Warping Behavior of Open and Closed Thin-Walled Sections with Restrained Torsion

Tesfaldet Gebre, Vera Galishnikova, Evgenia Tupikova

***Abstract*—The behavior of warping torsion of thin-walled members of open and closed cross sections is studied using the classical theories presented by Timoshenko and Benscoter. Based on their studies, the axial shear displacement modes and torsional warping functions are determined. In addition, the warping behavior of thin-walled sections of a bar with a non-uniform warping including the effects of torsion and shear forces using the governing equation for torsion are considered for different section types. Warping is extensively applied to different types of thin-walled steel cross section subjected to concentrated torsional loading, and to the most general torsional boundary conditions, by considering both torsional warping and shear deformation effects (primary shear deformation due to Saint-Venant torsion and secondary shear deformation due to restrained torsion). Warping functions indicate the non-plane warping deformations of thin-walled cross sections and sections with warping functions can warp, and normal stresses and strains from torsion (bi-moment) are proportional to the sectorial coordinate diagrams. Finally, the effects of warping on the open and closed thin-walled beams with restrained torsion are investigated, and it is shown that torsional warping is included as exceptional cases of deformations.**

***Index Terms*— thin-walled sections, warping, torsion restrained, open and closed section, resultant shear**

I. Introduction

THIN -walled beams are structural elements with three characteristic dimensions of different orders of magnitude: the thickness is small when compared to the dimensions of the cross-section, which in turn are small when compared to the beam length[1]. Thin-walled member with open and closed cross sections have been extensively used in a variety of structures that require high strength-to-weight, stiffness-to-weight ratios and as its one dimension much larger than the other two dimensions commonly referred to as cross-sectional dimensions. If a very thin column is designed then it will likely fail because of premature local buckling [2].

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T.H. Gebre, PhD candidate, Department of civil engineering, Engineering Academy, Peoples' Friendship University of Russia, Miklukho Maklaya, 6, 117198, Moscow, Russia (phone: +7 (495) 434-53-00, email: tesfaldethg@gmail.com)

V. V. Galishnikova, DSc, Professor, Vice-Rector of Moscow State University of Civil Engineering, Yaroslavskoye, 26, 129337 Moscow, Russia (e-mail: GalishnikovaVV@mgsu.ru

E. M. Tupikova, Associate Professor, Department of Civil Engineering, Engineering Academy Peoples' Friendship University of Russia, Miklukho Maklaya, 6, 117198, Moscow, Russia (e-mail: tupikova_em@pfur.ru).

The behavior of warping of open and closed thin-walled cross-sections was reported in [2],[3] with the assumption that the in-plane deformations of the section are negligible, the shear strain of the middle surface was neglected, and the shear deformation cannot be considered. The analysis of shear stiffness, warping constant, shear center and restrained warping, must be considered separately for open and closed sections. In addition to obtain these expressions, we must consider the shear flow caused by the shear force [4]. The torsion of a bar leads to two types of shear stress. If the shear stresses are in equilibrium without axial stress, the torsion is called uniform (St. Venant torsion) which leads to axial displacements that are called warping of the section [5]. The normal stresses due to warping restraint are derived from the strains that are caused by the warping restraint [4]. The shear stresses due to warping restraint are determined with the equilibrium equations [6]. For arbitrary profiles, loading cases and boundary conditions, the De Saint Venant torsional theory is not acceptable since the axial stresses are neglected [7-9]. If the axial displacements are restrained, torsion leads to longitudinal stresses which vary over the length of the bar. Equilibrium in the bar under the longitudinal stresses requires additional shear stresses, which cause additional torsional moments of the axis of rotation. Torsion that causes significant longitudinal stresses is called nonuniform torsion [7][6][10]-[13]. In both cases, it will be shown that the support conditions, the shape of the section and the load distribution must be considered.

Extensive research exists on the study of non-uniform torsion with or without consideration of warping effects for both open and closed thin-walled sections [13]-[19]. Most of the studies were performed with full consideration of open thin-walled beam sections. There are much more factors resisting torsion in closed thin-walled beam than in open thin-walled beam subjected to torsional load, and the torsion analysis of closed thin-walled beam become more elaborate, because the problem is statically indeterminate [20]. The analysis of uniform and nonuniform torsion shows that the warping behaviour of bars depends on the shape of their thin-walled sections. Due to numerical efficiency considerations, in many analyses, the warping components are modelled using a complete set of pre-defined warping shape functions [21]. Prismatic bars whose section is a full circle, a circular tube, a square tube, or a thin-walled section with a single interior vertex (for example an angle, a tee or a cross) do not warp. If the displacement of bars with such sections is restrained in the axial direction, their uniform torsional behaviour is not affected. Methods for analysis of open and closed cross-sections are generalized to include distortional displacement modes [22].

In this paper, the warping behavior of thin-walled sections of a bar with restrained torsion using the governing equation for homogeneous cross-section made of isotropic elastic material subjected to torsion is studied, and the examples are focused on open and closed sections of thin-walled structures. The study presented in this paper is extensively applied to different types of thin-walled steel cross section subjected to concentrated torsional loading and to the most general torsional boundary conditions. Warping functions indicate the non-plane warping deformations of thin-walled cross sections and sections with warping functions can warp, and normal stresses and strains from torsion (bimoment) are proportional to the sectorial coordinate diagrams.

