A New Algorithm of Approximating Trivariate Matrix Functions with Generalized Inverse Newton-Thiele Formula

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Abstract—**This work introduces a new algorithm of approximating trivariate matrix functions with generalized inverse Newton-Thiele formula. The approximation is of the form combined with continued fraction and Newton expansion by using the generalized matrix inverse. A recursive method for the approximation is derived. We also discuss some algebraic properties. In the end we take two examples to indicate the efficiency of the method which is better compared to the existing one.**

Index Terms—**Generalized inverse, Continued fraction, Newton-Thiele-type, Trivariate approximation.**

I. Introduction

MATRIX rational approximation theory has many practical applications in fields like digital filter design, circuit theory, control theory as well as elementary particle physics [1-6]. As we all know, the continued fraction with the Thiele-type is one of the significant approaches to solve matrix rational approximation problems [7-10]. Bodnarcuk and Skorobogatko gave the branched continued fractions (BCF) form in [11] which was not very convenient for practical use. There were two practical cases of BCF. One was given by Murphy, O'Donohoe [12] and Kutschminskaja [13]. The other was given in [14]. However, the two BCFs became much more complicated for the case of multivariate functions because of the decompositions for the three- or more-variate power series into suitable components. Therefore, Siemaszko in [15] presented a modified version of BCF and showed how to use the obtained BCF to approximate the bi-variable functions. Cuyt and Verdonk proposed multivariate reciprocal differences and limiting values which were different from the ones mentioned in [15], then obtained the branched continued fraction expansion in [16]. Wynn in [17] raised the question of rational vector-interpolation. He gave exact results via Samelson inverses using ε-algorithm to vector valued quantities. In [9] Graves-Morris established several principles for generalized inverse rational interpolation. For the vector valued continued fraction, many open problems are also left in [9], one of which is whether the scalar three-term relation could be generalized to vector condition. For the vector valued continued fraction, Zhao et al. in [18] constructed a practical so-called backward three-term recurrence relation. Gu introduced the definition of generalized inverse and proposed the methods of two-variable matrix rational interpolation and approximation via generalized inverse [19-23]. This method involves no multiplication of matrices. Therefore, we need not to define the left interpolations or the right ones. In [24] Qin et al. presented a new C^2 piecewise bivariate rational interpolation scheme with bi-quadratic denominator. Qin and Zhu constructed a class of piecewise bivariate rational interpolation surface scheme with bi-cubic denominator and four parameters in a rectangular domain, using two new kinds of Hermite-type interpolation basis functions[25].

In this paper, a new algorithm of approximating trivariate matrix functions with generalized inverse Newton-Thiele formula is introduced. The construction of the approximation, motivated by [10], consists the Newton expansion and bivariate continued fractions. The approach of the approximation uses generalized inverse and an efficient algorithm is inspired by [26]. In Section 2, we first introduce the definition of generalized matrix inverse and some notations $\varphi_{i,t,u}(x_0,x_1,\ldots,x_i;y_0,\ldots,y_t;z_0,\ldots,z_u)$, then we give the algorithm of trivariate Newton-Thiele matrix rational formula in Algorithm 2.6. In Section 3, we discuss the algebraic properties such as divisibility and characterization, and with their help we develop TGMRA$_{NT}$ of a matrix function $f(x,y,z)$ and its corresponding error estimate. In Section 4, an example is used to illustrate that the proposed method is superior to trivariate generalized inverse matrix rational interpolation (TGMRI) in [27]. In the final section, we give some conclusions which contains a brief comment.

II. Algorithm of NTMRF

Define $\mathbf{C}^{m\times n}$ be all the matrices which are consisted of m rows and n columns where the elements are complex numbers. Let $A=(a_{ij}), B=(b_{ij}),\ A,B\in\mathbf{C}^{m\times n}$.
Definition 2.1 [22]. Define the product as

$$A\cdot B=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}b_{ij}=tr(AB^*),$$

with B^* denotes $B's$ transpose.

We all know that the Euclidean norm of A is

$$\|A\|=(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{1/2}. \quad (1)$$

then

$$A\cdot\bar{A}=\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2=tr(A\bar{A}^*), \quad (2)$$

Manuscript received July 28, 2021; revised February 10, 2022.
This work was supported by the Natural Science Foundation of Jiangsu Province under Grant No. 16KJD110006.
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in (2) $\bar{A}$ is matrix $A's$ complex conjugate matrix. Bearing equations (1) and (2) in mind, we define

$$A_r^{-1} = \frac{1}{A} = \frac{\bar{A}}{\|A\|^2},\ A \neq 0,\ A \in \mathbf{C}^{m\times n}. \tag{3}$$

as the generalized matrix inverse of A.

In particular, for $A \in \mathbf{R}^{m\times n}$, $A_r^{-1} = \frac{1}{A} = \frac{A}{\|A\|^2}$, $A \neq 0$.

Suppose $\Omega = (a,b)\times(c,d)\times(e,f) \subset R^3$ be cubic domain in three-dimensional space. Then the three sequences $\{x_0, x_1, \ldots\} \subset (a,b)$,$\{y_0, y_1, \ldots\} \subset (c,d)$, $\{z_0, z_1, \ldots\} \subset (e,f)$ are of distinct numbers. We denote $\Omega_{x,y,z} = \{(x_i, y_j, z_k) : i,j,k = 0,1,\ldots\} \subset \Omega$, $b+1$ nodes $x_{a_0}, \ldots, x_{a_{b-2}}, x_c, x_d$ using the notation $x_{a_{b-2}}^{a_0}, x_c, x_d$, the $r+1$ nodes $y_{q_0}, \ldots, y_{q_{r-2}}, y_t, y_l$ using the notation $y_{q_{r-2}}^{q_0}, y_t, y_l$ and the $s+1$ nodes $z_{p_0}, \ldots, z_{p_{s-2}}, z_v, z_w$ using the notation $z_{p_{s-2}}^{p_0}, z_v, z_w$.

In Ω given a trivariate matrix function $f(x,y,z) \in \mathbf{C}^{m\times n}$. For $b \geq 2, r \geq 2, s \geq 2$,

$$\varphi_{0,0,0}(x_i; y_j; z_k) = f(x_i, y_j, z_k), \tag{4}$$

$$\varphi_{1,0,0}(x_c, x_d; y_j; z_k) = \frac{\varphi_{0,0,0}(x_d; y_j; z_k) - \varphi_{0,0,0}(x_c; y_j; z_k)}{x_d - x_c}, \tag{5}$$

$$\varphi_{b,0,0}(x_{a_{b-2}}^{a_0}, x_c, x_d; y_j; z_k) = \frac{\varphi_{b-1,0,0}(x_{a_{b-2}}^{a_0}, x_d; y_j; z_k) - \varphi_{b-1,0,0}(x_{a_{b-2}}^{a_0}, x_c; y_j; z_k)}{x_d - x_c}, \tag{6}$$

$$\varphi_{0,1,0}(x_i; y_t, y_l; z_k) = \frac{y_l - y_t}{\varphi_{0,0,0}(x_i; y_l; z_k) - \varphi_{0,0,0}(x_i; y_t; z_k)}, \tag{7}$$

