Oscillation for a Class of Fractional Differential Equations with Damping Term in the Sense of the Conformable Fractional Derivative

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Abstract—**In this paper, we are concerned with oscillation for a class of fractional differential equations with damping term, where the fractional derivative is defined in the sense of the conformable fractional derivative. By certain inequality and integration average technique, some new oscillatory criteria for the equations are established. We also present one application for the results established.**

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Index Terms—**Oscillation; Fractional differential equations; Conformable fractional derivative; Damping term**

I. Introduction

In the research of qualitative properties for differential equations, research of existence, stability and oscillation has gained much attention by many authors in the last few decades [1-5]. Also some numerical methods have been presented so far [6-10]. In [11-24], oscillation of solutions of various differential equations and systems as well as dynamic equations on time scales were researched, and a lot of new oscillation criteria for these equations have been established therein. In these investigations, we notice that relatively less attention has been paid to the research of oscillation of fractional differential equations.

In [25], Chen researched oscillation of the following fractional differential equation:

$$[r(t)(D_-^\alpha y(t))^\eta]' - q(t)f(\int_t^\infty (v-t)^{-\alpha}y(v)dv) = 0,\ t>0,$$

where $r,\ q$ are positive-valued functions, η is the quotient of two odd positive numbers, $\alpha \in (0,1)$, $D_-^\alpha y(t)$ denotes the Liouville right-sided fractional derivative of order α of y, and $D_-^\alpha y(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty(\xi-t)^{-\alpha}y(\xi)d\xi$. Then in [26], under similar conditions to [25], some new oscillatory criteria are established for the following fractional differential equation with damping term:

$$D_-^{1+\alpha}y(t) - p(t)D_-^\alpha y(t) + q(t)f(\int_t^\infty (v-t)^{-\alpha}y(v)dv) = 0,\ t>0,$$

In [27], Han et al. investigated oscillation of a class of fractional differential equations as follows

$$[r(t)g((D_-^\alpha y)(t))]' - p(t)f(\int_t^\infty (s-t)^{-\alpha}y(s)ds) = 0,\ t>0,\ \alpha\in(0,1).$$

For the research mentioned above, we note that the fractional differential equations concerned are all defined in Liouville right-sided fractional derivative.

In [28-30], the authors researched oscillation of several classes of fractional differential equations as follows

$$D_a^\alpha x + f_1(t,x) = v(t) + f_2(t,x),$$

$$D_a^\alpha x(t) + q(t)f(x(t)) = 0,$$

$$(D_{0+}^{1+\alpha}y)t + p(t)(D_{0+}^\alpha y)t + q(t)f(y(t)) = 0,$$

where the fractional derivative is defined by the Riemann-Liouville derivative.

Recently, Khalil et al. proposed a new definition for fractional derivative named conformable fractional derivative [31]. The fractional derivative is defined as follows

$$D^\alpha f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

and satisfies the following properties:

(i). $D^\alpha[af(t)+bg(t)] = aD^\alpha f(t) + bD^\alpha g(t)$.

(ii). $D^\alpha(t^\gamma) = \gamma t^{\gamma-\alpha}$.

(iii). $D^\alpha[f(t)g(t)] = f(t)D^\alpha g(t) + g(t)D^\alpha f(t)$.

(iv). $D^\alpha C = 0$, where C is a constant.

(v). $D_t^\alpha f[g(t)] = f_g'[g(t)]D_t^\alpha g(t)$.

(vi). $D_t^\alpha(\frac{f}{g})(t) = \frac{g(t)D^\alpha f(t) - f(t)D^\alpha g(t)}{g^2(t)}$.

(vii). $D_t^\alpha f(t) = t^{1-\alpha}f'(t)$.

Note that the properties above can be easily proved due to the definition of the conformable fractional derivative. Afterwards, many authors investigated various applications of the conformable fractional derivative [32-37].

