Fractional Order Chebyshev Cardinal Functions for Solving Two Classes of Fractional Differential Equations

Linna Li, Yuze Li, and Qiongdan Huang

Abstract—**In this paper, based on the operation matrix of the fractional order Chebyshev cardinal function(FOCCF), the accurate and effective method for solving two different forms of fractional delay differential equations(FDDEs) is proposed. Firstly, the new operational matrices(OMs) are presented for the fractional integration of the FOCCFs. Secondly, based on FOCCFs, a new direct method for is computing7 such problems is proposed. By making use of these OMs, the problems of the two different forms of FDDEs can be transformed to a set of simpler algebraic equations(AEs), which can be resolved by Newton's iterative method. Illustrative examples are introduced by data and tables, which can prove the effectiveness and applicability of this new technology to two different FDDEs.**

Index Terms—**The fractional order Chebyshev cardinal function(FOCCF); delay differential equation(DDE); Caputo fractional derivative, numerical method**

I. Introduction

THE problems of fractional differential equations (FDEs)[1] have been a hot topic for the past few years, since which can be putted to use in a lot of different areas, such as computer science and technology, management science and engineering, etc. However, most FDEs do not have analytical solutions, or analytical solutions are difficult to derive. Therefore, the study of numerical methods are significant because numerical methods can be used to solve such equations without analytical solutions.

There several numerical methods were proposed for the problems of FDEs, such as finite-difference method [2], variational iteration method [3], Legendre wavelet method [4], homotopy method [5], and so on. In these numerical methods, a class of efficient numerical technique was proposed to deal with various problems, which by expanding the desired solution into elements of the interpolate approximate basis functions. The main advantage of this method is that these complicated problems can be reducced to a set of simple AEs.

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However, the singular solutions appear in FDEs. The FDEs have the following disadvantages. First of all, the regularity of the solutions to related problems at the end point is limited. Secondly, fractional derivative(FD) is non-local operator. Thirdly, the kernel function of the FD is singular. Therfore, numerical solution using classical basis functions will result in slow convergence. In order to overcome these problems, we apply fractional order functions into the Chebyshev polynomials. Recently, there are some fractional order functions put to use in the basic functions for solving the FDEs. The fractional Riccati differential equation was calculated by Rahimkhani, Ordokhani and Babolian using the fractional order Bernoulli functions [6]. The fractional order collocation method for solving rational Bessel functions of Thomas-Fermi equation on semi-infinite field was proposed [7]. In [8], the FDEs was worked out by the fractional Bernstein polynomial. The fractional variational problems and fractional optimal control problems were figured out by the fractional-order Gegenbauer functions [9]. Mohammadi and Mohyud-Din [10] proposed an approach that the fractional Bagley-Torvik equations figured out by the fractional-order shifted Legendre polynomials. And some other academics worked out problems using the different fractional-order functions, such as fractional order Chebyshev orthogonal functions [11], fractional order Taylor functions [12], fractional Jacobi polynomials [13], Bloch equations [14] and so on.

In recent years, Chebyshev cardinal functions(CCFs) were constructed to solve various dynamic problems, and which has achieved good results. Lakestani and Mehdi [15] proposed a method for solving a parabolic partial differential equation with time-varying coefficients with additional measures based on CCF. In [16], Heydari, Mahmoudi, Shakibac and Avazzadeh studied a method to solve a class of nonlinear stochastic differential equations driven by fractional Brownian motion based on Chebyshev cardinal wavelet. In [17], the method of solving Duffing-harmonic oscillator was illustrated by using the mixed function composed of block-pulse and CCFs. Zahra, Saeed and Esmail [18] considered the numerical method of solving the Foredoom integral equations with CCFs. Using the CCFs, Heydari proposed a numerical method to solve a class of variable-order fractional optimal control problems. [19] In [20], Thanon, Sanoe, and Khomsan studied solving differential static beam problems based on Chebyshev polynomials. Other academics also pouted forward some techniques based on CCFs to solve different equations, such as the fourth-order integro-differential equations [21], second order one dimensional linear hyperbolic equation [22], singular boundary value problems [23], fractional nonlinear integro-differential equation [24], FDEs [25], the nonlinear age-structured population models [26], the Riccati differential equation [27] and weakly singular fredholm integral equations [28].

