Parameter Estimation for Hyperbolic Model with Small Noises Based on Discrete Observations

Chao Wei

Abstract—**In this paper, we consider the discretely observed hyperbolic diffusion model with two types of small noises. The least square method is utilized to derive the drift parameter estimator. The consistency and asymptotic distribution of the estimator are obtained under condition of two types of small noises. Finally, some numerical calculus examples are given.**

Index Terms—**Hyperbolic diffusion; small noises; drift parameter estimation; consistency; asymptotic distribution.**

I. Introduction

Research on stochastic differential equations has a long tradition([2], [3], [8], [12], [19], [21]). However, parameters in stochastic differential equations are always unknown. This seems to be a common problem in stochastic model. In the past several decades, some authors studied parameter estimation for financial models. For example, Tunaru and Zheng([17]) utilized Bayesian method to discuss parameter estimation risk in financial modelling. Wei et al.([18]) applied maximum likelihood method to study the parameter estimation for Cox-Ingersoll-Ross model. Yang et al.([22]) used α-path method to estimate the unknown parameter of uncertain differential equation from discretely sampled data. But, some financial processes show discontinuous sample paths and heavy tailed properties. These features cannot be captured by Brownian motion([5], [6], [9], [16]). Hence, the financial process has been described by Lévy process. In the last few years, some authors investigated parameter estimation for stochastic models driven by Lévy noises([1], [4], [11], [14]). For example, Long([10]) studied parameter estimation for a class of stochastic differential equations driven by small stable noises from discrete observations. Shen et al.([15]) analyzed parameter estimation for the discretely observed Vasicek model with small fractional Lévy noise. Wei([20]) used least squares method to obtain estimators of stochastic Lotka-Volterra model driven by small α-stable noises, discussed the consistency and asymptotic distribution of estimators. Zhao and Zhang([23]) investigated minimum distance estimate for stochastic nonlinear differential equations with small α-stable noises.

As we all know, parameter estimation for hyperbolic diffusion has been studied by some authors. For example, Protassov([13]) proposed a simple EM-based maximum likelihood estimation procedure to estimate parameters of the distribution when the subclass is known regardless of the dimensionality. Kuang and Xie([7]) investigated the properties of a sequential maximum likelihood estimator of the unknown parameter for the hyperbolic diffusion process.

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Chao Wei is a Professor of School of Mathematics and Statistics, An'yang Normal University, An'yang, 455000, China (email: chaowei0806@aliyun.com).

However, there are few literature about the parameter estimation for hyperbolic diffusion driven by Lévy noises. In this paper, we investigate the parameter estimation problem for hyperbolic diffusion driven by two types of small Lévy noises from discrete observations. The contrast function is introduced to obtain the least squares estimator. The consistency and asymptotic distribution of the estimator are derived under the condition of two types of small Lévy noises by using Cauchy-Schwarz inequality, Gronwall's inequality, B-D-G inequality and dominated convergence.

This paper is organized as follows. In Section 2, parameter estimation for the hyperbolic diffusion driven by small Lévy noises is studied. In Section 3, parameter estimation for the hyperbolic diffusion driven by small α-stable noises is discussed. In Section 4, some simulation results are given. The conclusion is given in Section 5.

II. Parameter Estimation for the Hyperbolic Diffusion Driven by Lévy Noises

We consider the following hyperbolic diffusion model:

$$\begin{cases} dX_t = \alpha \dfrac{X_t}{\sqrt{1+X_t^2}}dt + \varepsilon\sigma dL_t, \quad t\in[0,1], \varepsilon\in(0,1) \\ X_0 = x_0, \end{cases} \tag{1}$$

where α is unknown, $(L_t, t\geq 0)$ is a Lévy noise with decomposition as follows:

$$L_t = W_t + \int_0^t \int_{|z|>1} zN(ds,dz) + \int_0^t \int_{|z|\leq 1} z\widetilde{N}(ds,dz), \tag{2}$$

where $(W_t, t\geq 0)$ is a Brownian motion, $N(ds,dz)$ is a Poisson random measure independent of $(W_t, t\geq 0)$ with characteristic measure $dt\nu(dz)$, $\widetilde{N}(ds,dz) = N(ds,dz) - \nu(dz)$ is a martingale measure. It is assumed that $\nu(dz)$ is a Lévy measure on $\mathbb{R}\backslash\{0\}$ satisfying $\int(|z|^2\wedge 1)\nu(dz) < \infty$.