Motivated by the analysis above, in this paper, we are concerned with oscillation of a class of fractional differential equations with damping term as follows:

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$$D_t^\alpha[r(t)(D_t^\alpha x(t))^\gamma]+p(t)(D_t^\alpha x(t))^\gamma+q(t)x(t)=0,$$

$$t\geq t_0>0,\ 0<\alpha<1, \tag{1}$$

where $D_t^\alpha(.)$ denotes the conformable fractional derivative with respect to the variable t, the function $r\in C^\alpha([t_0,\infty),R_+)$, $p,\ q\in C([t_0,\infty),R_+)$, and C^α denotes continuous derivative of order α, γ is the ratio of two positive integers.

As usual, a solution $x(t)$ of Eq. (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Eq. (1) is called oscillatory if all its solutions are oscillatory.

We organize the next of this paper as follows. In Section 2, we establish some new oscillatory criteria for Eq. (1). In Section 3, we present some applications for them. Some conclusions are presented at the end of this paper.

For the sake of convenience, in the next of this paper, we denote $\xi=\frac{1}{\alpha}t^\alpha,\ \xi_i=\frac{1}{\alpha}t_i^\alpha,\ i=0,1,2,3,\ R_+=(0,\infty)$, $r(t)=\widetilde{r}(\xi),\ p(t)=\widetilde{p}(\xi),\ q(t)=\widetilde{q}(\xi)$, and $A(\xi)=exp(\int_{\xi_0}^{\xi}\frac{\widetilde{p}(\tau)}{\widetilde{r}(\tau)}d\tau)$.

II. Main results

Lemma 1 [38, Theorem 41]. Assume that A and B are nonnegative real numbers. Then

$$\lambda XY^{\lambda-1}-X^\lambda\leq(\lambda-1)Y^\lambda \text{ for all } \lambda>1.$$

Theorem 2. Let $h_1,\ h_2,\ \widehat{H}\in C([\xi_0,\infty),R)$ satisfying $\widehat{H}(\xi,\xi)=0,\widehat{H}(\xi,s)>0,\ \xi>s\geq\xi_0$, and H has continuous partial derivatives $\widehat{H}'_\xi(\xi,s)$ and $\widehat{H}'_s(\xi,s)$ on $[\xi_0,\infty)$. Assume that

$$\int_{\xi_0}^{\infty}\frac{1}{[A(s)\widetilde{r}(s)]^{\frac{1}{\gamma}}}ds=\infty, \tag{2}$$

and for any sufficiently large $T\geq\xi_0$, there exist $\phi\in C^1([t_0,\infty),R_+)$ and $\varphi\in C^1([t_0,\infty),[0,\infty))$, and $a,\ b,\ c$ with $T\leq a<c<b$ satisfying

$$\begin{aligned}&\frac{1}{\widehat{H}(b,c)}\int_c^b\widehat{H}(b,s)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\&\quad+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds\\&+\frac{1}{\widehat{H}(c,a)}\int_a^c\widehat{H}(s,a)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)\\&\quad+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds\\&>\frac{1}{\widehat{H}(b,c)}\int_c^b\widehat{H}(b,s)\{\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^\gamma(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}\\&\quad\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)\\&+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}'_s(b,s)\}^{\gamma+1}ds\\&+\frac{1}{\widehat{H}(c,a)}\int_a^c\widehat{H}(s,a)\{\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^\gamma(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}\\&\quad\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\\&\quad+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}'_s(s,a)\}^{\gamma+1}ds.\end{aligned} \tag{3}$$

where $\widetilde{\phi}(\xi)=\phi(t),\ \widetilde{\varphi}(\xi)=\varphi(t)$. Then every solution of Eq. (1) is oscillatory.

