilde{W}^{n+1}_i + \frac{1}{2}(U^n)^2\widetilde{W}^{n+1}_i = 0,\end{aligned} \tag{58}$$

Taking the inner product of (58) with $\widetilde{W}^{n+1}$, we have

$$\begin{aligned}&\left(\frac{1}{2\tau} - \frac{1}{2}\right)\|\widetilde{W}^{n+1}\|^2 + \frac{\gamma}{2}\|H\widetilde{W}^{n+1}_{x\bar{x}}\|^2\\ &+ \frac{1}{2}\|R\widetilde{W}^{n+1}_x\|^2 + \frac{1}{2}\langle (U^n)^2\widetilde{W}^{n+1}_i, \widetilde{W}^{n+1}\rangle = 0,\end{aligned} \tag{59}$$

where

$$\begin{aligned}&\langle H^2\widetilde{W}^{n+1}_{x\bar{x}x\bar{x}}, \widetilde{W}^{n+1}\rangle = \|H\widetilde{W}^{n+1}_{x\bar{x}}\|^2,\\ &\langle H\widetilde{W}^{n+1}_{x\bar{x}}, \widetilde{W}^{n+1}\rangle = -\|R\widetilde{W}^{n+1}_x\|^2\end{aligned}$$

are used from Lemmas 3. It will be easy to see that $\langle (U^n)^2\widetilde{W}^{n+1}_i, \widetilde{W}^{n+1}\rangle \ge 0$.

When $\tau < 1$, then $\frac{1}{2\tau} - \frac{1}{2} > 0$. That is, (59) only admits a zero solution, implying there exists unique U^{n+1} that satisfies the scheme in (43)-(46). This completes the proof of the existence and uniqueness of the new scheme.

Next, we define the grid function $u_i^n = u(x_i, t^n), u^n = (u_1^n, u_2^n, \cdots, u_J^n)^{\mathrm{T}}$. Then the truncation errors of compact difference scheme (43)-(46) $r^n = (r_1^n, r_2^n, \cdots, r_J^n)^{\mathrm{T}} \in R_{per}^J$ satisfy

$$\begin{aligned}&(u^n)_{\hat{t}} + \gamma H^2(\bar{u}^n)_{x\bar{x}x\bar{x}} - H(\bar{u}^n)_{x\bar{x}} - \bar{u}^n \\ &+ (u^n)^2\bar{u}^n = r^n, n = 1, 2, \cdots, N-1,\end{aligned} \tag{60}$$

$$\begin{aligned}&(u^0)_t + \gamma H^2(u^{\frac{1}{2}})_{x\bar{x}x\bar{x}} - H(u^{\frac{1}{2}})_{x\bar{x}} - u^{\frac{1}{2}} \\ &+ (u^{\frac{1}{2}})^3 = r^0,\end{aligned} \tag{61}$$

and

$$u_{i+M}^n = u_i^n, \quad i = 1, 2, \cdots, M, \quad n = 0, 1, \cdots, N, \tag{62}$$

$$u_i^0 = u_0(x_i), \quad i = 1, 2, \cdots, M. \tag{63}$$

According to the Taylor expansion, we have

$$|r_i^n| \le C(\tau^2 + h^4), i = 1, 2, \cdots, J, n = 0, 1, \cdots, N. \tag{64}$$

Theorem 8. Assume that u_0 is sufficiently smooth and $u(x,t) \in C_{x,t}^{8,3}([0,L] \times [0,T])$, If τ is sufficiently small, then the solution U^n of the compact finite difference scheme (43)-(46) converges to the solution of the problem (1)-(3) with the convergence rate of $O(\tau^2 + h^4)$ in the sense of $\|\cdot\|_\infty$ norms.