We give the following contrast function:

$$\rho_{n,\varepsilon}(\alpha) = \sum_{i=1}^n \frac{|X_{t_i} - X_{t_{i-1}} - \alpha\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}\Delta t_{i-1}|^2}{\varepsilon^2\sigma^2\Delta t_{i-1}}, \tag{3}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

We have

$$\widehat{\alpha}_{n,\varepsilon} = \frac{\sum_{i=1}^n \frac{(X_{t_i}-X_{t_{i-1}})X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}}{\frac{1}{n}\sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}}. \tag{4}$$

By the Euler-Maruyama scheme for (1), we obtain

$$X_{t_i} - X_{t_{i-1}} = \alpha_0 \frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}\Delta t_{i-1} + \varepsilon\sigma(L_{t_i} - L_{t_{i-1}}). \tag{5}$$

Then,

$$\begin{aligned}&\sum_{i=1}^{n}\frac{(X_{t_i}-X_{t_{i-1}})X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}\\&=\alpha_0\frac{1}{n}\sum_{i=1}^{n}\frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}\\&+\varepsilon\sigma\sum_{i=1}^{n}\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}(L_{t_i}-L_{t_{i-1}}).\end{aligned}\tag{6}$$

Substituting (6) into (4), we have

$$\widehat{\alpha}_{n,\varepsilon}-\alpha_0=\frac{\varepsilon\sigma\sum_{i=1}^{n}\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}(L_{t_i}-L_{t_{i-1}})}{\frac{1}{n}\sum_{i=1}^{n}\frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}}.\tag{7}$$

Consider the following ordinary differential equation:

$$dX_t^0=\alpha\frac{X_t^0}{\sqrt{1+(X_t^0)^2}}dt,\quad X_0^0=x_0>0.$$

Let $H_t^{n,\varepsilon}=X_{[nt]/n}$ where $[nt]$ denotes the integer part of nt.

Lemma 1: When $\varepsilon\to0$ and $n\to\infty$, the sequence $\{H_t^{n,\varepsilon}\}$ converges to $\{X_t^0\}$ uniformly in probability.

Proof: Since

$$X_t-X_t^0=\alpha_0\int_0^t\left(\frac{X_s}{\sqrt{1+X_s^2}}-\frac{X_s^0}{\sqrt{1+(X_s^0)^2}}\right)ds+\varepsilon\sigma L_t,$$

we obtain

$$\begin{aligned}&|X_t-X_t^0|^2\\\leq\ &2\alpha_0^2\left|\int_0^t\left(\frac{X_s}{\sqrt{1+X_s^2}}-\frac{X_s^0}{\sqrt{1+(X_s^0)^2}}\right)ds\right|^2+2\varepsilon^2\sigma^2L_t^2\\\leq\ &2t\alpha_0^2\int_0^t\left|\frac{X_s}{\sqrt{1+X_s^2}}-\frac{X_s^0}{\sqrt{1+(X_s^0)^2}}ds\right|^2+2\varepsilon^2\sigma^2L_t^2\\\leq\ &8t\alpha_0^2\int_0^t|X_s-X_s^0|^2+2\varepsilon^2\sigma^2L_t^2.\end{aligned}$$

By the Gronwall's inequality, we obtain

$$|X_t-X_t^0|^2\leq2\varepsilon^2\sigma^2e^{8t^2\alpha_0^2}L_t^2.$$

Then, we have

$$\sup_{0\leq t\leq T}|X_t-X_t^0|\leq\sqrt{2}\varepsilon\sigma e^{4T^2\alpha_0^2}\sup_{0\leq t\leq T}L_t.$$

Hence, for each $T>0$, we get

$$\sup_{0\leq t\leq T}|X_t-X_t^0|\xrightarrow{P}0.\tag{8}$$

When $n\to\infty$, $[nt]/n\to t$, it is derived that sequence $\{H_t^{n,\varepsilon}\}$ converges to $\{X_t^0\}$ uniformly in probability.

The proof is complete. ■

Lemma 2: As $\varepsilon\to0$ and $n\to\infty$,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}\xrightarrow{P}\int_0^1\frac{(X_t^0)^2}{1+(X_t^0)^2}dt.$$