$$\varphi_{0,r,0}(x_i; y_{q_{r-2}}^{q_0}, y_t, y_l; z_k) = \frac{y_l - y_t}{\varphi_{0,r-1,0}(x_i; y_{q_{r-2}}^{q_0}, y_l; z_k) - \varphi_{0,r-1,0}(x_i; y_{q_{r-2}}^{q_0}, y_t; z_k)}, \tag{8}$$

$$\varphi_{0,0,1}(x_i; y_j; z_v, z_w) = \frac{z_w - z_v}{\varphi_{0,0,0}(x_i; y_j; z_w) - \varphi_{0,0,0}(x_i; y_j; z_v)}, \tag{9}$$

$$\varphi_{0,0,s}(x_i; y_j; z_{p_{s-2}}^{p_0}, z_v, z_w) = \frac{z_w - z_v}{\varphi_{0,0,s-1}(x_i; y_j; z_{p_{s-2}}^{p_0}, z_w) - \varphi_{0,0,s-1}(x_i; y_j; z_{p_{s-2}}^{p_0}, z_v)}, \tag{10}$$

$$\varphi_{b,r,0}(x_{a_{b-2}}^{a_0}, x_c, x_d; y_{q_{r-2}}^{q_0}, y_t, y_l; z_k) = \frac{y_l - y_t}{\varphi_{b,r-1,0}(x_{a_{b-2}}^{a_0}, x_c, x_d; y_{q_{r-2}}^{q_0}, y_l; z_k) - \varphi_{b,r-1,0}(x_{a_{b-2}}^{a_0}, x_c, x_d; y_{q_{r-2}}^{q_0}, y_t; z_k)}, \tag{11}$$

$$\varphi_{0,r,s}(x_i; y_{q_{r-2}}^{q_0}, y_t, y_l; z_{p_{s-2}}^{p_0}, z_v, z_w) = \frac{z_w - z_v}{\varphi_{0,r,s-1}(x_i; y_{q_{r-2}}^{q_0}, y_t, y_l; z_{p_{s-2}}^{p_0}, z_w) - \varphi_{0,r,s-1}(x_i; y_{q_{r-2}}^{q_0}, y_t, y_l; z_{p_{s-2}}^{p_0}, z_v)}, \tag{12}$$

$$\varphi_{b,0,s}(x_{a_{b-2}}^{a_0}, x_c, x_d; y_j; z_{p_{s-2}}^{p_0}, z_v, z_w) = \frac{z_w - z_v}{\varphi_{b,0,s-1}(x_{a_{b-2}}^{a_0}, x_c, x_d; y_j; z_{p_{s-2}}^{p_0}, z_w) - \varphi_{b,0,s-1}(x_{a_{b-2}}^{a_0}, x_c, x_d; y_j; z_{p_{s-2}}^{p_0}, z_v)}, \tag{13}$$

$$\varphi_{b,r,s}(x_{a_{b-2}}^{a_0}, x_c, x_d; y_{q_{r-2}}^{q_0}, y_t, y_l; z_{p_{s-2}}^{p_0}, z_v, z_w) = \frac{z_w - z_v}{\varphi_{b,r,s-1}(x_{a_{b-2}}^{a_0}, x_c, x_d; y_{q_{r-2}}^{q_0}, y_t, y_l; z_{p_{s-2}}^{p_0}, z_w) - \varphi_{b,r,s-1}(x_{a_{b-2}}^{a_0}, x_c, x_d; y_{q_{r-2}}^{q_0}, y_t, y_l; z_{p_{s-2}}^{p_0}, z_v)}. \tag{14}$$

For simplicity, we denote $b+1$ nodes $x_{a_0}, \ldots, x_{a_{b-2}}, x_c, x_d$ by $x_0, \ldots, x_b$ using the notation x_b^0, the $r+1$ nodes $y_{q_0}, \ldots, y_{q_{r-2}}, y_t, y_l$ by $y_0, \ldots, y_r$ using the notation y_r^0 and the $s+1$ nodes $z_{p_0}, \ldots, z_{p_{s-2}}, z_v, z_w$ by $z_0, \ldots, z_s$ using the notation z_s^0.

From (4)-(14), we construct $R_{m,n,u}(x,y,z)$ which is a trivariate rational matrix function as follows:

$$R_{m,n,u}(x,y,z) = l_{0,n} + l_{1,n}(x - x_0) + \cdots + l_{m,n}(x - x_0)(x - x_1)\cdots(x - x_{m-1}) , \tag{15}$$

with

$$l_{i,n} = l_{i,n}(y,z) = h_{i,0}(z) + \frac{y - y_0}{h_{i,1}(z)} + \cdots + \frac{y - y_{n-1}}{h_{i,n}(z)}, \quad \text{for } i = 0,1,\ldots,m, \tag{16}$$

and

$$h_{i,t}(z) = \varphi_{i,t,0}(x_i^0; y_t^0; z_0) + \frac{z - z_0}{\varphi_{i,t,1}(x_i^0; y_t^0; z_1^0)} + \cdots + \frac{z - z_{u-1}}{\varphi_{i,t,u}(x_i^0; y_t^0; z_u^0)}, \quad \text{for } t = 0,\ldots,n. \tag{17}$$

To obtain Theorem 2.2, let

$$h_{i,t}^{(s)}(z) = \varphi_{i,t,s}(x_i^0; y_t^0; z_s^0) + \frac{z - z_s}{h_{i,t}^{(s+1)}(z)}, \quad \text{for } s = 0,\ldots,u,$$

where

$$h_{i,t}^{(u)}(z) = \varphi_{i,t,u}(x_i^0; y_t^0; z_u^0),\quad h_{i,t}^{(0)}(z) = h_{i,t}(z),$$

then $l_{i,n}^{(p)} = h_{i,p}(z) + \frac{y - y_p}{l_{i,n}^{(p+1)}}$,

where

$$l_{i,n}^{(n)} = h_{i,n}(z), l_{i,n}^{(0)} = l_{i,n}.$$

Theorem 2.2. Suppose that

(i) $\varphi_{i,t,p}(x_i^0; y_t^0; z_p^0), i = 0,\ldots,m, t = 0,\ldots,n, p = 0,\ldots,u$ exist and nonzero(except for $\varphi_{i,t,0}(x_i^0; y_t^0; z_0)$),

(ii) $h_{i,t}^{(s)}(z) = \varphi_{i,t,s}(x_i^0; y_t^0; z_s^0) + \frac{z - z_s}{h_{i,t}^{(s+1)}(z)}$ with $h_{i,t}^{(s+1)}(z_s) \neq 0, s = 0,\ldots,n-1$, and $l_{i,n}^{(p)} = h_{i,p}(z) + \frac{y - y_p}{l_{i,n}^{(p+1)}}$ with $l_{i,n}^{(p+1)}(y_p, z_s) \neq 0$, $s = 0,\ldots,u-1$,

so $R_{m,n,u}(x,y,z)$ mentioned in (15)(16)(17) exists such that $R_{m,n,u}(x_i,y_j,z_k) = f(x_i,y_j,z_k)$,where$(x_i,y_j,z_k) \in \Omega_{m,n,u}$.