II. Theoretical Basic Concepts

The torsion of bars with general sections, particularly thin-walled open sections, is associated with warping. As an example, as shown in Fig. 1, he displacements of an I-beam which is fork-supported and loaded with a twisting moment. The axis of the flanges displaces laterally because of the rotation of the sections about the axis of the bar. In addition,

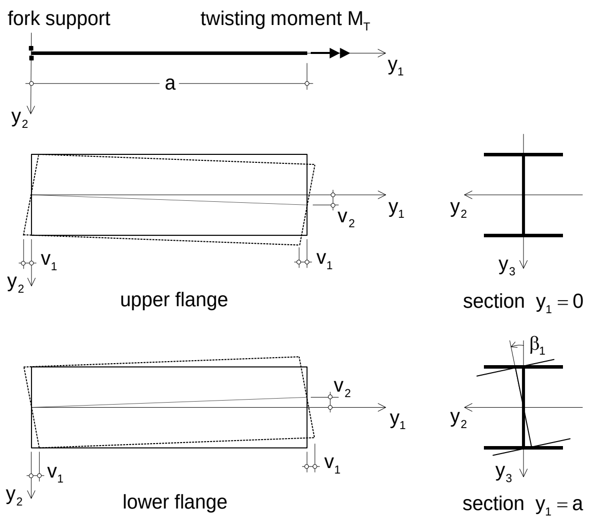


Fig. 1. Warping displacement and rotation displacement of an I-beam

there is a warping displacement in the longitudinal direction that varies linearly over the width of the flanges. The warping is constant over the length of the beam. The upper and lower flanges displace in opposite directions.

If the warping of a bar is restrained, the longitudinal displacement v_1 is no longer independent of the axial coordinate y_1 and the longitudinal strain is not null. The stresses σ_{11} affect both the displacements of the bar and its general state of stress, as a result it belongs to nonuniform torsion. The difference between the displacements and stresses due to uniform and nonuniform torsion is most pronounced for thin-walled sections. The governing equation for nonuniform torsion is therefore derived for bars with thin-walled sections, whose description is treated in Fig. 1. The local coordinate systems z_1, z_2, z_3 that have been introduced for the walls in Fig. 2 are also used in this section. The displacement coordinates w_1, w_2, w_3 and all stress coordinates are referred to the local coordinate system.

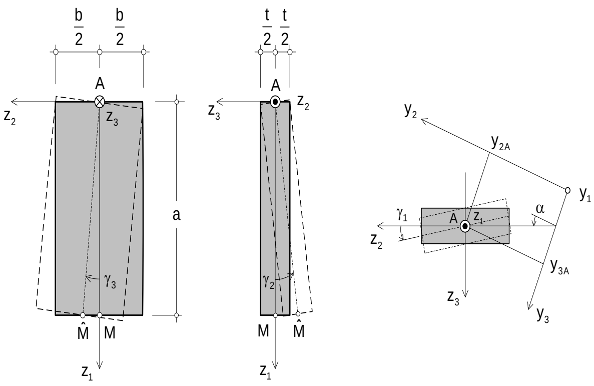


Fig. 2. local coordinate systems z_1, z_2, z_3 of wall segment in a section

Consider the wall in the section as depicted in Fig. 2 whose axis z_2 makes an angle α with axis y_2 of the section. If section $y_1 = a$ of the bar is twisted about the centre of

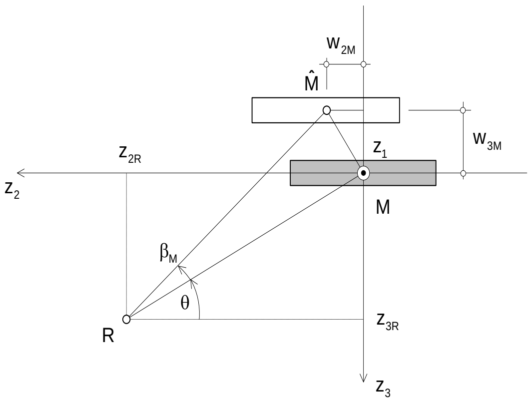


Fig. 3. The wall segment in a section with rigid body displacement

rotation R, the axis of the wall is rotated so that it makes angles γ_1, γ_2 *and* γ_3 with axes z_1, z_2 and z_3. Let the angle of rotation of the section at $y_1 = a$ about the center of rotation R be β_M as shown in fig. 2

The total displacement of the wall due to the twisting of the bar about the center of rotation R is decomposed into a rigid body motion and a twisting deformation. The rigid body motion rotates the axis of the wall from AM to $A\hat{M}$, but does not rotate the rectangle about axis z_1. The rigid body displacement of a point $P := (z_1, z_2, z_3)$ in section z_1 equals the rigid body displacement of the midpoint of the rectangle at the section: z

The displacement of point P due to the twisting of the wall about axis $A\hat{M}$ is given by the theory of uniform torsion:

$$w_{V1} = -\omega_V(z_2, z_3)\frac{d\beta_1}{dy_1} \tag{2}$$

$$w_{V2} = z_3\,\beta_1 \tag{3}$$

$$w_{V3} = -z_2\,\beta_1 \tag{4}$$

w_{V1} - displacement due to uniform torsion of the wall,

ω_V - warping function for uniform torsion of the wall (St. Venant).
The warping function $\omega_V(z_2, z_3)$ for uniform torsion of the wall is given by:

$$\omega_V(z_2, z_3) = -z_2 z_3 \tag{5}$$

It can be shown that the warping of the wall due to twisting about axis z_1 can be neglected relative to the warping due to rigid body motion. Consider the flange of an I-beam with height h, flange width b and flange thickness t. The maximum warping due to rotation about the midpoint of the flange follows from expressions (2) and (5):

$$w_{V1} = -\frac{bt}{4}\frac{d\beta_1}{dy_1}$$

The warping due to the rigid body motion follows from expression (1):

$$w_{R1} = (-z_2 z_{3R} + z_3 z_{2R})\frac{d\beta_1}{dy_1} = -\frac{b}{2}\left(\frac{h-t}{2}\right)\frac{d\beta_1}{dy_1}$$