Proof: According to Lemma 1, we obtain

$$\begin{aligned}&\sup\left|\frac{1}{n}\sum_{i=1}^{n}\frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}-\int_0^1\frac{(X_t^0)^2}{1+(X_t^0)^2}dt\right|\\=\ &\sup\left|\int_0^1\frac{(H_t^{n,\varepsilon})^2}{1+(H_t^{n,\varepsilon})^2}dt-\int_0^1\frac{(X_t^0)^2}{1+(X_t^0)^2}dt\right|\\\leq\ &\sup\int_0^1\left|\frac{(H_t^{n,\varepsilon})^2}{1+(H_t^{n,\varepsilon})^2}-\frac{(X_t^0)^2}{1+(X_t^0)^2}\right|dt\\\leq\ &\sup\int_0^1|(H_t^{n,\varepsilon})^2-(X_t^0)^2|dt\\\leq\ &\sup\int_0^1|H_t^{n,\varepsilon}+X_t^0||H_t^{n,\varepsilon}-X_t^0|dt\\\leq\ &\sup\int_0^1(|H_t^{n,\varepsilon}|+|X_t^0|)|M_t^{n,\varepsilon}-X_t^0|dt\\\leq\ &(\sup_{0\leq t\leq1}|X_t|+\sup_{0\leq t\leq1}|X_t^0|)\sup_{0\leq t\leq1}|H_t^{n,\varepsilon}-X_t^0|\\\xrightarrow{P}\ &0.\end{aligned}$$

Hence,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}\xrightarrow{P}\int_0^1\frac{(X_t^0)^2}{1+(X_t^0)^2}dt.$$

The proof is complete. ■

Lemma 3: When $\varepsilon\to0$ and $n\to\infty$,

$$\sum_{i=1}^{n}\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}(L_{t_i}-L_{t_{i-1}})\xrightarrow{P}\int_0^1\frac{X_t^0}{\sqrt{1+(X_t^0)^2}}dL_t.$$

Proof: According to Lemma 1, we obtain

$$\begin{aligned}&\left|\sum_{i=1}^{n}\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}(L_{t_i}-L_{t_{i-1}})\right.\\&\left.-\int_0^1\frac{X_t^0}{\sqrt{1+(X_t^0)^2}}dL_t\right|\\=\ &\left|\int_0^1\frac{H_t^{n,\varepsilon}}{\sqrt{1+(H_t^{n,\varepsilon})^2}}dL_t-\int_0^1\frac{X_t^0}{\sqrt{1+(X_t^0)^2}}dL_t\right|\\\leq\ &2\left|\int_0^1(H_t^{n,\varepsilon}-X_t^0)dL_t\right|\\=\ &2\left|\int_0^1(H_t^{n,\varepsilon}-X_t^0)dB_t\right.\\&+\int_0^1\int_{|z|>1}(H_t^{n,\varepsilon}-X_t^0)zN(dt,dz)\\+\ &\left.\int_0^1\int_{|z|\leq1}(H_t^{n,\varepsilon}-X_t^0)z\widetilde{N}(dt,dz)\right|\\\leq\ &2\left|\int_0^1(H_t^{n,\varepsilon}-X_t^0)dB_t\right|\\&+2\left|\int_0^1\int_{|z|>1}(H_t^{n,\varepsilon}-X_t^0)zN(dt,dz)\right|\\+\ &2\left|\int_0^1\int_{|z|\leq1}(H_t^{n,\varepsilon}-X_t^0)z\widetilde{N}(dt,dz)\right|.\end{aligned}$$

When $\varepsilon \to 0$ and $n \to \infty$,

$$\begin{aligned}
& |\int_0^1 \int_{|z|>1} (H_t^{n,\varepsilon} - X_t^0) z N(dt,dz)| \\
\leq \quad & \int_0^1 \int_{|z|>1} |H_t^{n,\varepsilon} - X_t^0||z| N(dt,dz) \\
\leq \quad & \sup_{0\leq t\leq 1} |H_t^{n,\varepsilon} - X_t^0| \int_0^1 \int_{|z|>1} |z| N(dt,dz) \\
\xrightarrow{P} \quad & 0.
\end{aligned} \tag{9}$$

By the B-D-G inequality, dominated convergence and Lemma 1, for any given $\eta > 0$, when $\varepsilon \to 0$ and $n \to \infty$, it follows that

$$\begin{aligned}
& P(|\int_0^1 (H_t^{n,\varepsilon} - X_t^0) dB_t| > \eta) \\
& \leq \frac{4\mathbb{E}\int_0^1 (H_t^{n,\varepsilon} - X_t^0)^2 dt}{\eta^2} \\
& \to 0,
\end{aligned}$$

and

$$\begin{aligned}
& P(|\int_0^1 \int_{|z|\leq 1} (H_t^{n,\varepsilon} - X_t^0) z \widetilde{N}(dt,dz)| > \eta) \\
& \leq \frac{\mathbb{E}(\int_0^1 \int_{|z|\leq 1} (H_t^{n,\varepsilon} - X_t^0) z \widetilde{N}(dt,dz))^2}{\eta^2} \\
& \leq \frac{\int_0^1 \mathbb{E}(H_t^{n,\varepsilon} - X_t^0)^2 dt \int_{|z|\leq 1} |z|^2 \nu(dz)}{\eta^2} \\
& \to 0.
\end{aligned}$$