The delay differential equation(DDE) is a difficult problem to solve. This problem not only relate to the current state, but also closely refer to the former state. In the past few years, there are many different numerical methods to solve the DDEs, especially the DDEs with integer order, such as shifted Chebyshev polynomial [29], variational iteration method [30], CCFs method [31] and so on. However, there were few works about FDDEs. The calculation of FDDEs is complicated than the integer order DDEs. It is very difficult to get the accurate analytical solution of FDDEs. Several numerical methods were presented to solve FDDEs, such as Chelyshkov wavelet [32], Müntz-Legendre [33], Bernoulli wavelet [34], shifted Jacobi polynomials [35], etc.

In this paper, the OMs of fractional integration based on the FOCCFs are derived to solve FDDEs. On the basis of the OMs of fractional integration, a direct numerical method is presented. This method using OMs reduces the FDDE to a group of AEs. To solve the unknown coefficients of the FOCCF, we can get the numerical approximate solutions of the FDDE. We think over the following two FDDEs:

Problem 1

$$\begin{cases} D^\alpha x(t) = f(t, x(t), x(t-\tau)), & \\ x^{(i)}(0) = \zeta_i, & \\ x(t) = \gamma(t), & t \leq 0. \end{cases} \tag{1}$$

Problem 2

$$\begin{cases} D^\alpha x(t) = f(t, x(t), x(\tau t)) \\ x^{(i)}(0) = \eta_i \end{cases} . \tag{2}$$

where $1 \geq t \geq 0, n-1 < \alpha < n, 0 < \tau < 1$ and $i = 0, 1, \cdots, n-1, n \in N$.

In section 2, some basic definitions of fractional operators, the FOCCFs and the CCFs and are proposed. We derive some new results of the FOCCFs in section 3. In section 4, the direct numerical method is described specifically for solving the two classes of FDDEs. The numerical results and discussions are presented in the form of graphs in section 5. Eventually, in section 6, the conclusion is drawn.

II. Preliminaries

In this section, the definition of the fractional calculus and the most details of CCFs are introduced systematically.

A. Basic definitions of fractional calculus

In this part, the important attributes of score calculations that will be used in this article are introduced. At first give the definition of FD, which detailed description as follows:

Definition 2.1([36]). The Riemann-Liouville(R-L) fractional integral operator of order $\alpha \geq 0$ is defined as

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t) & \alpha > 0 \\ f(t) & \alpha = 0 \end{cases},$$

where $t > 0$, $t^{\alpha-1} * f(t)$ is the convolution product of $t^{\alpha-1}$ and $f(t)$.

Suppose that λ and μ are real constants, for the R-L fractional integral(R-L FI), we have [36]

$1. I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\alpha+\beta}, \beta > -1,$
$2. I^\alpha(\lambda f(t) + \mu g(t)) = \lambda I^\alpha f(t) + \mu I^\alpha g(t).$

Definition 2.2([1][36]). The FD D^α, in the Caputo sense for $n-1 < \alpha \leq n, n \in N$, is defined by

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt.$$

The Caputo derivative has the following properties [1]
$1. D^\alpha I^\alpha x(t) = x(t),$
$2. I^\alpha D^\alpha x(t) = x(t) - \sum_{i=0}^{n-1} x^{(i)}(0) \frac{t^i}{i!}.$

B. The fractional order Chebyshev functions

Thus, the singular Sturm-Liouville differential equation of the classical Chebyshev polynomial is defined as [11] :

$$\frac{\sqrt{\eta^\alpha - t^\alpha}}{t^{\frac{\alpha}{2}-1}} \frac{d}{dt} [\frac{\sqrt{\eta^\alpha - t^\alpha}}{t^{\frac{\alpha}{2}-1}} \frac{d}{dt} \ {}_\eta FT_n^\alpha(t)] + n^2 \alpha^2 \ {}_\eta FT_n^\alpha(t) = 0. \tag{3}$$

where $t \in [0, \eta]$. The ${}_\eta FT_n^\alpha(t)$ can be obtained using recursive relation as follows:

$${}_\eta FT_0^\alpha(t) = 1,$$

$${}_\eta FT_1^\alpha(t) = 1 - 2(\frac{t}{\eta})^\alpha,$$

$${}_\eta FT_{n+1}^\alpha(t) = (2 - 4(\frac{t}{\eta})^\alpha) \ {}_\eta FT_n^\alpha(t) - \ {}_\eta FT_{n-1}^\alpha(t).$$