Proof. Assume (1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. Let $x(t)=\widetilde{x}(\xi)$. Then by use of the property (ii) we obtain $D_t^\alpha\xi(t)=1$, and furthermore by use of the property (v), we have

$$D_t^\alpha r(t)=D_t^\alpha\widetilde{r}(\xi)=\widetilde{r}'(\xi)D_t^\alpha\xi(t)=\widetilde{r}'(\xi), \tag{4}$$

and

$$D_t^\alpha x(t)=D_t^\alpha\widetilde{x}(\xi)=\widetilde{x}'(\xi)D_t^\alpha\xi(t)=\widetilde{x}'(\xi). \tag{5}$$

So Eq. (1) can be transformed into the following form:

$$[\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma]'+\widetilde{p}(\xi)(\widetilde{x}'(\xi))^\gamma+\widetilde{q}(\xi)\widetilde{x}^\gamma(\xi)=0,\ \xi\geq\xi_0>0, \tag{6}$$

Since $x(t)$ is a eventually positive solution of (1), then $\widetilde{x}(\xi)$ is a eventually positive solution of Eq. (6), and there exists $\xi_1>\xi_0$ such that $\widetilde{x}(\xi)>0$ on $[\xi_1,\infty)$. Furthermore, we have

$$\begin{aligned}&[A(\xi)\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma]'\\&=A(\xi)[\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma]'+A(\xi)\widetilde{p}(\xi)(\widetilde{x}'(\xi))^\gamma\\&\leq-KA(\xi)\widetilde{q}(\xi)\widetilde{x}^\gamma(\xi)<0,\end{aligned} \tag{7}$$

Then $A(\xi)\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma$ is strictly decreasing on $[\xi_1,\infty)$, and thus $\widetilde{x}'(\xi)$ is eventually of one sign. We claim $\widetilde{x}'(\xi)>0$ on $[\xi_2,\infty)$, where $\xi_2>\xi_1$ is sufficiently large. Otherwise, assume there exists a sufficiently large $\xi_3>\xi_2$ such that $\widetilde{x}'(\xi)<0$ on $[\xi_3,\infty)$. Then for $\xi\in[\xi_3,\infty)$ we have

$$\begin{aligned}\widetilde{x}(\xi)-\widetilde{x}(\xi_3)&=\int_{\xi_3}^{\xi}\widetilde{x}'(s)ds\\&=\int_{\xi_3}^{\xi}\frac{[A(s)\widetilde{r}(s)]^{\frac{1}{\gamma}}\widetilde{x}'(s)}{[A(s)\widetilde{r}(s)]^{\frac{1}{\gamma}}}ds\\&\leq[A(\xi_3)\widetilde{r}(\xi_3)]^{\frac{1}{\gamma}}\widetilde{x}'(\xi_3)\int_{\xi_3}^{\xi}\frac{1}{[A(s)\widetilde{r}(s)]^{\frac{1}{\gamma}}}ds.\end{aligned}$$

By (2) we deduce that $\lim_{\xi\to\infty}\widetilde{x}(\xi)=-\infty$, which contradicts to the fact that $\widetilde{x}(\xi)$ is a eventually positive solution of Eq. (6). So it holds that $\widetilde{x}'(\xi)>0$ on $[\xi_2,\infty)$.

Define the generalized Riccati transformation function:

$$\omega(t)=\widetilde{\phi}(\xi)\{\frac{A(\xi)\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma}{\widetilde{x}^\gamma(\xi)}+\widetilde{\varphi}(\xi)\}.$$

Let $\omega(t)=\widetilde{\omega}(\xi)$. Then $D_t^\alpha w(t)=\widetilde{w}'(\xi)$, and $D_t^\alpha\phi(t)=\widetilde{\phi}'(\xi),\ D_t^\alpha\varphi(t)=\widetilde{\varphi}'(\xi)$. So for $\xi\in[\xi_2,\infty)$, we have

$$\widetilde{\omega}'(\xi)=\widetilde{\phi}'(\xi)\frac{A(\xi)\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma}{\widetilde{x}^\gamma(\xi)}$$

$$-\widetilde{\phi}(\xi)\frac{\gamma A(\xi)\widetilde{x}^{\gamma-1}(\xi)\widetilde{r}(\xi)(\widetilde{x}'(\xi))^{\gamma+1}}{\widetilde{x}^{2\gamma}(\xi)}+\widetilde{\phi}(\xi)\frac{[A(\xi)\widetilde{r}(\xi)(\widetilde{x}'(\xi))^{\gamma}]'}{\widetilde{x}^{\gamma}(\xi)}$$