Then, we obtain

$$|\int_0^1 (H_t^{n,\varepsilon} - X_t^0) dB_t| \xrightarrow{P} 0, \tag{10}$$

and

$$|\int_0^1 \int_{|z|\leq 1} (H_t^{n,\varepsilon} - X_t^0) z \widetilde{N}(dt,dz)| \xrightarrow{P} 0. \tag{11}$$

According to (9), (10) and (11), we have

$$\sum_{i=1}^n \frac{X_{t_{i-1}}}{\sqrt{1 + X_{t_{i-1}}^2}} (L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_0^1 \frac{X_t^0}{\sqrt{1 + (X_t^0)^2}} dL_t.$$

■

Now, we introduce the main results below.

Theorem 1: When $\varepsilon \to 0$ and $n \to \infty$, the least squares estimator $\widehat{\alpha}_{n,\varepsilon}$ is consistent in probability, namely

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha_0.$$

Proof: According to (7), Lemma 2 and Lemma 3, when $\varepsilon \to 0$ and $n \to \infty$,

$$\widehat{\alpha}_{n,\varepsilon} - \alpha_0 \xrightarrow{P} 0. \tag{12}$$

The proof is complete. ■

Theorem 2: When $\varepsilon \to 0$ and $n \to \infty$,

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} \alpha_0) \xrightarrow{d} \frac{\sigma \int_0^1 \frac{X_t^0}{\sqrt{1+(X_t^0)^2}} dL_t}{\int_0^1 \frac{(X_t^0)^2}{1+(X_t^0)^2} dt}.$$

Proof: According to (7), we obtain

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} \alpha_0) = \frac{\sigma \sum_{i=1}^n \frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}} (L_{t_i} - L_{t_{i-1}})}{\frac{1}{n} \sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}}.$$

From Lemma 2 and Lemma 3, we have

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} \alpha_0) \xrightarrow{d} \frac{\sigma \int_0^1 \frac{X_t^0}{\sqrt{1+(X_t^0)^2}} dL_t}{\int_0^1 \frac{(X_t^0)^2}{1+(X_t^0)^2} dt}. \tag{13}$$

The proof is complete. ■

III. Parameter Estimation for the Hyperbolic Diffusion Driven by α-Stable Noises

We investigate parameter estimation for following α-stable hyperbolic diffusion:

$$\begin{cases} dX_t = \gamma \frac{X_t}{\sqrt{1 + X_t^2}} dt + \varepsilon\sigma dZ_t, \quad t \in [0,1] \\ X_0 = x_0, \end{cases} \tag{14}$$

where γ is unknown, Z is a strictly symmetric α-stable motion with $\alpha \in (1,2)$.

We assume that the process $\{X_t, t \geq 0\}$ can be observed at discrete point $\{t_i = i\Delta, i = 0,1,2,...,n\}$ with $\Delta > 0$. Consider the following contrast function:

$$\rho_{n,\varepsilon}(\gamma) = \sum_{i=1}^n \frac{|X_{t_i} - X_{t_{i-1}} - \gamma \frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}} \Delta t_{i-1}|^2}{\varepsilon^2 \sigma^2 \Delta t_{i-1}}, \tag{15}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

It can be obtained that

$$\widehat{\gamma}_{n,\varepsilon} = \frac{\sum_{i=1}^n \frac{(X_{t_i} - X_{t_{i-1}}) X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}}{\frac{1}{n} \sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}}. \tag{16}$$

Since

$$X_{t_i} - X_{t_{i-1}} = \gamma_0 \int_{t_{i-1}}^{t_i} \frac{X_s}{\sqrt{1 + (X_s)^2}} ds + \varepsilon\sigma \int_{t_{i-1}}^{t_i} dZ_s. \tag{17}$$

Substituting (17) into (16), we have

$$\begin{aligned}
& \widehat{\gamma}_{n,\varepsilon} \\
& = \frac{\gamma_0 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}\sqrt{1+X_s^2}} ds}{\frac{1}{n} \sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}} \\
& + \frac{\varepsilon\sigma \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}} dZ_s}{\frac{1}{n} \sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}}.
\end{aligned} \tag{18}$$

Consider the following ordinary differential equation:

$$dX_t^* = \gamma_0 \frac{X_t^*}{\sqrt{1 + (X_t^*)^2}} dt, \quad X_0^* = x_0.$$

Firstly, we give some important lemmas.