The analytical form of ${}_\eta FT_n^\alpha(t)$ of degree $N\alpha$ given by

$${}_\eta FT_n^\alpha(t) = \sum_{j=0}^n (-1)^j \frac{n 2^{2j} (n+j-1)!}{(n-j)!(2j)!} (\frac{t}{\eta})^{\alpha j} = \sum_{j=0}^n \beta_{n,j,\eta,\alpha} \cdot t^{\alpha j}, \tag{4}$$

where

$$\beta_{n,j,\eta,\alpha} = (-1)^j \frac{n 2^{2j} (n+j-1)!}{(n-j)!(2j)! \eta^{\alpha j}}, t \in [0, \eta],$$

and

$$\beta_{0,j,\eta,\alpha} = 1.$$

Note that ${}_\eta FT_n^\alpha(0) = 1$ and ${}_\eta FT_n^\alpha(\eta) = (-1)^n$.

The generalized fractional order of the Chebyshev orthogonal functions(GFCFs) are respect to the weight function $w(t) = \frac{t^{\frac{\alpha}{2}-1}}{\sqrt{\eta^\alpha - t^\alpha}}$ in the interval $[0, \eta]$:

$$\int_0^\eta \ {}_\eta FT_n^\alpha(t) \ {}_\eta FT_n^\alpha(t) w(t) dt = \frac{\pi}{2\alpha} c_n \delta_{mn},$$

where δ_{mn} is Kronecher delta, $c_0 = 2$, and $c_n = 1$ for $n \geq 1$.

Theorem 2.1([11]). The GFCF ${}_\eta FT_n^\alpha(t)$, has precisely n real zeros on interval $(0, \eta)$ in the form

$$t_j = \eta(\frac{1 - \cos(\frac{(2j-1)\pi}{2n})}{2})^{\frac{1}{\alpha}}, \quad j = 1, 2, \ldots, n. \tag{5}$$

Moreover, $\frac{d}{dt} \ {}_\eta FT_n^\alpha(t)$ has precisely $n-1$ real zeros on interval $(0, \eta)$ in the following points:

$$t'_j = \eta(\frac{1 - \cos(\frac{j\pi}{n})}{2})^{\frac{1}{\alpha}}, \quad j = 1, 2, \ldots, n-1.$$

C. The FOCCF

We consider the CCFs and present some important properties which are used in the following sections.

The CCFs [37] of order N on the interval $[0,\eta]$ are defined as

$$\phi_j = \frac{{}_\eta FT^\alpha_{N+1}(t)}{({}_\eta FT^\alpha_{N+1})'(t_j)(t-t_j)}, \qquad j=1,2,\ldots,N+1, \quad (6)$$

where ${}_\eta FT^\alpha_{N+1}(t)$ can be obtained as follows,

$${}_\eta FT^\alpha_{N+1}(t) = \sum_{j=0}^{N+1} \beta_{N+1,j,\eta,\alpha} \cdot t^{\alpha j}, t\in[0,\eta], \quad (7)$$

and

$$\beta_{N+1,j,\eta,\alpha} = (-1)^j \frac{(N+1)2^{2j}(N+j)!}{(N+1-j)!(2j)!\eta^{\alpha j}}, \quad \beta_{0,j,\eta,\alpha}=1.$$

Now any functions on $[0,\eta]$ can be similar to [19], [31]

$$f(t) = \sum_{j=1}^{N+1} f(t_j)\phi_j(t) = F^\top \Phi_N(t) \quad (8)$$

where

$$F = [f(t_1), f(t_2), \cdots, f(t_{N+1})]^\top, \quad (9)$$

and

$$\Phi_N(t) = [\phi_1(t), \phi_2(t), \cdots, \phi_{N+1}(t)]^\top, \quad (10)$$

then we choose t_j, $j=1,2,\cdots,N+1$ according to the principle of $t_1<t_2<\cdots<t_{N+1}$. Then t_j, $j=1,2,\cdots,N+1$ can be written as follows

$$t_j = \eta\Big(\frac{1-\cos(\frac{(2j-1)\pi}{2(N+1)})}{2}\Big)^{\frac{1}{\alpha}}, \quad j=1,2,\ldots,N+1. \quad (11)$$