Proof If conditions (i) and (ii) hold, then

$$h_{i,t}(z_k) = \varphi_{i,t,0}(x_i^0; y_t^0; z_0) + \frac{z_k - z_0}{\varphi_{i,t,1}(x_i^0; y_t^0; z_1^0)} + \cdots + \frac{z_k - z_{k-1}}{\varphi_{i,t,k}(x_i^0; y_t^0; z_k^0)} + \frac{z_k - z_k}{h_{i,t}^{(k+1)}(z_k)},$$

$$i = 0, \ldots, m, t = 0, \ldots, n,$$

since $h_{i,t}^{(k+1)}(z_k) \neq 0$, then $\frac{z_k - z_k}{h_{i,t}^{(k+1)}(z_k)} = 0$ we get

$$\begin{aligned}
h_{i,t}(z_k) &= \varphi_{i,t,0}(x_i^0; y_t^0; z_0) + \frac{z_k - z_0}{\varphi_{i,t,1}(x_i^0; y_t^0; z_1^0)} + \cdots + \frac{z_k - z_{k-1}}{\varphi_{i,t,k}(x_i^0; y_t^0; z_k^0)} \\
&= \varphi_{i,t,0}(x_i^0; y_t^0; z_0) + \frac{z_k - z_0}{\varphi_{i,t,1}(x_i^0; y_t^0; z_1^0)} + \cdots + \frac{z_k - z_{k-2}}{\varphi_{i,t,k-1}(x_i^0; y_t^0; z_{k-1}^0)} \\
&+ \frac{z_k - z_{k-1}}{\frac{z_k - z_{k-1}}{\varphi_{i,t,k-1}(x_i^0; y_t^0; z_{k-2}^0, z_k) - \varphi_{i,t,k-1}(x_i^0; y_t^0; z_{k-1}^0)}} \\
&= \varphi_{i,t,0}(x_i^0; y_t^0; z_0) + \frac{z_k - z_0}{\varphi_{i,t,1}(x_i^0; y_t^0; z_1^0)} + \cdots + \frac{z_k - z_{k-2}}{\varphi_{i,t,k-1}(x_i^0; y_t^0; z_{k-2}^0, z_k)} \\
&= \cdots = \varphi_{i,t,0}(x_i^0; y_t^0; z_0) + \frac{z_k - z_0}{\varphi_{i,t,1}(x_i^0; y_t^0; z_0, z_k)} \\
&= \varphi_{i,t,0}(x_i^0; y_t^0; z_k),
\end{aligned} \tag{18}$$

$$\begin{aligned}
l_{i,n}(y_j, z_k) &= h_{i,0}(z_k) + \frac{y_j - y_0}{h_{i,1}(z_k)} + \cdots + \frac{y_j - y_{j-1}}{h_{i,j}(z_k)} \\
&= \varphi_{i,0,0}(x_i^0; y_0; z_k) + \frac{y_j - y_0}{\varphi_{i,1,0}(x_i^0; y_1^0; z_k)} + \cdots + \frac{y_j - y_{j-1}}{\varphi_{i,j,0}(x_i^0; y_j^0; z_k)} \\
&= \varphi_{i,0,0}(x_i^0; y_0; z_k) + \frac{y_j - y_0}{\varphi_{i,1,0}(x_i^0; y_1^0; z_k)} + \cdots + \frac{y_j - y_{j-2}}{\varphi_{i,j-1,0}(x_i^0; y_{j-1}^0; z_k)} \\
&+ \frac{y_j - y_{j-1}}{\frac{y_j - y_{j-1}}{\varphi_{i,j-1,0}(x_i^0; y_{j-2}^0, y_j; z_k) - \varphi_{i,j-1,0}(x_i^0; y_{j-1}^0; z_k)}} \\
&= \varphi_{i,0,0}(x_i^0; y_0; z_k) + \frac{y_j - y_0}{\varphi_{i,1,0}(x_i^0; y_1^0; z_k)} + \cdots + \frac{y_j - y_{j-2}}{\varphi_{i,j-1,0}(x_i^0; y_{j-2}^0, y_j; z_k)} \\
&= \cdots = \varphi_{i,0,0}(x_i^0; y_0; z_k) + \frac{y_j - y_0}{\varphi_{i,1,0}(x_i^0; y_0, y_j; z_k)} \\
&= \varphi_{i,0,0}(x_i^0; y_j; z_k)
\end{aligned} \tag{19}$$

By using(6), (7) and (14), we have
$R_{m,n,u}(x_i, y_j, z_k) = l_{0,n}(y_j, z_k) + l_{1,n}(y_j, z_k)(x_i - x_0) + \cdots + l_{i,n}(y_j, z_k)(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1}) = f(x_i, y_j, z_k)$.

Algorithm 2.3 summarizes the trivariate Newton-Thiele matrix rational formula(NTMRF) $R_{m,n,u}(x, y, z)$.

Algorithm 2.3 (NTMRF):

Input: $\{(x_i, y_j, z_k), f(x_i, y_j, z_k)\}$
$(i = 0, 1, \cdots, m; j = 0, 1, \ldots, n; k = 0, 1, \ldots, u)$.

Output: $\hat{R}_m(x, y, z)$.

Step 1: For all $(x_i, y_j, z_k) \in \Lambda_{m,n,u}$, let $\varphi(x_i; y_j; z_k) = f(x_i, y_j, z_k)$.

Step 2: For $j = 0, 1, \ldots, n, k = 0, \ldots, u$, $p = 1, \ldots, m, i = p, \ldots, m$, caculate

$$\varphi_{p,0,0}(x_{p-1}^0, x_i; y_j; z_k) = \frac{\varphi_{p-1,0,0}(x_i^0; y_{p-2}^0, x_i; y_j; z_k) - \varphi_{p-1,0,0}(x_i^0; y_{p-1}^0; y_j; z_k)}{x_i - x_{p-1}}.$$

Step 3: For $i = 0, \ldots, m, q = 1, \ldots, n, j = q, \ldots, n$, $k = 0, \ldots, u$,calculate

$$\begin{aligned}
&\varphi_{i,q,0}(x_i^0; y_{q-1}^0, y_j; z_k) \\
&= \frac{y_j - y_{q-1}}{\varphi_{i,q-1,0}(x_i^0; y_{q-2}^0, y_j; z_k) - \varphi_{i,q-1,0}(x_i^0; y_{q-1}^0; z_k)} \\
&= (y_j - y_{q-1}) \\
&\cdot \frac{\overline{(\varphi_{i,q-1,0}(x_i^0; y_{q-2}^0, y_j; z_k) - \varphi_{i,q-1,0}(x_i^0; y_{q-1}^0; z_k))}}{\|\varphi_{i,q-1,0}(x_i^0; y_{q-2}^0, y_j; z_k) - \varphi_{i,q-1,0}(x_i^0; y_{q-1}^0; z_k)\|^2}.
\end{aligned}$$