Lemma 4: When $\varepsilon \to 0$ and $n \to \infty$,

$$\sup_{0\leq t\leq 1} |X_t - X_t^*| \xrightarrow{P} 0.$$

Proof: Since

$$\begin{aligned}&X_t - X_t^*\\&= \gamma_0 \int_0^t (\frac{X_s}{\sqrt{1+X_s^2}} - \frac{X_s^*}{\sqrt{1+(X_s^*)^2}})ds + \varepsilon\sigma \int_0^t dZ_s,\end{aligned}$$

we obtain

$$\begin{aligned}&|X_t - X_t^*|^2\\ \leq\ & 2\gamma_0^2 |\int_0^t (\frac{X_s}{\sqrt{1+X_s^2}} - \frac{X_s^*}{\sqrt{1+(X_s^*)^2}})ds|^2\\ &+2\varepsilon^2\sigma^2|\int_0^t dZ_s|^2\\ \leq\ & 2t\gamma_0^2 \int_0^t |\frac{X_s}{\sqrt{1+X_s^2}} - \frac{X_s^*}{\sqrt{1+(X_s^*)^2}}ds|^2\\ &+2\varepsilon^2\sigma^2|\int_0^t dZ_s|^2\\ \leq\ & 8t\gamma_0^2 \int_0^t |X_s - X_s^*|^2 + 2\varepsilon^2\sigma^2|\int_0^t dZ_s|^2.\end{aligned}$$

By Gronwall's inequality, one has

$$|X_t - X_t^*|^2 \leq 2\varepsilon^2\sigma^2 e^{8t^2\gamma_0^2}|\int_0^t dZ_s|^2.$$

Then,

$$\sup_{0\leq t\leq 1} |X_t - X_t^*| \leq \sqrt{2}\varepsilon\sigma e^{4\gamma_0^2} \sup_{0\leq t\leq 1} |\int_0^t dZ_s|.$$

For any given $\eta > 0$, when $\varepsilon \to 0$ and $n \to \infty$,

$$\begin{aligned}&P(|\sqrt{2}\varepsilon\sigma e^{4\gamma_0^2} \sup_{0\leq t\leq 1} |\int_0^t dZ_s|| > \delta)\\ \leq\ & \delta^{-1}\sqrt{2}\varepsilon\sigma e^{4\gamma_0^2}\mathbb{E}[\sup_{0\leq t\leq 1} |\int_0^t dZ_s|]\\ \leq\ & C\delta^{-1}\sqrt{2}\varepsilon\sigma e^{4\gamma_0^2}\\ \to\ & 0,\end{aligned}$$

where C is a constant.

Therefore,

$$\sup_{0\leq t\leq 1} |X_t - X_t^*| \xrightarrow{P} 0. \tag{19}$$

■

Lemma 5: When $\varepsilon \to 0$ and $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2} \xrightarrow{P} \int_0^1 \frac{(X_t^*)^2}{1+(X_t^*)^2}dt.$$

Proof:

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}\\ &= \frac{1}{n}\sum_{i=1}^n \frac{(X_{t_{i-1}}^*)^2}{1+(X_{t_{i-1}}^*)^2}\\ &+\frac{1}{n}\sum_{i=1}^n (\frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2} - \frac{(X_{t_{i-1}}^*)^2}{1+(X_{t_{i-1}}^*)^2}).\end{aligned} \tag{20}$$

When $\varepsilon \to 0$ and $n \to \infty$,

$$\begin{aligned}&|\frac{1}{n}\sum_{i=1}^n (\frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2} - \frac{(X_{t_{i-1}}^*)^2}{1+(X_{t_{i-1}}^*)^2})|\\ =\ & |\frac{1}{n}\sum_{i=1}^n \frac{X_{t_{i-1}}^2 - (X_{t_{i-1}}^*)^2}{(1+X_{t_{i-1}}^2)(1+(X_{t_{i-1}}^*)^2)}|\\ \leq\ & \frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}^2 - (X_{t_{i-1}}^*)^2|\\ =\ & \frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}} + X_{t_{i-1}}^*||X_{t_{i-1}} - X_{t_{i-1}}^*|\\ \leq\ & \frac{1}{n}\sum_{i=1}^n (|X_{t_{i-1}}| + |X_{t_{i-1}}^*|)|X_{t_{i-1}} - X_{t_{i-1}}^*|\\ \leq\ & \sup_{0\leq t\leq 1}(|X_t| + |X_t^*|)|X_t - X_t^*|\\ \xrightarrow{P}\ & 0.\end{aligned}$$