The ratio of the warping due to twisting and the total warping of the flange is thus:

$$r_F = \frac{w_{V1}}{w_{V1} + w_{R1}} = \frac{t}{h}$$

For typical structural sections, the ratio r_F lies in the range from 0.02 to 0.05. The warping w_{V1} will therefore be neglected. It will be assumed that the total warping of the wall is caused by the rigid body motion of the wall. The warping displacement is determined from the condition that the rigid body motion does not cause shear strain:

$$\varepsilon_{12} = \frac{\partial w_1}{\partial z_2} + \frac{\partial w_2}{\partial z_1} = 0$$

$$\frac{\partial w_1}{\partial z_2} = -\frac{\partial w_2}{\partial z_1} \tag{6}$$

The displacement w_2 at section y_1 defined as follow:

$$w_2 = z_{3R}\,\beta_1$$

The above expression is substituted into equation (6). The warping function of the section for twisting of the bar about axis y_1 passing through the centre of rotation R is called ω_R and is determined with the definition of equation (2):

$$\frac{\partial w_1}{\partial z_2} = -\frac{d\beta_1}{dy_1} z_{3R}$$

$$\frac{\partial \omega_R}{\partial z_2} = z_{3R} \tag{7}$$

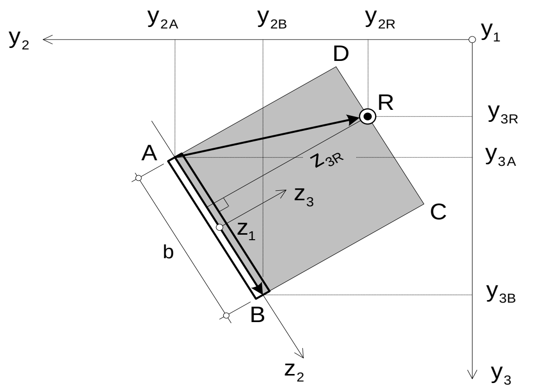


Fig. 4. The contribution of a wall AB to the warping function (ω_R)

The warping function ω_R is set to null at an arbitrary vertex. Equation (7) is integrated for each wall of the bar as:

$$\triangle\omega_{R(i)} = \int_{-0.5b_i}^{0.5b_i} z_{3R(i)}\,dz_2, \tag{8}$$

where $\triangle\omega_{R(i)}$ - contribution of wall I to the warping of the section
$z_{3R(i)}$ - coordinate z_3 of the centre of rotation R in the coordinate system of wall i.

The integral in expression (10) equals the area of rectangle ABCD in Fig. 4. Its value is determined with the cross product of vectors AB and AR:

$$AB = \boxed{\begin{array}{c} y_{2B} - y_{2A} \\ \hline y_{3B} - y_{3A} \end{array}} \qquad AR = \boxed{\begin{array}{c} y_{2R} - y_{2A} \\ \hline y_{3R} - y_{3A} \end{array}}$$

$$\triangle\omega_{Ri} = AB \times AR$$

$$= (y_{2B} - y_{2A})(y_{3R} - y_{3A}) - (y_{2R} - y_{2A})(y_{3B} - y_{3A})$$

The section is traversed in the direction of the walls, starting at vertex 0. When vertex k is reached, the value $\omega_{R(k)}$ of the warping function at vertex k is already known. The value of the warping function at the end vertex of each wall i leaving node k is determined by adding $\triangle\omega_{R(i)}$ to $\omega_{R(k)}$.

Once the value of the warping function ω_R is known at the vertices of the section, the warping displacement of the vertices can be computed with:

$$w_{1(k)} = -\omega_{R(k)}\frac{d\beta_1}{dy_1}, \tag{9}$$

where $w_{1(k)}$ - warping displacement at node k,
$\omega_{R(k)}$ - warping function at node k.

It follows from equation (7) that the warping varies linearly between the vertices of the section. For uniform torsion, the twisting rate $d\beta_1 / dy_1$ is constant over the length of the bar. The warping displacement in equation (9) is therefore constant over the length of a bar that is subject to uniform torsion.

III. Longitudinal stress, Shear Stress and Shear Flow

It is assumed that the distribution of the warping displacement w_1 for nonuniform torsion over the section is the same as that for uniform torsion in expression:

$$w_1(z_1, z_2, z_3) = -\omega(z_2, z_3)\frac{d\beta_1(z_1)}{dz_1}, \qquad (10)$$

where $\omega(z_2, z_3)$ - warping function for uniformed torsion.