Step 4: For $i = 0, \ldots, m, j = 0, \ldots, n, t = 1, \ldots, u$, $s = t, \ldots, u$, compute

$$\begin{aligned}
&\varphi_{i,j,t}(x_i^0; y_j^0; z_{t-2}^0, z_s) \\
&= \frac{(z_s - z_{t-1})}{\varphi_{i,j,t-1}(x_i^0; y_j^0; z_{t-2}^0, z_s) - \varphi_{i,j,t-1}(x_i^0; y_j^0; z_{t-1}^0)} \\
&= (z_s - z_{t-1}) \\
&\cdot \frac{\overline{(\varphi_{i,j,t-1}(x_i^0; y_j^0; z_{t-2}^0, z_s) - \varphi_{i,j,t-1}(x_i^0; y_j^0; z_{t-1}^0))}}{\|\varphi_{i,j,t-1}(x_i^0; y_j^0; z_{t-2}^0, z_s) - \varphi_{i,j,t-1}(x_i^0; y_j^0; z_{t-1}^0)\|^2}.
\end{aligned}$$

Step 5: For $i = 0, \ldots, m, j = 0, \ldots, n, t = 1, \ldots, u$, $s = t, \ldots, u$, compute $A_{i,j,k} = \varphi_{i,j,k}(x_i^0; y_j^0; z_k^0)$.

Step 6: For $i = 0, \ldots, m, j = 1, \ldots, n, k = 0, \ldots, u$, judge if $A_{i,j,k} \neq 0$, if yes go to step 7, otherwise exit and show "the procedure is unsuccessful".

Step 7: For $i = 0, \ldots, m, j = 0, \ldots, n, k = u - 2, \ldots, 0$,

$$\begin{cases}
Q_{i,j,u} = 1, P_{i,j,u} = A_{i,j,u}, Q_{i,j,u-1} = \|A_{i,j,u}\|^2, \\
P_{i,j,u-1} = A_{i,j,u-1}Q_{i,j,u-1} + (z - z_{u-1})\overline{P_{i,j,u}}, \\
Q_{i,j,k} = \|A_{i,j,k+1}\|^2 Q_{i,j,k+1} \\
\qquad + 2(z - z_{k+1})tr(A_{i,j,k+1}P_{i,j,k+2}{}^*) \\
\qquad\qquad + (z - z_{k+1})^2 Q_{i,j,k+2}, \\
P_{i,j,k} = A_{i,j,k}Q_{i,j,k} + (z - z_k)\overline{P_{i,j,k+1}}.
\end{cases}$$

Step 8: For $i = 0, \ldots, m, j = 0, \ldots, n, k = 0, \ldots, u$, let $B_{i,j,k}(z) = P_{i,j,k}/Q_{i,j,k}$, if $B_{i,j,k}(z_k) \neq 0$, then go to the next step, if $B_{i,j,k}(z_k) = 0$ then exit and show "the procedure is unsuccessful".

Step 9: For $i = 0, \ldots, m, j = n - 2, \ldots, 0$,

$$\begin{cases}
QQ_{i,n,0} = 1, PP_{i,n,0} = B_{i,n,0}, QQ_{i,n-1,0} = \|B_{i,n,0}\|^2, \\
PP_{i,n-1,0} = B_{i,n-1,0}QQ_{i,n-1,0} + (y - y_{n-1})\overline{PP_{i,n,0}}, \\
QQ_{i,j,0} = \|B_{i,j+1,0}\|^2 QQ_{i,j+1,0} \\
\qquad + 2(y - y_{j+1})tr(B_{i,j+1,0}P_{i,j+2}{}^*) \\
\qquad\qquad + (y - y_{j+1})^2 QQ_{i,j+2,0}, \\
PP_{i,j,0} = B_{i,j,0}QQ_{i,j,0} + (y - y_j)\overline{PP_{i,j+1,0}}.
\end{cases}$$

Step 10: For $i = 0, \ldots, m, j = 0, \ldots, n$, let $l_{i,j,0}(y, z) = PP_{i,j,0}/QQ_{i,j,0}$, if $l_{i,j,0}(y_j) \neq 0$, then go to the next step, if $l_{i,j,0}(y_j) = 0$ then exist and show "the procedure is unsuccessful".

Step 11: For $i = 0, \ldots, m$,
let $\hat{B}_i(y, z) = l_{i,0,0}(y, z) = PP_{i,0,0}/QQ_{i,0,0}$.
Step 12: For $k = 1, \ldots, m$, let $\hat{R}_0(x, y, z) = \hat{B}_0(y, z)$, then compute
$\hat{R}_k(x, y, z)$
$= \hat{R}_{k-1}(x, y, z) + \cdots + (x - x_0) \cdots (x - x_{k-1})\hat{B}_k(y, z)$.

III. THE DEFINITION OF $TGMRA_{NT}$

Lemma 3.1 [Theorem 3.1, 23]. Define

$$l_{i,n} = l_{i,n}(y, z) = h_{i,0}(z) + \frac{y - y_0}{h_{i,1}(z)} + \cdots + \frac{y - y_{n-1}}{h_{i,n}(z)}$$

$$h_{i,t}(z) = \varphi_{i,t,0}(x_i^0; y_t^0; z_0) + \frac{z - z_0}{\varphi_{i,t,1}(x_i^0; y_j^0; z_1^0)} + \cdots + \frac{z - z_{u-1}}{\varphi_{i,t,u}(x_i^0; y_j^0; z_u^0)} \quad i = 0, \ldots, m, t = 0, \ldots, n.$$

With generalized inverse from tail to head, we can get the rational matrix function $l_{i,n}$ with matrix $\bar{N}_i = \bar{N}_i(y, z)$ and the polynomial $\bar{D}_i = \bar{D}_i(y, z)$ such that
(i) $l_{i,n} = \bar{N}_i/\bar{D}_i$, (ii) $\bar{D}_i \geq 0$,
(iii) $\bar{D}_i| \; \|\bar{N}_i(y, z)\|^2$, where "|" stands for divisibility.

Theorem 3.2 (Divisibility). Suppose that
$R_{m,n,u} = R_{m,n,u}(x, y, z)$

$$= l_{0,n} + l_{1,n}(x - x_0) + \cdots + l_{m,n}(x - x_0) \cdots (x - x_{m-1}). \tag{20}$$

where $l_{i,n} = l_{i,n}(y, z)$.

Using generalized inverse, we rationalize $l_{i,n}(y, z)$ *of* $R_{m,n,u}(x, y, z)$ as in (16) from tail to head, and suppose that all the intermediate denominators are nonzero, then we obtain a matrix $\hat{N} = \hat{N}(x, y, z)$ as well as a polynomial $\hat{D} = \hat{D}(x, y, z)$ which satisfy that
(i) $R_{m,n,u} = \hat{N}/\hat{D}$, (ii) $\hat{D} \geq 0$, (iii) $\hat{D}|\|\hat{N}\|^2$.

Proof. Consider the construction of $\hat{N}$, $\hat{D}$ and $R_{m,n,u}$.