Proof. Subtracting (43)-(46) from (60) to (63) and letting $e^n = u^n - U^n$, we obtain

$$\begin{aligned}&(e^n)_{\hat{t}} + \gamma H^2(\bar{e}^n)_{x\bar{x}x\bar{x}} - H(\bar{e}^n)_{x\bar{x}} - \bar{e}^n + (u^n)^2\bar{u}^n \\ &- (U^n)^2\bar{U}^n = r^n, n = 1, 2, \cdots, N-1,\end{aligned} \tag{65}$$

$$\begin{aligned}&(e^0)_t + \gamma H^2(e^{\frac{1}{2}})_{x\bar{x}x\bar{x}} - H(e^{\frac{1}{2}})_{x\bar{x}} - e^{\frac{1}{2}} \\ &+ (u^{\frac{1}{2}})^3 - (U^{\frac{1}{2}})^3 = r^0,\end{aligned} \tag{66}$$

and

$$e_{i+M}^n = e_i^n, \quad i = 1, 2, \cdots, M, \quad n = 0, 1, \cdots, N, \tag{67}$$

$$e_i^0 = 0, \quad i = 1, 2, \cdots, M. \tag{68}$$

From Theorem 4, we obtain

$$\|e^1\|_\infty \le C(\tau^2 + h^4). \tag{69}$$

Taking the inner product of (65) with $\bar{e}^n$, we obtain

$$\begin{aligned}&\frac{\|e^{n+1}\|^2 - \|e^{n-1}\|^2}{4\tau} + \gamma\|H\bar{e}_{x\bar{x}}^n\|^2 + \|R\bar{e}_x^n\|^2 = \\ &\|\bar{e}^n\|^2 - \langle (u^n)^2\bar{u}^n - (U^n)^2\bar{U}^n, \bar{e}^n\rangle + \langle r^n, \bar{e}^n\rangle.\end{aligned} \tag{70}$$

where

$$\begin{aligned}&\langle e_{\hat{t}}^n, \bar{e}^n\rangle = \frac{\|e^{n+1}\|^2 - \|e^{n-1}\|^2}{4\tau}, \\ &\langle H^2\bar{e}_{x\bar{x}x\bar{x}}^n, \bar{e}^n\rangle = \|H\bar{e}_{x\bar{x}}^n\|^2, \\ &\langle H\bar{e}_{x\bar{x}}^n, \bar{e}^n\rangle = -\|R\bar{e}_x^n\|^2\end{aligned}$$

are used. In addition,

$$\|\bar{e}^n\|^2 \le \frac{1}{2}(\|e^{n+1}\|^2 + \|e^{n-1}\|^2), \tag{71}$$

$$\begin{aligned}&\langle (u^n)^2\bar{u}^n - (U^n)^2\bar{U}^n, \bar{e}^n\rangle = \\ &\langle (u^n)^2\bar{e}^n, \bar{e}^n\rangle + \langle (u^n + U^n)\bar{U}^n e^n, \bar{e}^n\rangle \\ &\le C[\langle \bar{e}^n, \bar{e}^n\rangle + \langle e^n, \bar{e}^n\rangle] \\ &\le C(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2),\end{aligned} \tag{72}$$

and

$$\begin{aligned}&\langle r^n, \bar{e}^n\rangle \le \|r^n\| \cdot \|\bar{e}^n\| \le \frac{1}{2}\|r^n\|^2 + \frac{1}{2}\|\bar{e}^n\|^2 \\ &\le \frac{1}{2}\|r^n\|^2 + \frac{1}{4}(\|e^{n+1}\|^2 + \|e^{n-1}\|^2).\end{aligned} \tag{73}$$

Using (70)-(73), we can obtain that

$$\begin{aligned}&\|e^{n+1}\|^2 - \|e^{n-1}\|^2 \le \\ &C\tau(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2) + 2\tau\|r^n\|^2.\end{aligned}$$

Let $P^n = \|e^n\|^2 + \|e^{n-1}\|^2$, then

$$P^{n+1} - P^n \le 2\tau\|r^n\|^2 + C\tau(P^{n+1} + P^n),$$

which is equivalent to

$$P^{n+1} - P^n \le \frac{2C\tau}{1 - C\tau}P^n + \frac{2\tau}{1 - C\tau}\|r^n\|^2. \tag{74}$$