The longitudinal stress $\sigma_{11(\omega)}$ due the displacement of equation (10) is derived from the strain and is the longitudinal stress due to nonuniform torsion is given as:

$$\sigma_{11(\omega)} = E\,\varepsilon_{11(\omega)} = E\frac{\partial w_1}{\partial z_1} = -E\,\omega\frac{d^2\beta_1}{dz_1^2},$$

Generally, it is assumed that the shear stress σ_{13} is negligible compared to the shear stress σ_{12}. The shear stress $\sigma_{12(\omega)}$ that is associated with the longitudinal stress $\sigma_{11(\omega)}$ is determined with the equilibrium equation for direction z_1:

$$\frac{\partial\sigma_{11(\omega)}}{\partial z_1} + \frac{\partial\sigma_{12(\omega)}}{\partial z_2} = 0 \qquad (11)$$

Substitution of (14) into (15) yields the derivative of the shear flow as:

$$\begin{aligned}\frac{\partial\sigma_{12(\omega)}}{\partial z_2} &= E\,\omega\frac{d^3\beta_1}{dz_1^3}\\ \frac{\partial F_{\omega(i)}}{\partial z_2} &= E\,t_i\,\omega\frac{d^3\beta_1}{dz_1^3}\end{aligned} \qquad (12)$$

The increment $\triangle F_{\omega(z_2)}$ of the shear flow from the start vertex of wall i to point z_2 on the axis of wall i is determined by integration of (12):

$$\begin{aligned}\triangle F_{\omega(z_2)} &= E\frac{d^3\beta_1}{dz_1^3}\triangle S_{\omega(z_2)}\\ \triangle S_{\omega(z_2)} &= \int_{-0.5b_i}^{z_{2i}} t_i\,\omega\,dz_2\end{aligned} \qquad (13)$$

The following Expression shows that the warping function ω is a linear function of z_2:

$$\omega = \frac{z_2}{b_i}(\omega_{(i,e)} - \omega_{(i,s)}) + \frac{1}{2}(\omega_{(i,e)} + \omega_{(i,s)}) \qquad (14)$$

This expression (14) is substituted into (13). The integration yields:

$$\triangle S_{\omega(z_2)} = \frac{1}{2}b_i\,t_i\left(\left(\left(\frac{z_2}{b_i}\right)^2 - \frac{1}{4}\right)(\omega_{(i,e)} - \omega_{(i,s)}) + \left(\frac{z_2}{b_i} + \frac{1}{2}\right)(\omega_{(i,e)} + \omega_{(i,s)})\right)$$

The increment of the static moment of the warping function is obtained for $z_2 = 0.5b_i$:

$$\triangle S_{\omega(i)} = \frac{1}{2}b_i\,t_i\,(\omega_{(i,e)} + \omega_{(i,s)}),$$

where $\triangle S_{\omega(i)}$ - increment of static moment of the warping function in wall i.

The increments of the static moment of warping function in the wall are computed for all walls of the section. The shear flows in the walls will be determined so that the shear flows at each vertex are in equilibrium. Since the factor relating the shear flow to the static moment is the same for all walls of the section, the computations depend only on the properties of the section and not on the loading or supports of the bar. Open thin-walled beams are much used in civil, mechanical, and aerospace engineering and their cross-sections may be an open plane arc (middle line) with regularly varying thickness, ‘small’ with respect to its length [23]. Therefore, the warping function is qualitatively described by a function given beforehand, and the most frequently used is Saint-Venant’s function of free-warping for beams with closed cross- sections [8].

IV. Numerical Examples for Different Open and Closed Section Types

In this section, examples are given to illustrate the warping behaviour of open and closed thin-walled sections. The first examples illustrate open sections and the second is about closed section. Considering an open thin-walled section, the torsion constant, and the warping function of the I-beam are given in Fig. 5.

Example 1: For uniform and nonuniform torsion of an I-beam about its centroid

The walls and vertices of the section are numbered as depicted in Fig.5. The torsion constant is given below:

$$J = \frac{1}{3}\left(\frac{f}{2}t_f^3 + h t_w^3 + \frac{f}{2}t_f^3\right) = \frac{1}{3}(2\,f\,t_f^3 + h t_w^3)$$

The contributions of the walls to the warping function are

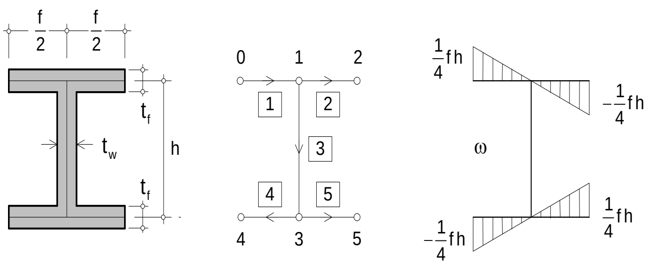


Fig. 5. The walls, vertices, and warping behavior of an I-beam

given as follow:

$$\begin{aligned}&wall\ 1: \triangle\omega_{R1} = -\frac{f}{2}\cdot\frac{h}{2} + \frac{f}{2}\cdot 0 = -\frac{1}{4}f\,h\\ &wall\ 2: \triangle\omega_{R2} = -\frac{f}{2}\cdot\frac{h}{2} - \frac{f}{2}\cdot 0 = -\frac{1}{4}f\,h\end{aligned}$$

$$wall\ 3: \triangle\omega_{R3} = -0\cdot\frac{f}{2}+0\cdot f = 0$$

$$wall\ 4: \triangle\omega_{R4} = -\frac{f}{2}\cdot\frac{h}{2}-0\cdot 0 = -\frac{1}{4}f\,h$$

$$wall\ 5: \triangle\omega_{R5} = \frac{f}{2}\cdot\frac{h}{2}+\frac{f}{2}\cdot 0 = \frac{1}{4}f\,h$$

The warping function is set to null at vertex 1. Since the direction from vertex 1 to vertex 0 is opposite to that of wall 1, the value of the warping function at vertex 0 is: $0-(-0.25\,f\,h)=0.25\,f\,h$. similarly,

The value at vertex 2 is: $0+(-0.25\,f\,h)=-0.25\,f\,h$.