First, let $\hat{D}_0 = 1$, and $\hat{N}_0 = l_{0,n}$, By Lemma 3.1, there are a matrix $\bar{N}_0$ and a polynomial $\bar{D}_0$ which satisfy
(i) $l_{0,n} = \bar{N}_0/\bar{D}_0$, (ii) $\bar{D}_0 \geq 0$, (iii) $\bar{D}_0| \; \|\bar{N}_0\|^2$, "|" means divisibility. Then let $S^{(j)}(x, y, z) = S^{(j-1)}(x, y, z) + l_{j,n}(x - x_0) \cdots (x - x_{j-1})$ with $l_{j,n} = \bar{N}_j/\bar{D}_j$, $j = 1, \ldots, m$.

We can get

$$\begin{aligned} &S^{(j+1)}(x, y, z) \\ &= S^{(j)}(x, y, z) + l_{j+1,n}(x - x_0) \cdots (x - x_j) \\ &= \frac{\hat{N}_j}{\hat{D}_j} + \frac{\bar{N}_{j+1}}{\bar{D}_{j+1}}(x - x_0) \cdots (x - x_j) \\ &= \frac{\hat{N}_j \bar{D}_{j+1} + \hat{D}_j \bar{N}_{j+1}(x - x_0) \cdots (x - x_j)}{\hat{D}_j \bar{D}_{j+1}} \\ &= \frac{\hat{N}_{j+1}}{\hat{D}_{j+1}}, \end{aligned}$$

where

$$\begin{aligned} \hat{N}_{j+1} &= \hat{N}_j \bar{D}_{j+1} + \hat{D}_j \bar{N}_{j+1}(x - x_0) \cdots (x - x_j), \\ \hat{D}_{j+1} &= \hat{D}_j \bar{D}_{j+1}, \end{aligned}$$

Obviously,
$\|\hat{N}_{j+1}\|^2$
$= \|\hat{N}_j\|^2 \bar{D}_{j+1}^2 + \hat{D}_j^2 \|\bar{N}_{j+1}\|^2 (x - x_0)^2 \cdots (x - x_j)^2$
$+(\overline{\hat{N}}_j \cdot \bar{N}_{j+1} + \hat{N}_j \cdot \overline{\bar{N}}_{j+1})\bar{D}_{j+1}\hat{D}_j(x - x_0) \cdots (x - x_j)$,
since $\hat{D}_j|\|\hat{N}_j\|^2$, $\bar{D}_{j+1}|\|\bar{N}_{j+1}\|^2$, so $\hat{D}_{j+1}|\|\hat{N}_{j+1}\|^2$.

So that $S^{(m)}(x, y, z) = R_{m,n,u}(x, y, z)$.

Definition 3.3. Define $R(x, y, z) = \hat{N}/\hat{D}$ be the type of $[t/w]$ if the degree of matrix $\{\hat{N}_{ij}\} \leq t$ for $1 \leq i \leq u, 1 \leq j \leq v$, for some (i, j) the degree of $\{\hat{N}_{ij}\} = t$ and the degree of polynomial $\{\hat{D}\} = w$ where $\hat{N} = (\hat{N}_{ij}) \in \mathbf{C}^{u \times v}$.

Lemma 3.4 [Theorem 3.4, 23]. Let $l_{i,n}(y, z)$ be defined in Lemma 3.1, and $l_{i,n} = \bar{N}_i/\bar{D}_i$, if n, u are even, $l_{i,n}$ is the type of $[r/r]$; if n, u are not both even, $U_0(y)$ is the type of $[r/r - 1]$, where $r = nu + n + u$.

Theorem 3.5 (Characterization).
Let $R_{m,n,u}(x, y, z) = \hat{N}/\hat{D}$ be
$R_{m,n,u}(x, y, z) = l_{0,n} + l_{1,n}(x - x_0) + \cdots$
$+ l_{m,n}(x - x_0)(x - x_1) \cdots (x - x_{m-1})$
where $l_{i,n} = \bar{N}_i/\bar{D}_i$ as in (16). Suppose that all $\bar{D}_i$ for $i = 0, \ldots, m$ have no common factors, then
(i)when n, u are both even, $R_{m,n,u}(x, y, z)$ is the type of $[(m + 1)r + m/(m + 1)r]$,
(ii)when n, u are not both even, $R_{m,n,u}(x, y, z)$ is the type of $[(m + 1)r/(m + 1)(r - 1)]$.

Proof. The proof is by induction on m. Assume that n, u are even integers and denote $R_{i,n,u} = R_{i,n,u}(x, y, z)$, $i = 0, \ldots, m$.

When $m = 0, R_{0,n,u} = l_{0,n} = \bar{N}_0/\bar{D}_0$, from Theorem 3.4 in [15], $R_{0,n,u}(x, y, z)$ is the type of $[r/r]$.

When $m = 1$, for $deg\{\bar{N}_0\} = deg\{\bar{D}_0\} = r$,
$deg\{\bar{N}_1\} = deg\{\bar{D}_1\} = r$,

$$\begin{aligned} R_{1,n,u} &= \frac{\hat{N}_1}{\hat{D}_1} = l_{0,n} + l_{1,n}(x - x_0) \\ &= \frac{\bar{N}_0}{\bar{D}_0} + \frac{\bar{N}_1}{\bar{D}_1}(x - x_0) = \frac{\bar{N}_0 \bar{D}_1 + \bar{N}_1 \bar{N}_0(x - x_0)}{\bar{D}_0 \bar{D}_1} \end{aligned}$$

is the type of $[2r + 1/2r]$.

When $m = 2$, we know that
$deg\{\hat{N}_1\} = 2r + 1$, $deg\{\hat{D}_1\} = 2r$,
$deg\{\bar{N}_2\} = r$, $deg\{\bar{D}_2\} = r$,

$$\begin{aligned} R_{2,n,u} &= \frac{\hat{N}_2}{\hat{D}_2} = \frac{\hat{N}_1}{\hat{D}_1} + \frac{\bar{N}_2}{\bar{D}_2}(x - x_0)(x - x_1) \\ &= \frac{\hat{N}_1 \bar{D}_2 + \bar{N}_2 \hat{D}_1(x - x_0)(x - x_1)}{\hat{D}_1 \bar{D}_2}, \end{aligned}$$

so $R_{2,n,u}$ is the type of $[3r + 2/3r]$.

Assume that $m = k$, $R_{k,n,u} = \dfrac{\hat{N}_k}{\hat{D}_k}$ is the type of $[(k + 1)r + k/(k + 1)r]$.

When $m = k + 1$, we have

$$\begin{aligned} R_{k+1,n,u} &= \frac{\hat{N}_k}{\hat{D}_k} + \frac{\bar{N}_{k+1}}{\bar{D}_{k+1}}(x - x_0) \cdots (x - x_k) \\ &= \frac{\hat{N}_k \bar{D}_{k+1} + \bar{N}_{k+1}\hat{D}_k(x - x_0) \cdots (x - x_k)}{\hat{D}_k \bar{D}_{k+1}} \\ &= \frac{\hat{N}_{k+1}}{\hat{D}_{k+1}}, \end{aligned}$$

where $deg\{N_{k+1}\} = (k+1)(r+1)+r = (k+2)r+(k+1)$, $deg\{D_{k+1}\} = (k + 1)r + r = (k + 2)r$.

Therefore, when n, u are both even, $R_{m,n,u}(x,y,z)$ is the type of $[(m+1)r+m/(m+1)r]$.