Summing (74) from 1 to $n-1$, we get

$$P^n \le P^1 + \frac{2C\tau}{1 - C\tau}\sum_{l=1}^{n-1} P^l + \frac{2\tau}{1 - C\tau}\sum_{l=1}^{n-1}\|r^l\|^2, \tag{75}$$

where $2 \le n \le N$. If τ, which is sufficiently small, satisfies $\tau \le \frac{1}{2C}$, then $1 - C\tau \ge \frac{1}{2}$ and (75) gives

$$P^n \le P^1 + 4C\tau\sum_{l=1}^{n-1} P^l + 4\tau\sum_{l=1}^{n-1}\|r^l\|^2, \tag{76}$$

where $2 \le n \le N$, and

$$\tau\sum_{l=0}^{n-1}\|r^l\|^2 \le n\tau \max_{1 \le l \le n-1}\|r^l\|^2 \le T \cdot C(\tau^2 + h^4)^2.$$

Since $e_i^0 = 0$, and the scheme (44) is used to compute U^1, we have $P^1 \le C(\tau^2 + h^4)^2$ followed by a simple analysis for the scheme (44). Therefore

$$P^n \le C(\tau^2 + h^4)^2 + 4C\tau\sum_{l=1}^{n-1}\|P^l\|^2, \quad 2 \le n \le N.$$

Using Lemma 3, we obtain

$$P^n \le C(\tau^2 + h^4)^2, \quad 2 \le n \le N.$$

Thus,

$$\|e^n\| \le C(\tau^2 + h^4), \quad 2 \le n \le N. \tag{77}$$

Now taking the inner product of (65) with $e_{\hat{t}}^n$, we obtain

$$\begin{aligned}&\|e_{\hat{t}}^n\|^2 + \gamma\frac{\|He_{x\bar{x}}^{n+1}\|^2 - \|He_{x\bar{x}}^{n-1}\|^2}{4\tau} \\ &+ \frac{\|Re_x^{n+1}\|^2 - \|Re_x^{n-1}\|^2}{4\tau} \\ &= \langle \bar{e}^n + (U^n)^2\bar{U}^n - (u^n)^2\bar{u}^n, e_{\hat{t}}^n\rangle + \langle r^n, e_{\hat{t}}^n\rangle,\end{aligned} \tag{78}$$

where

$$\begin{aligned}&\langle H^2\bar{e}_{x\bar{x}x\bar{x}}^n, e_{\hat{t}}^n\rangle = \frac{\|He_{x\bar{x}}^{n+1}\|^2 - \|He_{x\bar{x}}^{n-1}\|^2}{4\tau}, \\ &\langle H\bar{e}_{x\bar{x}}^n, e_{\hat{t}}^n\rangle = -\frac{\|Re_x^{n+1}\|^2 - \|Re_x^{n-1}\|^2}{4\tau}\end{aligned}$$

are used. In addition,

$$\gamma\frac{\|He_{x\bar{x}}^{n+1}\|^2-\|He_{x\bar{x}}^{n-1}\|^2}{4\tau}+\frac{\|Re_x^{n+1}\|^2-\|Re_x^{n-1}\|^2}{4\tau}$$
$$=-\|e_{\hat{t}}^n\|^2+\langle\bar{e}^n+(U^n)^2\bar{U}^n-(u^n)^2\bar{u}^n,e_{\hat{t}}^n\rangle+\langle r^n,e_{\hat{t}}^n\rangle$$
$$\le-\|e_{\hat{t}}^n\|^2+\frac{1}{2}\|\bar{e}^n+(U^n)^2\bar{U}^n-(u^n)^2\bar{u}^n\|^2+\frac{1}{2}\|e_{\hat{t}}^n\|^2$$
$$+\frac{1}{2}\|r^n\|^2+\frac{1}{2}\|e_{\hat{t}}^n\|^2$$
$$=\frac{1}{2}\|\bar{e}^n+(U^n)^2\bar{U}^n-(u^n)^2\bar{u}^n\|^2+\frac{1}{2}\|r^n\|^2$$
$$=\frac{1}{2}\|((u^n)^2-1)\bar{e}^n+(u^n+U^n)\bar{U}^ne^n\|^2+\frac{1}{2}\|r^n\|^2$$
$$\le C(\|\bar{e}^n\|^2+\|e^n\|^2)+\frac{1}{2}\|r^n\|^2$$
$$\le\frac{C}{2}(\|e^{n+1}\|^2+2\|e^n\|^2+(\|e^{n-1}\|^2)+\frac{1}{2}\|r^n\|^2,$$