$$+\widetilde{\phi}'(\xi)\widetilde{\varphi}(\xi)+\widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi)$$
$$=\frac{\widetilde{\phi}'(\xi)}{\widetilde{\phi}(\xi)}\omega(\xi)-\gamma\frac{(\omega(\xi)-\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi))^{\frac{\gamma+1}{\gamma}}}{[A(\xi)\widetilde{\phi}(\xi)\widetilde{r}(\xi)]^{\frac{1}{\gamma}}}$$
$$+\widetilde{\phi}(\xi)\frac{[A(\xi)[\widetilde{r}(\xi)(\widetilde{x}'(\xi))^{\gamma}]'+\widetilde{r}(\xi)(\widetilde{x}'(\xi))^{\gamma}A'(\xi)]}{\widetilde{x}^{\gamma}(\xi)}+\widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi)$$

$$=\frac{\widetilde{\phi}'(\xi)}{\widetilde{\phi}(\xi)}\omega(\xi)-\gamma\frac{(\omega(\xi)-\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi))^{\frac{\gamma+1}{\gamma}}}{[A(\xi)\widetilde{\phi}(\xi)\widetilde{r}(\xi)]^{\frac{1}{\gamma}}}$$
$$+\widetilde{\phi}(\xi)\frac{[A(\xi)[\widetilde{r}(\xi)(\widetilde{x}'(\xi))^{\gamma}]'+\widetilde{r}(\xi)(\widetilde{x}'(\xi))^{\gamma}A'(\xi)D_t^{\alpha}\xi]}{\widetilde{x}^{\gamma}(\xi)}$$
$$+\widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi)$$

$$=\frac{\widetilde{\phi}'(\xi)}{\widetilde{\phi}(\xi)}\omega(\xi)-\gamma\frac{(\omega(\xi)-\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi))^{\frac{\gamma+1}{\gamma}}}{[A(\xi)\widetilde{\phi}(\xi)\widetilde{r}(\xi)]^{\frac{1}{\gamma}}}$$
$$+\widetilde{\phi}(\xi)\frac{[A(\xi)[\widetilde{r}(\xi)(\widetilde{x}'(\xi))^{\gamma}]'+\widetilde{r}(\xi)(\widetilde{x}'(\xi))^{\gamma}A(\xi)\frac{\widetilde{p}(\xi)}{\widetilde{r}(\xi)}]}{\widetilde{x}^{\gamma}(\xi)}$$
$$+\widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi)$$

$$=\frac{\widetilde{\phi}'(\xi)}{\widetilde{\phi}(\xi)}\omega(\xi)-\gamma\frac{(\omega(\xi)-\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi))^{\frac{\gamma+1}{\gamma}}}{[A(\xi)\widetilde{\phi}(\xi)\widetilde{r}(\xi)]^{\frac{1}{\gamma}}}$$
$$+\widetilde{\phi}(\xi)\frac{[A(\xi)[\widetilde{r}(\xi)(\widetilde{x}'(\xi))^{\gamma}]'+\widetilde{p}(\xi)(\widetilde{x}'(\xi))^{\gamma}A(\xi)]}{\widetilde{x}^{\gamma}(\xi)}$$
$$+\widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi)$$

$$=-A(\xi)\widetilde{\phi}(\xi)\widetilde{q}(\xi)+\widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi)+\frac{\widetilde{\phi}'(\xi)}{\widetilde{\phi}(\xi)}\widetilde{\omega}(\xi)$$
$$-\gamma\frac{(\widetilde{\omega}(\xi)-\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi))^{\frac{\gamma+1}{\gamma}}}{[A(\xi)\widetilde{\phi}(\xi)\widetilde{r}(\xi)]^{\frac{1}{\gamma}}},\xi\geq\xi_2. \tag{8}$$