When n, u are not both even, $R_{m,n,u}(x,y,z)$ is the type of $[(m+1)r/(m+1)(r-1)]$, then it can be proved in a similar manner.

Definition 3.6 A rational matrix fraction $R_{m,n,u}(x,y,z)=\hat{N}/\hat{D}$ is a trivariate generalized inverse Newton-Thiele matrix rational approximation $(TGMRA_{NT})$ of a matrix function $f(x,y,z)$ on the cubic domain $\Omega_{m,n,u}$, if

(i)$R_{m,n,u}(x_i,y_j,z_k)=f(x_i,y_j,z_k),\ \ (x_i,y_j,z_k)\in\Omega_{m,n,u}$,

(ii)$\hat{D}(x_i,y_j,z_k)\neq 0,\ \ (x_i,y_j,z_k)\in\Omega_{m,n,u}$,

(iii) (a) when n,u are both even,

$$deg\{\hat{N}\}=(m+1)r+m,\ \ deg\{\hat{D}\}=(m+1)r,$$

(b) when n,u are not both even,

$$deg\{\hat{N}\}=(m+1)r,\ \ deg\{\hat{D}\}=(m+1)(r-1),$$

(iv) $\hat{D}|\|\hat{N}\|^2$, (v) $\hat{D}$ is a real polynomial, and $\hat{D}\geq 0$.

To obtain the remainder term of the $TGMRA_{NT}$, we first give lemma 3.7.

Lemma 3.7 Suppose

$$R^*_{m,n,u}(x,y,z)=l^*_{0,n}+l^*_{1,n}(x-x_0)+\cdots+l^*_{m,n}(x-x_0)(x-x_1)\cdots(x-x_{m-1})$$

with

$$l^*_{i,n}=h^*_{i,0}(z)+\frac{y-y_0}{h^*_{i,1}(z)}+\cdots+\frac{y-y_{n-1}}{h^*_{i,n}(z)}+\frac{y-y_n}{h^*_{i,n+1}(z)},$$

$$h^*_{i,j}(z)=\varphi_{i,j,0}(x_i^0;y_j^0;z_0)+\frac{z-z_0}{\varphi_{i,j,1}(x_i^0;y_j^0;z_1^0)}+\cdots+\frac{z-z_{u-1}}{\varphi_{i,j,u}(x_i^0;y_j^0;z_u^0)}+\frac{z-z_u}{\varphi_{i,j,u+1}(x_i^0;y_j^0;z_u^0,z)},$$

$$h^*_{i,n+1}(z)=\varphi_{i,n+1,0}(x_i^0;y_n^0,y;z_0)+\frac{z-z_0}{\varphi_{i,n+1,1}(x_i^0;y_n^0,y;z_1^0)}+\cdots+\frac{z-z_{u-1}}{\varphi_{i,n+1,u}(x_i^0;y_n^0;z_u^0)}+\frac{z-z_u}{\varphi_{i,n+1,u+1}(x_i^0;y_n^0,y;z_u^0,z)},$$

$$i=0,\ldots,m\ j=0,\ldots,n.$$

$$R^{**}_{m,n,u}(x,y,z)=l^{**}_{0,n}+l^{**}_{1,n}(x-x_0)+\cdots+l^{**}_{m,n}(x-x_0)(x-x_1)\cdots(x-x_{m-1}).$$

$$l^{**}_{i,n}=h^*_{i,0}(z)+\frac{y-y_0}{h^*_{i,1}(z)}+\cdots+\frac{y-y_{n-1}}{h^*_{i,n}(z)}.$$

Then $R^*_{m,n,u}(x,y,z)$ satisfies

$$R^*_{m,n,u}(x_i,y,z)=f(x_i,y,z),\qquad i=0,\ldots,m,$$

$$R^*_{m,n,u}(x,y_j,z)=R^{**}_{m,n,u}(x,y_j,z),\qquad j=0,\ldots,n,$$

$$R^{**}_{m,n,u}(x,y,z_k)=R_{m,n,u}(x,y,z_k),\qquad k=0,\ldots,u.$$

Remark: we can get the proof of the lemma easily for it is generalized from [15, P.143].

Motivated by [15, P.144], now we give the error term.

Let
$$R_{m,n,u}(x,y,z)=\frac{\hat{N}(x,y,z)}{\hat{D}(x,y,z)},$$

$$R^*_{m,n,u}(x,y,z)=\frac{\hat{N}^*(x,y,z)}{\hat{D}^*(x,y,z)},$$

$$R^{**}_{m,n,u}(x,y,z)=\frac{\hat{N}^{**}(x,y,z)}{\hat{D}^{**}(x,y,z)}.$$

Theorem 3.8 Suppose the points $\Omega_{m,n,u}=\{(x_l,y_k,z_t):l=0,\ldots,m,k=0,\ldots,n,t=0,\ldots,u,(x_l,y_k,z_t)\in R^3\}\subset S=\{a\leq x\leq b,c\leq y\leq d,e\leq z\leq f\}$. The trivariate matrix function $f(x,y,z)$ is L times continuously differentiable on S where L =max$(m+1,n+1,u+1)$, then for any point $(x,y,z)\in S$, a point $(\xi,\eta,\zeta)\in S$ exists such that

$$\begin{aligned}&f(x,y,z)-R_{m,n,u}(x,y,z)\\&=\frac{1}{\hat{D}\hat{D}^*\hat{D}^{**}}\{\frac{\omega_{m+1}(x)}{(m+1)!}\frac{\partial^{m+1}}{\partial x^{m+1}}\hat{E}_1(\xi,y,z)\\&+\frac{\varpi_{n+1}(y)}{(n+1)!}\frac{\partial^{n+1}}{\partial y^{n+1}}\hat{E}_2(x,\eta,z)\\&\qquad+\frac{\tilde{\omega}_{u+1}(z)}{(u+1)!}\frac{\partial^{u+1}}{\partial y^{u+1}}\hat{E}_3(x,y,\zeta)\}\end{aligned}$$

where

$$\hat{D}=\hat{D}(x,y,z),\hat{D}^*=\hat{D}^*(x,y,z),\hat{D}^{**}=\hat{D}^{**}(x,y,z),$$

$$\omega_{m+1}(x)=(x-x_0)(x-x_1)\cdots(x-x_m),$$

$$\varpi_{n+1}(y)=(y-y_0)(y-y_1)\cdots(y-y_n),$$

$$\tilde{\omega}_{u+1}(z)=(z-z_0)(z-z_1)\cdots(z-z_u).$$

$$\hat{E}_1=\hat{D}\hat{D}^*\hat{D}^{**}[f(x,y,z)-R^*_{m,n,u}(x,y,z)],$$

$$\hat{E}_2=\hat{D}\hat{D}^*\hat{D}^{**}[R^*_{m,n,u}(x,y,z)-R^{**}_{m,n,u}(x,y,z)],$$

$$\hat{E}_3=\hat{D}\hat{D}^*\hat{D}^{**}[R^{**}_{m,n,u}(x,y,z)-R_{m,n,u}(x,y,z)].$$