where Cauchy-Schwarz inequality and Theorem 6 are used. Thus,

$$\begin{aligned}&\gamma(\|He_{x\bar{x}}^{n+1}\|^2-\|He_{x\bar{x}}^{n-1}\|^2)\\&+(\|Re_x^{n+1}\|^2-\|Re_x^{n-1}\|^2)\\&\le2C\tau(\|e^{n+1}\|^2+2\|e^n\|^2+(\|e^{n-1}\|^2)+2\tau\|r^n\|^2.\end{aligned}\tag{79}$$

From (79) and $|r_i^n|\le C(\tau^2+h^4)$, we obtain that

$$\begin{aligned}&\gamma(\|He_{x\bar{x}}^{n+1}\|^2-\|He_{x\bar{x}}^{n-1}\|^2)+\|Re_x^{n+1}\|^2\\&-\|Re_x^{n-1}\|^2\le C(\tau^2+h^4)^2.\end{aligned}$$

Let $Q^n=\gamma(\|He_{x\bar{x}}^n\|^2+\|He_{x\bar{x}}^{n-1}\|^2)+\|Re_x^n\|^2+\|Re_x^{n-1}\|^2$, then,

$$Q^{n+1}-Q^n\le C(\tau^2+h^4)^2,\quad 1\le n\le N-1.\tag{80}$$

Summing (80) from 1 to $n-1$, we get

$$Q^n\le Q^1+C(\tau^2+h^4)^2.$$

Since $e_i^0=0$, and the scheme (44) is used to compute U^1, we have $Q^1\le C(\tau^2+h^4)^2$ followed by a simple analysis for the scheme (44). Therefore

$$Q^n\le C(\tau^2+h^4)^2,2\le n\le N.$$

Using Lemma 3, we obtain that

$$\|e_x^n\|\le\|Re_x^n\|\le C(\tau^2+h^4),2\le n\le N.\tag{81}$$

Combining (77) and (81), we obtain that

$$\|e^n\|_{H^1}\le C(\tau^2+h^4),2\le n\le N.$$

By Lemma 6, we can get that

$$\|e^n\|_\infty\le C(\tau^2+h^4),2\le n\le N.\tag{82}$$

This completes the proof.

IV. Numerical experiments

In this section, we present some numerical experiments to validate our theoretical results. For convenience, we denote the nonlinear compact difference scheme (5)-(7) as Scheme A and the linearized compact difference scheme (43)-(46) as Scheme B. We consider the following periodic initial value problem of the inhomogeneous EFK equation [7, 8]

$$\begin{aligned}&u_t+\gamma u_{xxxx}-u_{xx}-u+u^3=f(x,t),\\&x\in\Omega=(0,1),\quad 0<t\le T,\end{aligned}\tag{83}$$

$$u(x,0)=u_0(x),\quad x\in\Omega,\tag{84}$$

$$u(x+1,t)=u(x,t),\quad x\in\Omega,\quad 0<t\le T,\tag{85}$$

where $f(x,t)=e^{-t}\sin(2\pi x)[-2+4\pi^2+16\gamma\pi^4+\sin^2(2\pi x)e^{-2t}]$. The problem (83)-(85) has the following exact solution

$$u(x,t)=e^{-t}\sin(2\pi x).\tag{86}$$

In this case, we choose the parameter $\gamma=0.01$. To investigate the accuracy of the present schemes, we computed the $\|\cdot\|_\infty$ norm error of the numerical solutions (83)-(85). if τ is sufficiently small, then $e(h,\tau)=O(h^{q_1}+\tau^{q_1})\approx O(h^{q_1})$. Consequently, $e(2h,\tau)/e(h,\tau)\approx2^{q_1}$, hence, $q_1\approx\log_2[e(h,2\tau)/e(h,\tau)]$ is the convergence order with respect to h. Likewise, if h is sufficiently small, $q_2\approx\log_2[e(h,2\tau)/e(h,\tau)]$ is the convergence rate with respect to τ. In our computation, we calculated the convergence orders based on the following formula as [25, 31]

$$\text{Rate}_h=\log_2\Big(\frac{e(2h,\tau)}{e(h,\tau)}\Big),\qquad\text{Rate}_\tau=\log_2\Big(\frac{e(h,2\tau)}{e(h,\tau)}\Big).$$