Using the inequality ([39, Eq. (2.17)]) we obtain that

$$[\widetilde{\omega}(\xi)-\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi)]^{1+\frac{1}{\gamma}}\geq$$
$$\widetilde{\omega}^{1+\frac{1}{\gamma}}(\xi)+\frac{1}{\gamma}[\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi)]^{1+\frac{1}{\gamma}}-(1+\frac{1}{\gamma})[\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi)]^{\frac{1}{\gamma}}\widetilde{\omega}(\xi). \tag{9}$$

A combination of (8) and (9) yields:

$$\widetilde{\omega}'(\xi)\leq-\widetilde{q}(\xi)\widetilde{\phi}(\xi)A(\xi)$$
$$-\frac{\gamma}{[\widetilde{\phi}(\xi)\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}}\{\widetilde{\omega}^{1+\frac{1}{\gamma}}(\xi)+\frac{1}{\gamma}[\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi)]^{1+\frac{1}{\gamma}}$$
$$-(1+\frac{1}{\gamma})[\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi)]^{\frac{1}{\gamma}}\widetilde{\omega}(\xi)\}+\frac{\widetilde{\phi}'(\xi)}{\widetilde{\phi}(\xi)}\widetilde{\omega}(\xi)+\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi)$$

$$=-\widetilde{q}(\xi)\widetilde{\phi}(\xi)A(\xi)+\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi)-\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(\xi)\widetilde{\phi}(\xi)}{[\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}}$$

$$-\frac{\gamma}{[\widetilde{\phi}(\xi)\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}}\widetilde{\omega}^{1+\frac{1}{\gamma}}(\xi)+\{\frac{(\gamma+1)\widetilde{\varphi}^{\frac{1}{\gamma}}(\xi)}{[\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}}+\frac{\widetilde{\phi}'(\xi)}{\widetilde{\phi}(\xi)}\}\widetilde{\omega}(\xi)$$

$$=-\widetilde{q}(\xi)\widetilde{\phi}(\xi)A(\xi)+\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi)-\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(\xi)\widetilde{\phi}(\xi)}{[\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}}$$
$$-\frac{\gamma}{[\widetilde{\phi}(\xi)\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}}\widetilde{\omega}^{1+\frac{1}{\gamma}}(\xi)$$
$$+\{\frac{(\gamma+1)\widetilde{\phi}(\xi)\widetilde{\varphi}^{\frac{1}{\gamma}}(\xi)+\widetilde{\phi}'(\xi)[\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}}{\widetilde{\phi}(\xi)[\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}}\}\widetilde{\omega}(\xi). \tag{10}$$

Select $a,\ b,\ c$ arbitrarily in $[\xi_2,\infty)$ with $b>c>a$. Substituting ξ with s, multiplying both sides of (10) by $\widehat{H}(\xi,s)$ and integrating it with respect to s from c to ξ for $\xi\in[c,b)$, we get that

$$\int_c^{\xi}\widehat{H}(\xi,s)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$

$$\leq-\int_c^{\xi}\widehat{H}(\xi,s)\widetilde{w}'(s)ds$$
$$+\int_c^{\xi}\widehat{H}(\xi,s)\{-\frac{\gamma}{[\widetilde{\phi}(s)\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\widetilde{\omega}^{1+\frac{1}{\gamma}}(s)$$
$$+\{\frac{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}{\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}\widetilde{\omega}(s)\}ds$$

$$=\widehat{H}(\xi,c)\widetilde{w}(c)+\int_c^{\xi}\widehat{H}(\xi,s)\{-\frac{\gamma}{[\widetilde{\phi}(s)\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\widetilde{\omega}^{1+\frac{1}{\gamma}}(s)$$
$$+\frac{1}{\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)$$
$$+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}_s'(\xi,s)\}\widetilde{\omega}(s)\}ds. \tag{11}$$