Note that

$${}_\eta FT^\alpha_{N+1}(t) = \beta \times \prod_{j=1}^{N+1}(t-t_j), \quad (12)$$

$$({}_\eta FT^\alpha_{N+1})'(t_j) = \beta \times \prod_{j=1, j\neq k}^{N+1}(t_k-t_j), \quad (13)$$

where $\beta = \frac{2^{2N+1}}{\eta^{N+1}}$.

III. The operational matrix of the fractional integration

In this part, we briefly review the method of solving FDEs with the CCFs.

Theorem 3.1. The integration of the vector $\Phi_N(t)$ defined in (10) can be similar to

$$I^\alpha \Phi_N(t) \simeq P_N \Phi_N(t), \quad (14)$$

where P_N can be written as

$$P_N = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1(N+1)} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2(N+1)} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{(N+1)1} & \alpha_{(N+1)2} & \cdots & \alpha_{(N+1)(N+1)} \end{pmatrix}, \quad (15)$$

and α_{jk} is as follows,

$$\alpha_{jk} = \frac{\beta}{({}_\eta FT^\alpha_{N+1})'(t_j)} \cdot \frac{1}{\Gamma(\alpha)} \int_0^{t_k} (t_k-s)^{\alpha-1} \times \prod_{i=1, i\neq j}^{N+1}(s-t_i)ds, \quad j,k=1,2,\cdots,N+1. \quad (16)$$

Proof. Let

$$I^\alpha \Phi_N(t) = [I^\alpha\phi_1(t), I^\alpha\phi_2(t), \cdots, I^\alpha\phi_{N+1}(t)]^T. \quad (17)$$

Using (8), any function $I^\alpha\phi_j(t)$ can be expended as

$$I^\alpha\phi_j(t) \approx \sum_{k=1}^{N+1} \alpha_{jk}\phi_k(t), \quad (18)$$

and according to Definition 2.1

$$\alpha_{jk} = I^\alpha\phi_j(t) = \frac{\beta}{({}_\eta FT^\alpha_{N+1})'(t_j)} \cdot \frac{1}{\Gamma(\alpha)} \int_0^{t_k} (t_k-s)^{\alpha-1} \times \prod_{i=1, i\neq j}^{N+1}(s-t_i)ds, j,k=1,2,\cdots,N+1.$$

Comparing (14) and (18), we get

$$P_N = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1(N+1)} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2(N+1)} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{(N+1)1} & \alpha_{(N+1)2} & \cdots & \alpha_{(N+1)(N+1)} \end{pmatrix}.$$

IV. Description of the proposed method

In this part, the director method to solve the FDDEs are presented. Some properties of the CCFs we used are proposed in section 3.

A. Direct approach to Problem 1

Considering the FDDE in Problem 1,

$$D^\alpha x(t) = f(t, x(t), x(t-\tau)), \qquad t\in[0,1], \tau\in(0,1), \quad (19)$$

and the condition is

$$\begin{cases} x^{(i)}(0) = \zeta_i & i=0,1,\cdots,n-1, n\in N \\ x(t)=\gamma(t) & t<0 \end{cases}. \quad (20)$$

For the previous problem, let $D^\alpha x(t)$ be approximated by the CCFs as

$$D^\alpha x(t) \simeq C^\top \Phi_N(t), \quad (21)$$

where C is vectors with $N+1$ unknowns as follows

$$C = [c_1, c_2, \cdots, c_{N+1}]^\top. \quad (22)$$

Using (14), we obtain

$$\begin{aligned} x(t) &\simeq I^\alpha(C^\top \Phi_N(t)) + \sum_{i=0}^{n-1} \frac{\zeta_i}{i!} t^i \\ &\simeq C^\top P_N \Phi_N(t) + \sum_{i=0}^{n-1} \frac{\zeta_i}{i!} t^i \\ &\simeq C^\top P_N \Phi_N(t) + e_1^\top \Phi_N(t), \end{aligned} \quad (23)$$

where $e_1^\top = [\zeta_1, \zeta_2, \cdots, \zeta_i]$, $i = 1, 2, \cdots, n-1$.