Since

$$\frac{1}{n}\sum_{i=1}^n \frac{(X_{t_{i-1}}^*)^2}{1+(X_{t_{i-1}}^*)^2} \xrightarrow{P} \int_0^1 \frac{(X_t^*)^2}{1+(X_t^*)^2}dt, \tag{21}$$

we obtain

$$\frac{1}{n}\sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2} \xrightarrow{P} \int_0^1 \frac{(X_t^*)^2}{1+(X_t^*)^2}dt. \tag{22}$$

■

Now we introduce the main results.

Theorem 3: When $\varepsilon \to 0$, $n \to \infty$, and $\varepsilon n^{1-\frac{1}{\alpha}} \to 0$,

$$\widehat{\gamma}_{n,\varepsilon} \xrightarrow{P} \gamma_0.$$

Proof: Observe that

$$\begin{aligned}&\gamma_0 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}\sqrt{1+X_s^2}}ds\\ &\xrightarrow{P} \gamma_0 \int_0^1 \frac{X_t X_t^*}{\sqrt{1+X_t^2}\sqrt{1+(X_t^*)^2}}dt.\end{aligned} \tag{23}$$

When $\varepsilon \to 0$ and $n \to \infty$,

$$\begin{aligned}&\gamma_0 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}\sqrt{1+X_s^2}}ds\\ &\xrightarrow{P} \gamma_0 \int_0^1 \frac{(X_t^*)^2}{1+(X_t^*)^2}dt.\end{aligned} \tag{24}$$

For $\forall\delta > 0$, when $\varepsilon \to 0$, $n \to \infty$, and $\varepsilon n^{1-\frac{1}{\alpha}} \to 0$,

$$\begin{aligned}&P(|\varepsilon\sigma \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}dZ_s| > \delta)\\ &\leq \delta^{-1}\varepsilon\sigma \sum_{i=1}^n \mathbb{E}|\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}dZ_s|\\ &\leq 2K\delta^{-1}\varepsilon\sigma n^{1-\frac{1}{\alpha}} \sup_{0\leq t\leq 1} \mathbb{E}[(X_t^*)^{\frac{1}{2}}]\\ &\to 0,\end{aligned}$$

where K is constant.

Thus,

$$\varepsilon\sigma \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}dZ_s \xrightarrow{P} 0. \tag{25}$$

Therefore,

$$\widehat{\gamma}_{n,\varepsilon} \xrightarrow{P} \gamma_0. \tag{26}$$

■

Theorem 4: When $\varepsilon \to 0$, $n \to \infty$, and $n\varepsilon \to \infty$,

$$\varepsilon^{-1}(\widehat{\gamma}_{n,\varepsilon} - \gamma_0) \xrightarrow{d} \frac{\sigma(\int_0^1 (\frac{X_t^*}{\sqrt{1+(X_t^*)^2}})^\alpha dt)^{\frac{1}{\alpha}}}{(\int_0^1 \frac{(X_t^*)^2}{1+(X_t^*)^2} dt)} S_\alpha(1,0,0).$$

Proof: Since

$$\begin{aligned}
&\varepsilon^{-1}(\widehat{\gamma}_{n,\varepsilon} - \gamma_0) \\
&= \frac{\varepsilon^{-1}\gamma_0 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}\sqrt{1+X_s^2}} ds}{\frac{1}{n}\sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}} \\
&+ \frac{\sigma \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}} dZ_s}{\frac{1}{n}\sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}} \\
&- \frac{\varepsilon^{-1}\gamma_0 \frac{1}{n}\sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}}{\frac{1}{n}\sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}}.
\end{aligned}$$

We have

$$\varepsilon^{-1}\gamma_0 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}\sqrt{1+X_s^2}} ds \xrightarrow{P} 0, \tag{27}$$

and

$$\varepsilon^{-1}\gamma_0 \frac{1}{n}\sum_{i=1}^n \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2} \xrightarrow{P} 0. \tag{28}$$

As

$$\begin{aligned}
&\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}} dZ_s \\
&= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}^*}{\sqrt{1+(X_{t_{i-1}}^*)^2}} dZ_s \\
&+ \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}} - \frac{X_{t_{i-1}}^*}{\sqrt{1+(X_{t_{i-1}}^*)^2}}) dZ_s
\end{aligned} \tag{29}$$