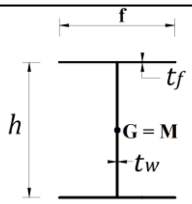
The value at vertex 3 is: $0+0=0$.

The value at vertex 4 is: $0+(-0.25\,f\,h)=-0.25\,f\,h$.

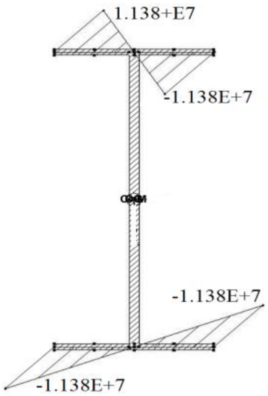
The value at vertex 5 is: $0+0.25\,f\,h=0.25\,f\,h$.

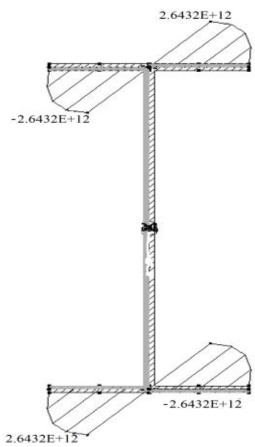
For the above expressions, the warping function distribution within the section is shown in Fig. 5 and it is checked using Shape-thin software for all parameters, which are, warping function, static moments, and warping omega as it is shown in the figure below. The warping functions about the shear center M are geometrical quantities and they are required to determine the longitudinal stress and shear stress stresses caused by warping restraint, and the warping statical moment is determined from the normalized warping functions.It is commonly known that the statical moments (1st moments of area) are registered with reference to the global axes or the principal axes of the section. The statical moment is defined as the product of *dA* and the distance from its centroid to a reference axis that lies in the plane of the section.

TABLE I
THE WARPING FUNCTION DISTRIBUTION OF AN I-SECTION ALONG Y_2 AND Y_3 AXES

I-SECTION	DIMENSIONS
	see below

h(mm)	*400*
f(mm)	*180*
t_f*(mm)*	*11*
t_w*(mm)*	*8*





ALONG Y_2 AXIS ALONG Y_3 AXIS

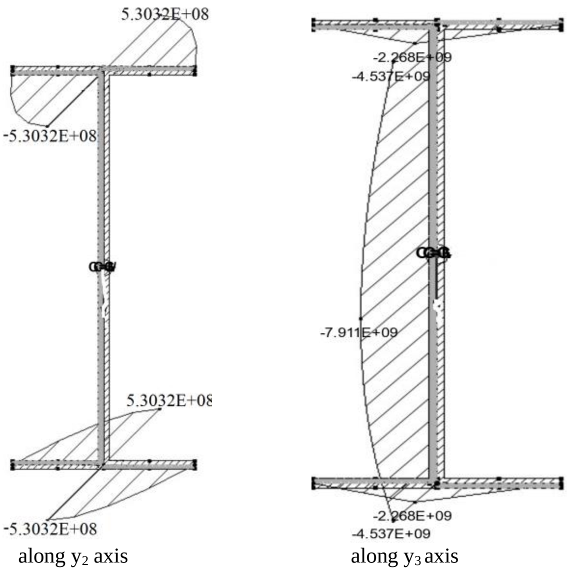


Fig. 6. The static moments along y2 and y3 axes of an I section.

Example 2: For uniform and nonuniform torsion of sections with one interior vertex.

Referring Fig. 7, it shows thin-walled sections with different numbers of walls that have only one interior vertex, where the walls meet. The interior vertex is the shear centre of the section. The shear centre is also the centre of rotation *R*. The warping functions for sections with a single internal vertex define as the plane is not warping cross section deformations. Thin-walled beams, which are formed by sections with exactly one interior vertex as shown in Fig.6, remain plane, do not warp, and do not have normal stresses under torsion[2]. The contributions of the walls to the warping functions of the sections are given by as follows:

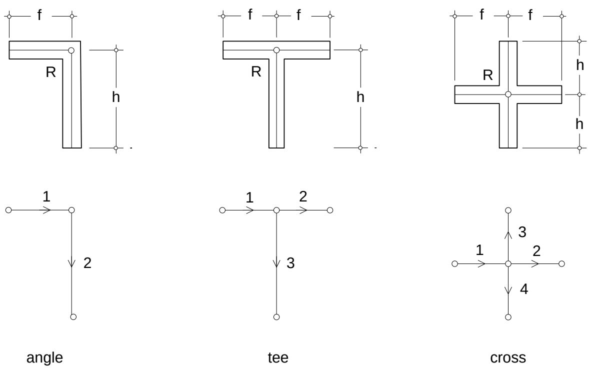


Fig. 7. Thin-walled Sections with a single interior vertex

$$\begin{array}{lll}
Angle: & wall\ 1 & \triangle\omega_{R1} = -f\cdot 0+f\cdot 0 = 0\\
 & wall\ 2 & \triangle\omega_{R2} = 0\cdot 0-0\cdot h = 0\\
Tee: & wall\ 1 & \triangle\omega_{R1} = -f\cdot 0+f\cdot 0 = 0\\
 & wall\ 2 & \triangle\omega_{R2} = -f\cdot 0-0\cdot 0 = 0\\
 & wall\ 3 & \triangle\omega_{R3} = 0\cdot 0-0\cdot h = 0\\
Cross: & wall\ 1 & \triangle\omega_{R1} = -f\cdot 0+f\cdot 0 = 0\\
 & wall\ 2 & \triangle\omega_{R2} = -f\cdot 0-0\cdot 0 = 0\\
 & wall\ 3 & \triangle\omega_{R3} = -0\cdot 0-0\cdot h = 0\\
 & wall\ 4 & \triangle\omega_{R3} = 0\cdot 0+0\cdot h = 0
\end{array}$$