Moreover, according to (23), we can get

$$x(t-\tau) \simeq (C^\top P_N + e_1^\top)\Phi_N(t-\tau). \quad (24)$$

So, we obtain

$$x(t-\tau) = \begin{cases} \gamma(t) & 0 \le t \le \tau, \\ (C^\top P_N + e_1^\top)\Phi_N(t-\tau) & \tau < t \le 1. \end{cases} \quad (25)$$

B. Direct approach to Problem 2

Considering the FDDE in Problem 2,

$$D^\alpha x(t) = f(t, x(t), x(\tau t)), \qquad t \in [0,1], \tau \in (0,1), \quad (26)$$

and the condition is

$$x^{(i)}(0) = \eta_i \qquad i = 0, 1, \cdots, n-1, n \in N. \quad (27)$$

For the previous problem, let $D^\alpha x(t)$ be approximated by the CCFs as

$$D^\alpha x(t) \simeq C^\top \Phi_N(t), \quad (28)$$

where C is vectors with $N+1$ unknowns as follows

$$C = [c_1, c_2, \cdots, c_{N+1}]^\top. \quad (29)$$

Using (14), we obtain

$$\begin{aligned} x(t) &\simeq I^\alpha(C^\top \Phi_N(t)) + \sum_{i=0}^{n-1} \frac{\eta_i}{i!} t^i \\ &\simeq C^\top P_N \Phi_N(t) + \sum_{i=0}^{n-1} \frac{\eta_i}{i!} t^i \\ &\simeq C^\top P_N \Phi_N(t) + e_2^\top \Phi_N(t), \end{aligned} \quad (30)$$

where $e_2^\top = [\eta_1, \eta_2, \cdots, \eta_i]$, $i = 1, 2, \cdots, n-1$.

Moreover, for any $0 < \tau < 1$, according to (30), we can get

$$x(\tau t) \simeq (C^\top P_N + e_2^\top)\Phi_N(\tau t). \quad (31)$$

Then, in conclusion, we obtain two sets of AEs for Problem 1 and Problem 2, which can be written as

Problem 1

$$C^\top \Phi_N(t) = \begin{cases} f(t, C^\top P_N \Phi_N(t) + e_1^\top \Phi_N(t), \gamma(t)), 0 \le t \le \tau \\ f(t, C^\top P_N \Phi_N(t) + e_1^\top \Phi_N(t), (C^\top P_N + e_1^\top) \cdot \\ \Phi_N(t-\tau)), \tau < t \le 1 \end{cases}$$

Problem 2

$$C^\top \Phi_N(t) = f(t, C^\top P_N \Phi_N(t) + e_1^\top \Phi_N(t), (C^\top P_N + e_2^\top) \cdot \Phi_N(\tau t)), \qquad 0 \le t \le 1.$$

This is the systems of AEs with $(N+1)$ unknowns and $(N+1)$ equations, which can be solved by Newton's iterative method to get the unknown vector C. Then using (23) or (30), we obtain the solutions $x(t)$ of the FDDEs.

V. Numerical examples

In this part, the approximate solution of the FDDE by using operating matrices of FOCCF is calculated. We can compare the exact and approximate solution by charts and graphics. These equations are calculated by maple in Windows 7(64bit).

Example 1 Consider the following DDE

$$D^\alpha x(t) = x(t-\tau) - x(t) + \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} + 2\tau t - \tau^2 - \tau,$$

where $0 \le t \le 1, 0 < \alpha \le 1$, and the condition is

$$x(t) = 0, \quad t \le 0,$$

The exact solution is $x(t) = t^2 - t$ for $\alpha = 1$. The approximate solutions are calculated by (21), and compared with the exact solution.