For $\forall\delta > 0$, when $\varepsilon \to 0$, $n \to \infty$ and $n^{1-\frac{1}{\alpha}} \to 0$,

$$\begin{aligned}
&P(|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}} - \frac{X_{t_{i-1}}^*}{\sqrt{1+(X_{t_{i-1}}^*)^2}}) dZ_s| \\
&> \delta) \\
\le\ & \delta^{-1}\sum_{i=1}^n \mathbb{E}[|\int_{t_{i-1}}^{t_i} (\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}} - \frac{X_{t_{i-1}}^*}{\sqrt{1+(X_{t_{i-1}}^*)^2}}) \\
&dZ_s|] \\
=\ & \delta^{-1}\sum_{i=1}^n \mathbb{E}[|\int_{t_{i-1}}^{t_i} \\
&\frac{X_{t_{i-1}}\sqrt{1+(X_{t_{i-1}}^*)^2} - X_{t_{i-1}}^*\sqrt{1+X_{t_{i-1}}^2}}{\sqrt{1+X_{t_{i-1}}^2}\sqrt{1+(X_{t_{i-1}}^*)^2}} dZ_s|] \\
\le\ & 2\delta^{-1}\sum_{i=1}^n \mathbb{E}\int_{t_{i-1}}^{t_i} |X_{t_{i-1}} - X_{t_{i-1}}^*| dZ_s \\
\le\ & 4C\delta^{-1}\sum_{i=1}^n \mathbb{E}(\int_{t_{i-1}}^{t_i} |X_{t_{i-1}} - X_{t_{i-1}}^*|^\alpha ds)^{\frac{1}{\alpha}} \\
\le\ & 4C\delta^{-1}\sum_{i=1}^n \mathbb{E}\sup_{t_{i-1}\le t\le t_i} |X_t - X_t^*| n^{-\frac{2}{\alpha}} \\
\le\ & 4C\delta^{-1}\sum_{i=1}^n \mathbb{E}\sup_{0\le t\le 1} |X_t - X_t^*| n^{-\frac{1}{\alpha}} \\
\to\ & 0.
\end{aligned}$$

Then, we have

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\frac{X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}} - \frac{X_{t_{i-1}}^*}{\sqrt{1+(X_{t_{i-1}}^*)^2}}) dZ_s \xrightarrow{P} 0. \tag{30}$$

Since

$$\begin{aligned}
&\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}^*}{\sqrt{1+(X_{t_{i-1}}^*)^2}} dZ_s \\
&= \int_0^1 \sum_{i=1}^n \frac{X_{t_{i-1}}^*}{\sqrt{1+(X_{t_{i-1}}^*)^2}} 1_{(t_{i-1},t_i]}(s) dZ_s \\
&= Z' \circ \int_0^1 \sum_{i=1}^n (\frac{X_{t_{i-1}}^*}{\sqrt{1+(X_{t_{i-1}}^*)^2}} 1_{(t_{i-1},t_i]}(s))^\alpha ds,
\end{aligned}$$

where $Z' \overset{d}{=} Z$.

We obtain

$$\begin{aligned}
&\int_0^1 \sum_{i=1}^n (\frac{X_{t_{i-1}}^*}{\sqrt{1+(X_{t_{i-1}}^*)^2}} 1_{(t_{i-1},t_i]}(s))^\alpha ds \\
&\xrightarrow{P} \int_0^1 (\frac{X_t^*}{\sqrt{1+(X_t^*)^2}})^\alpha dt.
\end{aligned} \tag{31}$$

Thus,

$$\begin{aligned}
&Z' \circ \int_0^1 \sum_{i=1}^n (\frac{X_{t_{i-1}}^*}{\sqrt{1+(X_{t_{i-1}}^*)^2}} 1_{(t_{i-1},t_i]}(s))^\alpha ds \\
&\xrightarrow{P} Z' \circ \int_0^1 (\frac{X_t^*}{\sqrt{1+(X_t^*)^2}})^\alpha dt.
\end{aligned} \tag{32}$$

Then,

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X^*_{t_{i-1}}}{\sqrt{1+(X^*_{t_{i-1}})^2}}dZ_s \xrightarrow{d} \left(\int_0^1\left(\frac{X^*_t}{\sqrt{1+(X^*_t)^2}}\right)^{\alpha}dt\right)^{\frac{1}{\alpha}}S_\alpha(1,0,0). \quad (33)$$