In closed single or multi-celled closed cross-sections to attain the characteristic distribution of the warping displacement, it is needed to introduce circulation shear force flows around the cells. In closed sections the longitudinal deformations are depends on both normal, shear stresses and warping function. Fig. 8 shows the dimensions of the section of a thin-walled rectangular tube. The value φ_{C1} of the stress function on the boundary of the cell, the shear stresses, the torsion constant, and the warping of the section are to be determined. The local coordinate systems of the walls are shown in Fig. 8:

Example 3: Single Cell Thin-Walled Rectangular Section.

The walls of the cell are numbered in the sequence AB, BC, CD, DA. The integral is evaluated as follows:

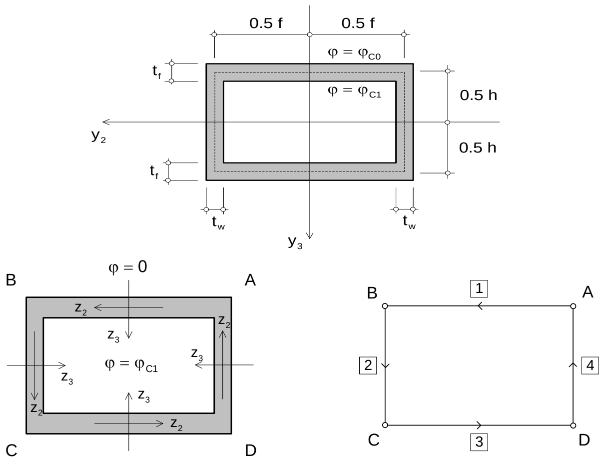


Fig.8. Local walls coordinate systems for a rectangular cell

$$\frac{\varphi_{C1}}{t_f} f + \frac{\varphi_{C1}}{t_w} h + \frac{\varphi_{C1}}{t_f} f + \frac{\varphi_{C1}}{t_w} h = 2 f h \tag{16}$$

$$\varphi_{C1} = \left(\frac{1}{h t_f} + \frac{1}{f t_w} \right)^{-1}$$

The contributions of the four walls of the section to the torsion constant for the torsion constant are:

$$wall\ 1: \quad b_1 = f \qquad y_{2A_1} = 0 \qquad y_{3A_1} = -\frac{h}{2}$$

$$\alpha = 0 \qquad \triangle J = f \varphi_{C1} \frac{h}{2}$$

$$wall\ 2: \quad b_2 = h \qquad y_{2A_1} = \frac{f}{2} \qquad y_{3A_1} = \ 0$$

$$\alpha = 90 \qquad \triangle J = h \varphi_{C1} \frac{f}{2}$$

$$wall\ 1: \quad b_1 = f \qquad y_{2A_1} = 0 \qquad y_{3A_1} = -\frac{h}{2}$$

$$\alpha = 0 \qquad \triangle J = f \varphi_{C1} \frac{h}{2}$$

$$wall\ 2: \quad b_2 = h \qquad y_{2A_1} = \frac{f}{2} \qquad y_{3A_1} = \ 0$$

$$\alpha = 90 \qquad \triangle J = h \varphi_{C1} \frac{f}{2}$$

Substitution of these contributions and (16) into the torsion constant yields:

$$J = 2 f h \left(\frac{1}{h t_f} + \frac{1}{f t_w} \right)^{-1} = \frac{2 f^2 h^2 t_f t_w}{f t_w + h t_f}$$

The shear stress in the walls is determined as follows:

$$walls\ 1\ and\ 3: \qquad \sigma_{12} = \frac{M_T}{J t_f} \varphi_{C1} = \frac{M_T}{2 f h t_f}$$

$$walls\ 2\ and\ 4: \qquad \sigma_{12} = \frac{M_T}{J t_w} \varphi_{C1} = \frac{M_T}{2 f h t_w}$$

The contributions of the walls to the warping function are given with the following expressions:

$$wall\ 1: \quad b_i = f \qquad t_i = t_f \qquad z_{3R} = \frac{h}{2}$$

$$\triangle\omega_{R(1)} = f \left(\frac{h}{2} - \frac{\varphi_{C1}}{t_f} \right) = 2\omega_0$$

$$wall\ 2: \quad b_i = h \qquad t_i = t_w \qquad z_{3R} = \frac{f}{2}$$

$$\triangle\omega_{R(2)} = h \left(\frac{f}{2} - \frac{\varphi_{C1}}{t_w} \right) = -2\omega_0$$

$$wall\ 3: \quad b_i = f \qquad t_i = t_f \qquad z_{3R} = \frac{h}{2}$$

$$\triangle\omega_{R(3)} = f \left(\frac{h}{2} - \frac{\varphi_{C1}}{t_f} \right) = 2\omega_0$$

$$wall\ 4: \quad b_i = h \qquad t_i = t_w \qquad z_{3R} = \frac{f}{2}$$

$$\triangle\omega_{R(4)} = h \left(\frac{f}{2} - \frac{\varphi_{C1}}{t_w} \right) = -2\omega_0$$

$$\omega_0 = \frac{1}{4} f h \frac{\mu - 1}{\mu + 1} \qquad with \qquad \mu := \frac{h}{t_w} \frac{t_f}{f}$$