Tables 1 and 2 give the comparison of error results and CPU times between the present schemes and the methods proposed in [7, 8]. From Tables 1 and 2, we can see that the convergence orders of the present schemes are equal to $O(\tau^2+h^4)$, which confirms the theoretical order of convergence obtained in Theorems 4 and 8. Furthermore, we observe that the errors from the present schemes are much smaller than that obtained based on the methods in [7, 8]. Also, the present schemes have relatively less computational cost than the methods in [7, 8]. Thus, we can conclude that the present two compact schemes are more effective than the schemes in [7, 8].

V. Conclusions

We have developed two conservative and fourth-order compact finite-difference schemes for the initial value problem of the EFK equation. Both schemes have been shown to be second-order convergent in time and fourth-order convergent in space. The existence, uniqueness, and unconditional stability of the numerical solutions are proved. Numerical experiments show that the present schemes provide accurate numerical solutions which coincide with the theoretical results.

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TABLE I
COMPARISON OF ERRORS AND SPATIAL CONVERGENCE ORDER WITH VARIOUS h AND $\tau = 0.00001$ AT $T = 1$.

h	Scheme A	Rate$_h$	CPU	Nonlinear [7]	Rate$_h$	CPU	Scheme B	Rate$_h$	CPU	Linear [8]	Rate$_h$	CPU
1/20	1.9838×10^{-5}	-	4.0916s	5.9270×10^{-5}	-	16.3407s	1.9838×10^{-5}	-	1.4577s	5.9270×10^{-5}	-	5.8484s
1/40	1.2362×10^{-6}	4.0043	7.0970s	3.7365×10^{-6}	3.9875	24.4892s	1.2362×10^{-6}	4.0043	2.9194s	3.7365×10^{-6}	3.9875	8.9232s
1/80	7.7202×10^{-8}	4.0012	12.2146s	2.3405×10^{-7}	3.9968	46.8838s	7.7189×10^{-8}	4.0014	4.7710s	2.3403×10^{-7}	3.9969	18.1076s
1/160	4.8024×10^{-9}	4.0068	23.5712s	1.4675×10^{-8}	3.9954	98.1835s	4.7782×10^{-9}	4.0139	8.7946s	1.4654×10^{-8}	3.9973	38.6512s

TABLE II
COMPARISON OF ERRORS AND SPATIAL CONVERGENCE ORDER WITH VARIOUS τ AND $h = 0.001$ AT $T = 1$.

τ	Scheme A	Rate$_\tau$	CPU	Nonlinear [7]	Rate$_\tau$	CPU	Scheme B	Rate$_\tau$	CPU	Linear [8]	Rate$_\tau$	CPU
1/20	1.1628×10^{-4}	-	0.6066s	1.1630×10^{-4}	-	3.9295s	4.6191×10^{-4}	-	0.2072s	4.6193×10^{-4}	-	1.0500s
1/40	2.9045×10^{-5}	2.0013	1.1944s	2.9128×10^{-5}	1.9974	6.6614s	1.1584×10^{-4}	1.9954	0.4037s	1.1585×10^{-4}	1.9954	1.9621s
1/80	7.2250×10^{-6}	2.0072	2.1807s	7.2831×10^{-6}	1.9998	12.2967s	2.8969×10^{-5}	1.9996	0.7913s	2.8982×10^{-5}	1.9990	3.6146s
1/160	1.7750×10^{-6}	2.0252	3.7662s	1.8181×10^{-6}	2.0021	21.9366s	7.2016×10^{-6}	2.0081	1.3829s	7.2443×10^{-6}	2.0002	6.3254s

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