Proof Let

$$\hat{E}_1=\hat{D}\hat{D}^*\hat{D}^{**}[f(x,y,z)-R^*_{m,n,u}(x,y,z)],$$

$$\hat{E}_2=\hat{D}\hat{D}^*\hat{D}^{**}[R^*_{m,n,u}(x,y)-R^{**}_{m,n,u}(x,y,z)],$$

and

$$\hat{E}_3=\hat{D}\hat{D}^*\hat{D}^{**}[R^{**}_{m,n,u}(x,y,z)-R_{m,n,u}(x,y,z)],$$

$$\hat{E}=\hat{E}_1+\hat{E}_2+\hat{E}_3.$$

where $\hat{E}_i=\hat{E}_i(x,y,z),\ \ i=1,2,3.$ from lemma 3.7, we know

$$\hat{E}_1(x_i,y,z)=0,\ i=0,\ldots,m,$$

which results in

$$\hat{E}_1=\frac{\omega_{m+1}(x)}{(m+1)!}\frac{\partial^{m+1}}{\partial x^{m+1}}\hat{E}_1|_{x=\xi},$$

where $\omega_{m+1}(x)=(x-x_0)(x-x_1)\cdots(x-x_m)$, and $\xi\in(a,b)$ depends on y,z.

Similarly, from$\hat{E}_2(x,y_j,z)=0,\ j=0,\ldots,n$, have

$$\hat{E}_2=\frac{\varpi_{n+1}(y)}{(n+1)!}\frac{\partial^{n+1}}{\partial y^{n+1}}\hat{E}_2|_{y=\eta},$$

where $\varpi_{n+1}(y)=(y-y_0)(y-y_1)\cdots(y-y_n)$, and $\eta\in(c,d)$ depends on x,z.

$$\hat{E}_3=0,\ k=0,\ldots,u,$$

TABLE I
THE NUMERICAL RESULTS OF $TGMRA_{NT}$

(x, y, z)	$f(x, y, z)$
$(0.05, 0.05, 0.05)$	$\begin{pmatrix} 0.98877108 & 1.16183424 \\ 0.15000000 & 0.09983342 \end{pmatrix}$
(x, y, z)	$TGMRA_{NT}$
$(0.05, 0.05, 0.05)$	$\begin{pmatrix} 0.98877149 & 1.16184470 \\ 0.15000006 & 0.09983963 \end{pmatrix}$
$\|f - TGMRA_{NT}\|_F$	$CPUtime$
$1.217430e-05$	$2.642s$
(x, y, z)	$f(x, y, z)$
$(0.15, 0.15, 0.15)$	$\begin{pmatrix} 0.90044710 & 1.56831219 \\ 0.45000000 & 0.29552021 \end{pmatrix}$
(x, y, z)	$TGMRA_{NT}$
$(0.15, 0.15, 0.15)$	$\begin{pmatrix} 0.90044875 & 1.56830481 \\ 0.45000004 & 0.29551631 \end{pmatrix}$
$\|f - TGMRA_{NT}\|_F$	$CPUtime$
$8.503429e-06$	$2.590s$
(x, y, z)	$f(x, y, z)$
$(0.25, 0.25, 0.25)$	$\begin{pmatrix} 0.73168887 & 2.11700002 \\ 0.75000000 & 0.47942554 \end{pmatrix}$
(x, y, z)	$TGMRA_{NT}$
$(0.25, 0.25, 0.25)$	$\begin{pmatrix} 0.73168440 & 2.11701354 \\ 0.74999969 & 0.47943216 \end{pmatrix}$
$\|f - TGMRA_{NT}\|_F$	$CPUtime$
$1.570661e-05$	$2.752s$

TABLE II
THE NUMERICAL RESULTS OF $TGMRI$

(x, y, z)	$f(x, y, z)$
$(0.05, 0.05, 0.05)$	$\begin{pmatrix} 0.98877108 & 1.16183424 \\ 0.15000000 & 0.09983342 \end{pmatrix}$
(x, y, z)	$TGMRI$
$(0.05, 0.05, 0.05)$	$\begin{pmatrix} 0.98876528 & 1.16184269 \\ 0.15000036 & 0.09984494 \end{pmatrix}$
$\|f - TGMRI\|_F$	$CPUtime$
$1.542720e-05$	$24.370s$
(x, y, z)	$f(x, y, z)$
$(0.15, 0.15, 0.15)$	$\begin{pmatrix} 0.90044710 & 1.56831219 \\ 0.45000000 & 0.29552021 \end{pmatrix}$
(x, y, z)	$TGMRI$
$(0.15, 0.15, 0.15)$	$\begin{pmatrix} 0.90044875 & 1.56830481 \\ 0.45000004 & 0.29551631 \end{pmatrix}$
$\|f - TGMRI\|_F$	$CPUtime$
$1.114339e-05$	$24.500s$
(x, y, z)	$f(x, y, z)$
$(0.25, 0.25, 0.25)$	$\begin{pmatrix} 0.73168887 & 2.11700002 \\ 0.75000000 & 0.47942554 \end{pmatrix}$
(x, y, z)	$TGMRI$
$(0.25, 0.25, 0.25)$	$\begin{pmatrix} 0.73166651 & 2.11700217 \\ 0.75000530 & 0.47943999 \end{pmatrix}$
$\|f - TGMRI\|_F$	$CPUtime$
$2.723745e-05$	$57.399s$

we can get that

$$\hat{E}_3 = \frac{\tilde{\omega}_{u+1}(z)}{(u+1)!}\frac{\partial^{u+1}}{\partial y^{n+1}}\hat{E}_3|_{z=\zeta},$$

where $\tilde{\omega}_{u+1}(z) = (z - z_0)(z - z_1)\cdots(z - z_u)$,
and $\zeta \in (c, d)$ depends on x, y.

Thus, we have

$\hat{E} = \hat{E}_1 + \hat{E}_2 + \hat{E}_3$
$= \{\frac{\omega_{m+1}(x)}{(m+1)!}\frac{\partial^{m+1}}{\partial x^{m+1}}\hat{E}_1(\xi, y, z)$
$+\frac{\varpi_{n+1}(y)}{(n+1)!}\frac{\partial^{n+1}}{\partial y^{n+1}}\hat{E}_2(x, \eta, z)+\frac{\tilde{\omega}_{u+1}(z)}{(u+1)!}\frac{\partial^{u+1}}{\partial y^{u+1}}\hat{E}_3(x, y, \zeta)\}.$

Therefore, the proof is completed.

IV. Numerical Example

Example 4.1 Let $f(x, y, z)$ be a matrix function of order 2 and

$$f(x, y, z) = \begin{pmatrix} cos(x + y + z) & e^{(x+y+z)} \\ x + y + z & sin(x + z) \end{pmatrix}$$

and we suppose $\{x_3^0\} = \{y_3^0\} = \{z_3^0\} = \{0, 0.1, 0.2, 0.3\}$. The numerical results $TGMRA_{NT}$ of Algorithm 2.3 and $TGMRI$ [27] are given in TABLE I and TABLE II.