The warping function is set to $-\omega_0$ at vertex A in Fig. 8. The warping function at the other vertices is computed by means of the increments:

$$\omega_A = -\omega_0, \quad \omega_B = \omega_A + \triangle\omega_{R(1)} = \omega_0,$$

$$\omega_C = \omega_B + \triangle\omega_{R(2)} = -\omega_0$$

$$\omega_D = \omega_C + \triangle\omega_{R(3)} = \omega_0,$$

$$\omega_A = \omega_D + \triangle\omega_{R(4)} = -\omega_0$$

The results show that the closed line integral of the warping function over the cell boundary is null, so that the warping is single valued. The variation of the warping function over the section is shown in Fig. 9.

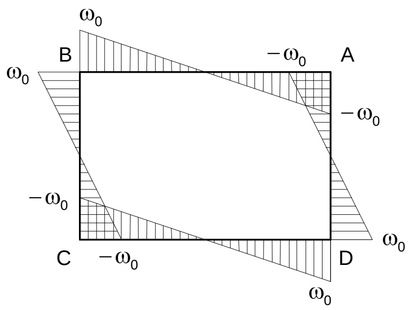
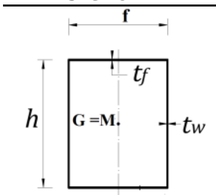


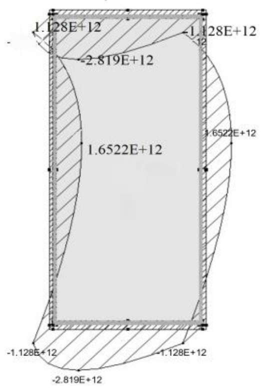
Fig. 9. Warping function for a single cell of rectangular section.

Similarly, for closed sections (rectangular hollow section), the warping function distribution within the section is shown in Fig. 9, and it checked using Shape-thin software for all parameters which are: warping function, static moments, and warping omega, as it is shown in the tables II and fig. 10.

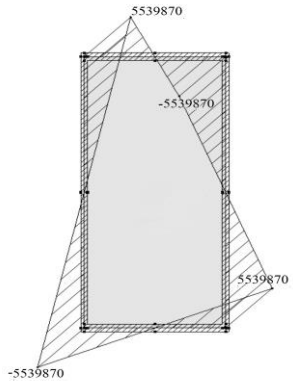
The warping functions about the shear center *M* are geometrical quantities, and they are required to determine the longitudinal stress and shear stress stresses caused by warping restraint, and the warping statical moment is determined from the normalized warping functions. It is commonly known that the statical moments (1st moments of area) are registered with reference to the global axes or the principal axes of the section. The statical moment is defined as the product of *dA* and the distance from its centroid to a reference axis that lies in the plane of the section. The static moments along *y2* and *y3* are shown below.

TABLE II
THE WARPING FUNCTION DISTRIBUTION OF A RECTANGULAR SECTION ALONG Y_2 AND Y_3 AXES

I-SECTION	DIMENSIONS	
	*h(mm)*	*400*
	f(mm)	*180*
	t_f(mm)	*11*
	t_w(mm)	*8*



ALONG Y_2 AXIS



ALONG Y_3 AXIS

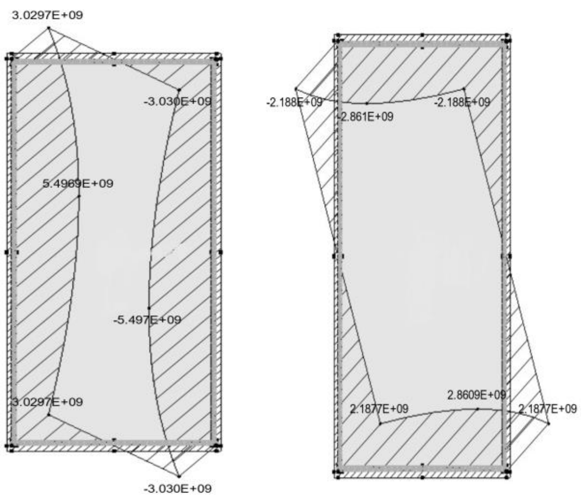


Fig. 10. The static moments along y2 and y3 axes of a rectangular section

Example 4: Thin-Walled Rectangular Section with Three Cells

Fig. 11 shows the dimensions of the section of a thin-walled rectangular bar with 3 cells. The values $\varphi_{C1}, \varphi_{C2}$ *and* φ_{C3} of the stress function on the boundaries of the cells and the torsion constant are to be determined. The local coordinate systems for the walls of each of the three cells are defined as shown in Fig. 11.