Setting

$$\lambda=1+\frac{1}{\gamma},\ X^{\lambda}=\frac{\gamma}{[\widetilde{\phi}(s)\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\widetilde{\omega}^{1+\frac{1}{\gamma}}(s),$$

$$Y^{\lambda-1}=\frac{\gamma^{\frac{1}{\gamma+1}}}{(\gamma+1)\widetilde{\phi}^{\frac{\gamma}{\gamma+1}}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma(\gamma+1)}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)$$
$$+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}_s'(\xi,s)\},$$

by a combination of Lemma 1 and (11) we get that

$$\int_c^{\xi}\widehat{H}(\xi,s)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$

$$\leq\widehat{H}(\xi,c)\widetilde{w}(c)+\int_c^{\xi}\widehat{H}(\xi,s)\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}$$
$$\{\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}$$
$$+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}_s'(\xi,s)\}^{\gamma+1}\}ds. \tag{12}$$

Dividing both sides of the inequality (12) by $\widehat{H}(\xi,c)$ and let $\xi\to b^-$, we obtain

$$\frac{1}{\widehat{H}(b,c)}\int_c^b\widehat{H}(b,s)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$

$$\le\widetilde{w}(c)+\frac{1}{\widehat{H}(b,c)}\int_c^b\widehat{H}(b,s)\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}'_s(b,s)\}^{\gamma+1}\}ds. \tag{13}$$

On the other hand, substituting ξ with s, multiplying both sides of (12) by $\widehat{H}(s,\xi)$ and integrating it with respect to s from ξ to c for $\xi\in(a,c]$, similar to (11)-(12) we get that

$$\int_\xi^c\widehat{H}(s,\xi)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$

$$\le-\widehat{H}(c,\xi)\widetilde{w}(c)+\int_\xi^c\widehat{H}(s,\xi)\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}'_s(s,\xi)\}^{\gamma+1}\}ds. \tag{14}$$

Dividing both sides of the inequality (14) by $\widehat{H}(c,\xi)$ and letting $\xi\to a^+$, we obtain

$$\frac{1}{\widehat{H}(c,a)}\int_a^c\widehat{H}(s,a)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$

$$\le-\widetilde{w}(c)+\frac{1}{\widehat{H}(c,a)}\int_a^c\widehat{H}(s,a)\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}'_s(s,a)\}^{\gamma+1}\}ds. \tag{15}$$

A combination of (13) and (15) yields

$$\frac{1}{\widehat{H}(b,c)}\int_c^b\widehat{H}(b,s)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$

$$+\frac{1}{\widehat{H}(c,a)}\int_a^c\widehat{H}(s,a)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$

$$\le\frac{1}{\widehat{H}(b,c)}\int_c^b\widehat{H}(b,s)\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}'_s(b,s)\}^{\gamma+1}\}ds$$

$$+\frac{1}{\widehat{H}(c,a)}\int_a^c\widehat{H}(s,a)\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}'_s(s,a)\}^{\gamma+1}\}ds. \tag{16}$$

which contradicts to (3). So the proof is complete.

Theorem 3. If (2) holds, and for any sufficiently large $l\ge\xi_0$,

$$\lim_{\xi\to\infty}\sup\int_l^\xi\widehat{H}(s,l)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}-\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}'_s(s,l)\}^{\gamma+1}\}ds>0, \tag{17}$$

and

$$\lim_{\xi\to\infty}\sup\int_l^\xi\widehat{H}(\xi,s)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}-\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}'_s(\xi,s)\}^{\gamma+1}\}ds>0, \tag{18}$$

then Eq. (1) is oscillatory.

Proof: For any sufficiently large $T\ge\xi_0$, let $a=T$. In (17) we choose $l=a$. Then there exists $c>a$ such that

$$\int_a^c\widehat{H}(s,a)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}-\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}'_s(s,a)\}^{\gamma+1}\}ds>0. \tag{19}$$

In (18) we choose $l=c>a$. Then there exists $b>c$ such that

$$\int_c^b\widehat{H}(b,s)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}-\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\widehat{H}'_s(b,s)\}^{\gamma+1}\}ds>0. \tag{20}$$

Combining (19) and (20) we obtain (3). So according to Theorem 2, Eq. (1) is oscillatory.