Therefore,

$$\varepsilon^{-1}(\widehat{\gamma}_{n,\varepsilon}-\gamma_0)\xrightarrow{d}\frac{\sigma(\int_0^1(\frac{X^*_t}{\sqrt{1+(X^*_t)^2}})^{\alpha}dt)^{\frac{1}{\alpha}}}{(\int_0^1\frac{(X^*_t)^2}{1+(X^*_t)^2}dt)}S_\alpha(1,0,0). \quad (34)$$

■

IV. Simulation

We use the discrete sample $(X_{t_i})_{i=0,1,\ldots,n}$ to compute the estimator $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\gamma}_{n,\varepsilon}$. In Table 1 and Figure 1, $\sigma=0.8$, $x_0=0.1$, $\varepsilon=0.001$. In Table 2 and Figure 2, $x_0=0.5$, $\alpha=1.6$, $\varepsilon=0.002$. Two tables list the value of least squares estimator "$\widehat{\alpha}_{n,\varepsilon}$", "$\widehat{\gamma}_{n,\varepsilon}$", and the absolute errors (AE)"$|\alpha_0-\widehat{\alpha}_{n,\varepsilon}|$", "$|\gamma_0-\widehat{\gamma}_{n,\varepsilon}|$".

Two tables provide that when n is large enough and ε is small enough, the estimator is very close to the true parameter value. Two figures illustrate that If we let n converge to the infinity and ε converge to zero, the estimator will converge to the true value.

TABLE I
Least squares estimator simulation results of α_0

True	Aver		AE
α_0	Size n	$\widehat{\alpha}_{n,\varepsilon}$	$\|\alpha_0-\widehat{\alpha}_{n,\varepsilon}\|$
	10000	1.1452	0.1452
1	30000	1.0328	0.0328
	50000	1.0007	0.0007
	10000	2.1639	0.1639
2	30000	2.0581	0.0581
	50000	2.0006	0.0006

TABLE II
Least squares estimator simulation results of γ_0

True	Aver		AE
γ_0	Size n	$\widehat{\gamma}_{n,\varepsilon}$	$\|\gamma_0-\widehat{\gamma}_{n,\varepsilon}\|$
	10000	0.8572	0.1428
1	30000	1.0431	0.0431
	50000	1.0005	0.0005
	10000	2.1309	0.1309
2	30000	2.0608	0.0608
	50000	2.0010	0.0010

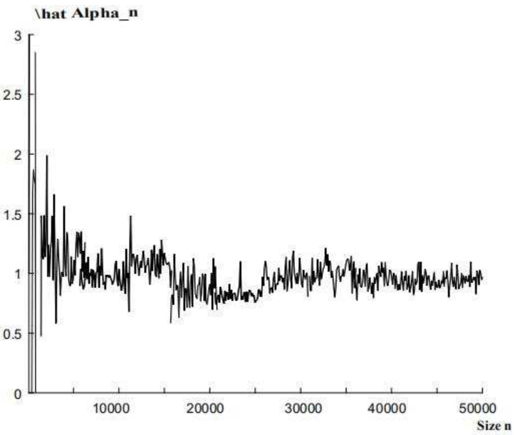

Fig. 1. The simulation of the estimator $\widehat{\alpha}_{n,\varepsilon}$ with $\alpha_0=1$

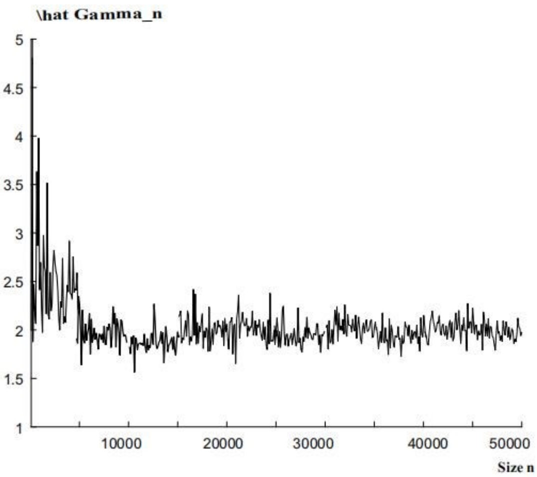

Fig. 2. The simulation of the estimator $\widehat{\gamma}_{n,\varepsilon}$ with $\gamma_0=2$

V. Conclusion

The aim of this paper is to estimate the parameter of hyperbolic diffusion driven by small Lévy noises from discrete observation. The least squares estimation has been used to obtain the parameter estimator. The consistency and asymptotic distribution of the estimator have been derived. Further research tops will include parameter estimation for stochastic differential equation driven by fractional Lévy noises.

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