Using the contribution of the wall of the cell and applying it for the evaluation of each of the three cells, a system of three linear equations for the unknown cell stress functions φ_{Ci} can be written in matrix form as:

$$\frac{\varphi_{C1}}{t_f} f + \frac{\varphi_{C1}}{t_w} h + \frac{\varphi_{C1}}{t_f} f + \frac{\varphi_{C1} - \varphi_{C2}}{t_w} h = 2 f h$$

$$\frac{\varphi_{C2}}{t_f} f + \frac{\varphi_{C2} - \varphi_{C1}}{t_w} h + \frac{\varphi_{C2}}{t_f} f + \frac{\varphi_{C2} - \varphi_{C3}}{t_w} h = 2 f h$$

$$\frac{\varphi_{C3}}{t_f} f + \frac{\varphi_{C3} - \varphi_{C2}}{t_w} h + \frac{\varphi_{C3}}{t_f} f + \frac{\varphi_{C3}}{t_w} h = 2 f h$$

$$\begin{bmatrix} c_1 & c_2 & 0 \\ c_2 & c_1 & c_2 \\ 0 & c_2 & c_1 \end{bmatrix} \begin{bmatrix} \varphi_{C1} \\ \varphi_{C2} \\ \varphi_{c3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (17)$$

$$c_1 = \frac{1}{h t_f} + \frac{1}{f t_w} \qquad c_2 = -\frac{1}{2 f t_w}$$

Generally, these equations are solved numerically. Consider the special case of a section with dimensions $h = f$ and $t_f = t_w = t$. The solution of equation (17) becomes $\varphi_1 = \varphi_3 = 5t f / 7$ and $\varphi_2 = 6t f / 7$.

The torsion constant *J* is determined based on the contribution of cells of the thin-walled section. The walls of the cells make the following contributions to this expression:

$$cell\ 1: \quad \varphi_{C1} (f \frac{h}{2} + h\frac{3f}{2} + f\frac{h}{2} - h\frac{f}{2}) = 2 f h \varphi_{C1}$$

$$cell\ 2: \quad \varphi_{C2} (f \frac{h}{2} + h\frac{f}{2} + f\frac{h}{2} + h\frac{f}{2}) = 2 f h \varphi_{C2}$$

$$cell\ 3: \quad \varphi_{C3} (f \frac{h}{2} - h\frac{f}{2} + f\frac{h}{2} + h\frac{3f}{2}) = 2 f h \varphi_{C3}$$

The contributions of the cells are added to provide the torsion constant of the section:

$$J = 2 f h (\varphi_{C1} + \varphi_{C2} + \varphi_{C3})$$

For the special case of a section with dimensions $h = f$ and $t_f = t_w = t,$ the vales of the stress function are substituted as:

$$J = 2 f h (\frac{5t f}{7} + \frac{6t f}{7} + \frac{5t f}{7}) = \frac{32}{7} t f^3$$

The values of the stress function on the cell boundaries and of the torsion constant can also be compared to the corresponding values $\bar{\varphi}_{C1}$ and $\bar{J}$ for a rectangular tube with dimensions $3 f \times f$ and wall thickness t, but without the inner walls (single cell). The stress function and the torsion constant are computed as follows:

$$\bar{\varphi}_{C1} = \left(\frac{1}{3\,f\,t} + \frac{1}{f\,t} \right)^{-1} = \frac{3}{4}\,f\,t$$

$$\bar{J} = \frac{2(3\,f)^2\,f^2 t^2}{3\,f\,t + f\,t} = \frac{9}{2}\,f^3 t$$

The ratio of the torsion constants for sections with and without inner walls is 64/63 = 1.0159. The influence of the inner walls on the torsional rigidity of the bar is negligible. The shear stress in the outer wall of the middle cell in Fig. 10 is determined with the following expression:

$$\sigma_{12} = \frac{M_T}{J\,t_i}(\varphi_i^+ - \varphi_i^-) = \frac{M_T}{t}\frac{7}{32\,f^3 t}\frac{6}{7}t\,f = \frac{3}{16}\frac{M_T}{f^2\,t}$$

The corresponding value for the bar without walls is:

$$\bar{\sigma}_{12} = \frac{M_T}{J\,t_i}(\varphi_i^+ - \varphi_i^-) = \frac{M_T}{t}\frac{2}{9\,f^3 t}\frac{3}{4}t\,f = \frac{1}{6}\frac{M_T}{f^2\,t}$$

The ratio of the maximum shear stresses in the sections with and without inner walls is 18/16 =1.125. The section with the inner walls has higher shear stresses in the outer walls of the middle cell compared to the section without inner walls.

V. Conclusion

The behavior of warping torsion of thin-walled members with open and closed cross sections were considered using the first-order torsion theory, as the difference between the displacements and stresses due to uniform and nonuniform torsion is most pronounced for thin-walled sections. Numerical examples were shown to demonstrate the behaviour of the warping torsion of the open and closed thin-walled sections. The analysis of uniform and nonuniform torsion showed that the warping behaviour of bars depends on the shape of their thin-walled sections. Prismatic bars whose section is a full circle, a circular tube, a square tube, or a thin-walled section with a single interior vertex (for example an angle, a tee or a cross) do not warp. For the use of single- or multi-celled closed cross-sections to attain the characteristic distribution of the warping displacement, it is needed to introduce circulation shear force flows around the cells, and the longitudinal deformations are dependent on both normal, shear stresses and warping functions. The warping function distribution within the section for open and closed sections were checked using Shape-thin software for all parameters, which are: warping function, static moments and warping omega, and the results were similar. The warping function about the shear centre M are geometrical quantities and they are required to determine the longitudinal stress and shear stresses caused by warping restraint, and the warping statical moment is determined from the normalized warping functions.

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