In Theorems 2-3, if we choose $\widehat{H}(\xi,s)=(\xi-s)^{\lambda},\ \xi\geq s\geq\xi_0$, where $\lambda>1$ is a constant, then we obtain the following two corollaries.

Corollary 4. Under the conditions of Theorem 2, if for any sufficiently large $T\geq\xi_0$, there exist $a,\ b,\ c$ with $T\leq a<c<b$ satisfying

$$\frac{1}{(c-a)^{\lambda}}\int_a^c(s-a)^{\lambda}\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$

$$+\frac{1}{(b-c)^{\lambda}}\int_c^b(b-s)^{\lambda}\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$

$$>\frac{1}{(c-a)^{\lambda}}\int_a^c(s-a)^{\lambda}\{\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\lambda(s-a)^{\lambda-1}\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}ds$$

$$+\frac{1}{(b-c)^{\lambda}}\int_c^b(b-s)^{\lambda}\{\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}-\lambda(b-s)^{\lambda-1}\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}ds,\qquad(21)$$

then Eq. (1) is oscillatory.

Corollary 5. Under the conditions of Theorem 3, if for any any sufficiently large $l\geq\xi_0$,

$$\limsup_{\xi\to\infty}\int_l^{\xi}(s-l)^{\lambda}\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}-\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+\lambda(s-l)^{\lambda-1}\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}\}ds>0\qquad(22)$$

and

$$\limsup_{\xi\to\infty}\int_l^{\xi}(\xi-s)^{\lambda}\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}-\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}-\lambda(\xi-s)^{\lambda-1}\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}\}ds>0,\qquad(23)$$

then Eq. (1) is oscillatory.

Theorem 6. Under the conditions of Theorem 2, furthermore, suppose (2) does not hold. If for any $T\geq\xi_0$, there exist $a,\ b$ with $b>a\geq T$ such that for any $u\in C[a,b]$, $u'(t)\in L^2[a,b]$, $u(a)=u(b)=0$, the following inequality holds:

$$\int_a^b u^2(s)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}-\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+2u(s)u'(s)\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}\}ds>0,\qquad(24)$$

where $\widetilde{\phi},\ \widetilde{\varphi}$ are defined as in Theorem 2, then Eq. (1) is oscillatory.

Proof: Assume (1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. Let $x(t)=\widetilde{x}(\xi)$, and $\omega(t)$, $\widetilde{\omega}(\xi)$ be defined as in Theorem 2. Then similar to the proof of Theorem 2, it holds that $\widetilde{x}'(\xi)>0$ on $[\xi_2,\infty)$, where ξ_2 is sufficiently large, and we can obtain (10). Select $a,\ b$ arbitrarily in $[\xi_2,\infty)$ with $b>a$ such that $u(a)=u(b)=0$. Substituting ξ with s, multiplying both sides of (10) by $u^2(s)$, integrating it with respect to s from a to b, we get that

$$\int_a^b u^2(s)[A(s)\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}]ds$$

$$\leq-\int_a^b u^2(s)\widetilde{w}'(s)ds+\int_a^b u^2(s)\{-\frac{\gamma}{[\widetilde{\phi}(s)\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\widetilde{\omega}^{1+\frac{1}{\gamma}}(s)+\{\frac{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}{\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}\widetilde{\omega}(s)\}ds$$

$$=\int_a^b u^2(s)\{-\frac{\gamma}{[\widetilde{\phi}(s)\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\widetilde{\omega}^{1+\frac{1}{\gamma}}(s)+\frac{1}{\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+2u(s)u'(s)\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}\widetilde{\omega}(s)\}ds.\qquad(25)$$

Setting

$$\lambda=1+\frac{1}{\gamma},\ X^{\lambda}=\frac{\gamma}{[\widetilde{\phi}(s)\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\widetilde{\omega}^{1+\frac{1}{\gamma}}(s),$$

$$Y^{\lambda-1}=\frac{\gamma^{\frac{1}{\gamma+1}}}{(\gamma+1)\widetilde{\phi}^{\frac{\gamma}{\gamma+1}}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma(\gamma+1)}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}+2u(s)u'(s)\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\},$$

by a combination of Lemma 1 and (25) we get that

$$\int_a^b u^2(s)\{A(s)\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$

$$\le \int_c^\xi u^2(s)\{\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^\gamma(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}$$

$$\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s) + \widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}$$

$$+2u(s)u'(s)\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}ds. \quad (26)$$

which contradicts to (24). So every solution of Eq. (1) is oscillatory, and the proof is complete.