In the Tab. I, we present the absolute errors for different values of N, t, τ by fractional order of the CCF to solve FDDE when $\alpha = 1$. We can find the relation between τ, t and the approximate solutions. Also, in the Tab. II, the values of the approximate solutions and the residual error $R_N(t)$ are presented. The residual error can be calculated as follows,

$$\begin{aligned} R_N(t) &= |D^\alpha x(t) - f(t, x(t), x(t-\tau))| \\ R_N(t) &= |D^\alpha x(t) - f(t, x(t), x(t\tau))|. \end{aligned} \quad (32)$$

As is shown in Tab. I, we conclude that the larger the values of t, the lower the absolute error. Under the same N, t condition, we can see that the smaller the value of τ, the lower the absolute errors. The other case is that the closer t is to 1, the lower the absolute error under the same conditions of N, τ. In general, we find the absolute error for the different values of τ, t and N between approximate and exact solution are in the acceptable ranges. From Tab. II, this method has the residual errors.

TABLE I
THE ABSOLUTE ERRORS OF DIFFERENT τ VALUES WHEN $\alpha = 1$

N	t	$\tau = 0.0001$	$\tau = 0.001$	$\tau = 0.01$
2	0.2	1.600×10^{-5}	1.601×10^{-4}	1.607×10^{-3}
	0.4	2.400×10^{-5}	2.403×10^{-4}	2.430×10^{-3}
	0.6	2.401×10^{-5}	2.407×10^{-4}	2.468×10^{-3}
	0.8	1.601×10^{-5}	1.613×10^{-4}	1.722×10^{-3}
	1	1.940×10^{-8}	1.991×10^{-6}	1.922×10^{-4}
5	0.2	1.599×10^{-5}	1.601×10^{-4}	1.607×10^{-3}
	0.4	2.400×10^{-5}	2.403×10^{-4}	2.430×10^{-3}
	0.6	2.400×10^{-5}	2.407×10^{-4}	2.468×10^{-3}
	0.8	1.601×10^{-5}	1.613×10^{-4}	1.722×10^{-3}
	1	4.010×10^{-8}	2.002×10^{-6}	1.922×10^{-4}

TABLE II
THE VALUES OF DIFFERENT α ON THE $R_N(t)$ WHEN $N=5$ AND $\tau=0.01$

t	$\alpha=0.50$		$\alpha=0.90$	
	$x_N(t)$	$R_N(t)$	$x_N(t)$	$R_N(t)$
0.2	-0.14818	1.163×10^{-3}	-0.16167	1.092×10^{-3}
0.4	-0.24253	1.246×10^{-4}	-0.24028	1.117×10^{-4}
0.6	-0.27022	2.550×10^{-5}	-0.24042	2.234×10^{-5}
0.8	-0.12496	2.637×10^{-4}	-0.16141	2.280×10^{-4}
1.0	-0.07486	4.002×10^{-3}	-0.00321	3.429×10^{-3}

Example 2 Consider the following DDE

$$\begin{cases} D^{\alpha}x(t)=-x(t)-x(t-0.3)+e^{-t+0.3} & 0\le t\le 1 \\ x(0)=1, x'(0)=-1, x''(0)=1 \\ x(t)=e^{-t}, \quad t\le 0, 2<\alpha\le 3. \end{cases}$$

The exact solution is $x(t)=e^{-t}$ for $\alpha=3$. The approximate solutions are calculated by (21), and compared with the exact solution by charts and graphics.

In the Tab. III, we illustrate the absolute errors for different values of N,t by the CCF to solve FDDE when $\alpha=3$. We can find the relation between t and the approximate solutions.

TABLE III
THE ABSOLUTE ERRORS OF DIFFERENT N VALUES WHEN $\alpha=3,\tau=0.3$

t	$N=2$	$N=5$
0	3.943×10^{-3}	3.655×10^{-5}
0.2	4.130×10^{-3}	3.778×10^{-4}
0.4	1.812×10^{-3}	2.457×10^{-4}
0.6	2.000×10^{-3}	6.635×10^{-4}
0.8	3.315×10^{-3}	1.297×10^{-3}

As is shown in Tab. III, we see that the larger the values of N, the lower the absolute error at the same condition of α,t. In general, the absolute error is in acceptable ranges for the different values of t and N between approximate and exact solution.

Example 3 Consider the following DDE

$$\begin{cases} D^{\alpha}x(t)=-x(t)+\frac{\tau}{2}x(\tau t)-\frac{\tau}{2}e^{-\tau t} & 0\le t\le 1, \\ x(0)=1, & 0<\alpha\le 1. \end{cases}$$

The exact solution is $x(t)=e^{-t}$ for $\tau=0.2$ when $\alpha=1$. The approximate solutions are calculated by (28), and compared with the exact solution by graphics and tables.

In the Tab. IV, we give the absolute errors of different N and t values by the CCF when $\alpha=1,\tau=0.2$ to solve FDDE. We find the relation between N,t and the approximate solutions. The values of approximate solutions and $R_N(t)$ are presented in the Tab. V. $R_N(t)$ can be obtained by (32).

TABLE IV
THE ABSOLUTE ERRORS OF DIFFERENT N VALUES WHEN $\alpha=1,\tau=0.2$

t	$N=2$	$N=5$
2^{-2}	3.810×10^{-3}	8.524×10^{-4}
2^{-3}	2.341×10^{-3}	4.859×10^{-4}
2^{-4}	1.751×10^{-4}	2.591×10^{-4}
2^{-5}	1.361×10^{-3}	1.337×10^{-4}
2^{-6}	2.254×10^{-3}	6.776×10^{-5}

TABLE V
THE VALUES OF DIFFERENT α ON THE $R_N(t)$ WHEN $N=5$ AND $\tau=0.1$

t	$\alpha=0.70$		$\alpha=0.80$	
	$x_N(t)$	$R_N(t)$	$x_N(t)$	$R_N(t)$
0.1	0.80277	6.40×10^{-9}	0.84531	1.91×10^{-9}
0.3	0.65283	1.35×10^{-8}	0.67706	1.83×10^{-9}
0.5	0.54701	1.00×10^{-8}	0.56472	1.12×10^{-9}
0.7	0.46654	2.36×10^{-8}	0.47758	2.85×10^{-8}
0.9	0.43739	1.68×10^{-7}	0.41877	4.51×10^{-9}

Example 4 Consider the following FDDE [32], [33], [34]

$$\begin{cases} D^{\alpha}x(t)=\mu\sin(x(t-\tau))-\lambda x(t) & 0<t<T, \\ x(t)=\sigma_0, & -\tau\le t\le 0. \end{cases}$$

where $0<\alpha(t)\le 1,\mu>0$ and $\lambda>0$ are the parameters. The approximate solutions are calculated by (21).

In the Tab. VI, we present the approximate solutions of different t,α and τ values by the CCF to solve FDDE when $\lambda=0.4,\mu=0.4,N=2$ and $T=1$. We can find the relation between α,τ and the approximate solutions.

TABLE VI
THE APPROXIMATE SOLUTIONS OF DIFFERENT α AND τ VALUES WHEN $\lambda=0.4,\mu=0.4,N=2(\tau_1=0.01,\tau_2=0.001)$

t	$\alpha=0.35,\tau_1$	$\alpha=0.5,\tau_1$	$\alpha=0.95,\tau_2$	$\alpha=1,\tau_2$
0.1	1.48495	1.51293	1.54577	1.54850
0.2	1.42919	1.47590	1.52338	1.52708
0.3	1.38763	1.44518	1.50196	1.50636
0.4	1.36028	1.42079	1.48151	1.48631
0.5	1.34712	1.40272	1.46203	1.46696
0.6	1.34815	1.39098	1.44352	1.44829
0.7	1.36339	1.38556	1.42597	1.43030
0.8	1.39283	1.38646	1.40940	1.41300
0.9	1.43646	1.39368	1.39379	1.39638
1.0	1.49430	1.40722	1.37915	1.38045

VI. CONCLUSIONS

In this paper, we discuss the approximate solutions of two classes of FEEDs by using OMs based on the CCFs. The OMes of fractional integration are derived. Using this method, the FDDEs are transformed into a set of AEs by

OMs. The solutions are presented in the form of data and tables. Results show that the approximate solution got by this direct method is close to the exact solution.In conclusion, this direct method is an available and convenient method of solving FDDEs approximate solution.

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