III. Applications

Consider the following fractional differential equation with damping term:

$$D_t^\alpha[(D_t^\alpha x(t))^{\frac{1}{3}}] + t^{\frac{2}{3}} D_t^\alpha x(t) + \frac{t^{2\alpha}}{\alpha^2} x^{\frac{1}{3}}(t) = 0,$$

$$t \ge 5,\ 0 < \alpha < 1. \quad (27)$$

In Eq. (1), if we set $t_0 = 5,\ \gamma = \frac{1}{3},\ r(t) \equiv 1,\ p(t) = t^{\frac{2}{3}},\ q(t) = \frac{t^{2\alpha}}{\alpha^2}$, then we obtain (27). So $\widetilde{r}(\xi) \equiv 1,\ \widetilde{q}(\xi) = q(t) = \frac{t^{2\alpha}}{\alpha^2} = \xi^2$. Furthermore, since $A(\xi) = exp(\int_{\xi_0}^\xi \frac{\widetilde{p}(\tau)}{\widetilde{r}(\tau)}d\tau) \ge 1$, so in (22)-(23), after letting $\widetilde{\phi}(\xi) \equiv 1,\ \widetilde{\varphi}(\xi) = 0,\ \lambda = 2$, considering $\widetilde{q}(s) \equiv 1$, for any sufficiently large l we obtain

$$\lim_{\xi\to\infty} \sup \int_l^\xi (s-l)^\lambda\{A(s)\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}$$

$$-\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^\gamma(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)$$

$$+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}} + \lambda(s-l)^{\lambda-1}\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}\}ds$$

$$= \lim_{\xi\to\infty} \sup \int_l^\xi A(s)(s-l)^2 \left[s^2 - (\frac{3}{2})^{\frac{4}{3}}(s-l)^{\frac{4}{3}}\right] ds$$

$$\ge \lim_{\xi\to\infty} \sup \int_l^\xi (s-l)^2 \left[s^2 - (\frac{3}{2})^{\frac{4}{3}}(s-l)^{\frac{4}{3}}\right] ds = \infty$$

and

$$\lim_{\xi\to\infty} \sup \int_l^\xi (\xi-s)^\lambda\{A(s)\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\varphi}^{1+\frac{1}{\gamma}}(s)\widetilde{\phi}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}$$

$$-\frac{1}{(\gamma+1)^{\gamma+1}\widetilde{\phi}^\gamma(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\{(\gamma+1)\widetilde{\phi}(s)\widetilde{\varphi}^{\frac{1}{\gamma}}(s)$$

$$+\widetilde{\phi}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}} - \lambda(\xi-s)^{\lambda-1}\widetilde{\phi}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}\}ds$$

$$= \lim_{\xi\to\infty} \sup \int_l^\xi A(s)(\xi-s)^2 \left[s^2 - (\frac{3}{2})^{\frac{4}{3}}(\xi-s)^{\frac{4}{3}}\right] ds$$

$$\ge \lim_{\xi\to\infty} \sup \int_l^\xi (\xi-s)^2 \left[s^2 - (\frac{3}{2})^{\frac{4}{3}}(\xi-s)^{\frac{4}{3}}\right] ds = \infty.$$

So according to Corollary 5 we deduce that Eq. (27) is oscillatory.

IV. Conclusions

We have investigated oscillation for a class of fractional differential equations with damping term, where the fractional derivative is defined in the sense of the conformable fractional derivative derivative. Some new oscillatory criteria were presented. The validation of the main results have been verified by one application.

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