Example 4.2 Let $f(x, y, z)$ be a 2×3 matrix function and

$$f(x, y, z) = \begin{pmatrix} cos(x + y + z) & sin(x + y + z) & e^x \\ x + y + z & e^y & x + y \end{pmatrix}$$

and we suppose $\{x_3^0\} = \{y_3^0\} = \{z_3^0\} = \{0, 0.2, 0.4, 0.6\}$. The numerical results $TGMRA_{NT}$ of Algorithm 2.3 and $TGMRI$ [27] are given in TABLE III and TABLE IV.

TABLE III
THE NUMERICAL RESULTS OF $TGMRA_{NT}$

(x, y, z)	$f(x, y, z)$
$(0.1, 0.1, 0.1)$	$\begin{pmatrix} 0.955336 & 0.295520 & 1.105171 \\ 0.300000 & 1.105171 & 0.200000 \end{pmatrix}$
(x, y, z)	$TGMRA_{NT}$
$(0.1, 0.1, 0.1)$	$\begin{pmatrix} 0.955321 & 0.295628 & 1.105252 \\ 0.300001 & 1.105215 & 0.200001 \end{pmatrix}$
$\|f - TGMRA_{NT}\|_F$	$CPUtime$
$1.428951e-04$	$3.175s$
(x, y, z)	$f(x, y, z)$
$(0.3, 0.3, 0.3)$	$\begin{pmatrix} 0.621610 & 0.783327 & 1.349859 \\ 0.900000 & 1.349859 & 0.600000 \end{pmatrix}$
(x, y, z)	$TGMRA_{NT}$
$(0.3, 0.3, 0.3)$	$\begin{pmatrix} 0.621641 & 0.783264 & 1.349808 \\ 0.899998 & 1.349826 & 0.599998 \end{pmatrix}$
$\|f - TGMRA_{NT}\|_F$	$CPUtime$
$9.301701e-05$	$2.972s$
(x, y, z)	$f(x, y, z)$
$(0.5, 0.5, 0.5)$	$\begin{pmatrix} 0.070737 & 0.997495 & 1.648721 \\ 1.500000 & 1.648721 & 1.000000 \end{pmatrix}$
(x, y, z)	$TGMRA_{NT}$
$(0.5, 0.5, 0.5)$	$\begin{pmatrix} 0.070649 & 0.997583 & 1.648809 \\ 1.500004 & 1.648789 & 1.000004 \end{pmatrix}$
$\|f - TGMRA_{NT}\|_F$	$CPUtime$
$1.671671e-04$	$2.955s$

From the two examples, we notice that the error using $TGMRA_{NT}$ is less than $TGMRI$ mentioned in [27], and the CPU time needed is much less than the time of $TGMRI$ method.

V. Conclusions

We introduce a new method of approximating trivariate matrix functions with generalized inverse Newton-Thiele formula and obtain a recursive algorithm as well as some crucial properties. According to the numerical example, we can easily find that the method using $TGMRA_{NT}$ is much better than the one using $TGMRI$ [27].

TABLE IV
THE NUMERICAL RESULTS OF $TGMRI$

(x, y, z)	$f(x, y, z)$
$(0.1, 0.1, 0.1)$	$\begin{pmatrix} 0.955336 & 0.295520 & 1.105171 \\ 0.300000 & 1.105171 & 0.200000 \end{pmatrix}$
(x, y, z)	$TGMRI$
$(0.1, 0.1, 0.1)$	$\begin{pmatrix} 0.955205 & 0.295685 & 1.105216 \\ 0.300003 & 1.105215 & 0.200003 \end{pmatrix}$
$\Vert f - TGMRI \Vert_F$	$CPUtime$
$2.203323e-04$	$51.531s$
(x, y, z)	$f(x, y, z)$
$(0.3, 0.3, 0.3)$	$\begin{pmatrix} 0.621610 & 0.783327 & 1.349859 \\ 0.900000 & 1.349859 & 0.600000 \end{pmatrix}$
(x, y, z)	$TGMRI$
$(0.3, 0.3, 0.3)$	$\begin{pmatrix} 0.621725 & 0.783260 & 1.349824 \\ 0.899995 & 1.349826 & 0.599995 \end{pmatrix}$
$\Vert f - TGMRI \Vert_F$	$CPUtime$
$1.418878e-04$	$52.009s$
(x, y, z)	$f(x, y, z)$
$(0.5, 0.5, 0.5)$	$\begin{pmatrix} 0.070737 & 0.997495 & 1.648721 \\ 1.500000 & 1.648721 & 1.000000 \end{pmatrix}$
(x, y, z)	$TGMRI$
$(0.5, 0.5, 0.5)$	$\begin{pmatrix} 0.070509 & 0.997526 & 1.648796 \\ 1.500012 & 1.648789 & 1.000012 \end{pmatrix}$
$\Vert f - TGMRI \Vert_F$	$CPUtime$
$2.522200e-04$	$54.551s$

REFERENCES

[1] G.A. Baker, P.R. Graves-Morris, "Padé Approximants: Basic Theory," Addison-Wesley, Reading, MA, 1981.

[2] A.C. Antoulas, J.A. Ball, J. Kang, J.C. Willems, "On the solution of the minimal rational interpolation problem," *Linear Algebra and its Applications*, vol. 137-138, pp. 511-573, 1990.

[3] A.C. Antoulas, "Rational interpolation and the Euclidean algorithm," *Linear Algebra and its Applications*, vol. 108, pp. 157-171, 1988.

[4] B.D.O. Anderson, A.C. Antoulas, "Rational interpolation and state-variable realizationas," *Linear Algebra and its Applications*, vol. 137-138, pp. 479-509, 1990.

[5] B. Beckermann, G.Labahn, "Recursiveness in matrix rational interpolation problems," *Journal of Computational and Applied Mathematics*, vol. 77, pp. 5-34, 1997.

[6] T. Boros, A.H. Sayed, T. Kailath,"A recursive method for solving unconst rained tangential interpolation problems," *IEEE Transactions on Automatic Control*, vol. 44, pp. 454-470, 1999.

[7] C. Gu, "On matrix valued osculatory rational interpolant," *Numerical Mathematics A Journal of Chinese Universities*, vol. 18, pp. 135-141, 1996.

[8] P.R. Graves-Morris, T.P. Hopkins, "Reliable rational interpolation,"*Numerische Mathematik*, vol. 36, pp. 111-128, 1980.

[9] P.R. Graves-Morris, "Vector valued rational interpolants I," *Numerische Mathematik*, vol. 42, pp. 331-348, 1983.

[10] J. Tan and Y. Fang, "Newton-Thieles rational interpolants," *Numerical Algorithms*, vol. 24, pp. 141-157, 2000.

[11] P.I. Bodnarcuk and W.Ja. Skorobogatko, "Branched Continued Fractions and Applications," Naukowaja Dumka, Kiev 1974.

[12] J.A. Murphy and M.R. ODonohoe, "A two-variable generalization of the Stieltjes-type continued fractions," *Journal of Computational and Applied Mathematics*, vol. 4, pp. 181-190, 1978.

[13] Kh. J. Kutschminskaja, "Corresponding and associated branching continued fractions for the double power series," *Dokl. Akad. Nauk USSR,*, vol. 7, pp. 614-617, 1978.

[14] W. Siemaszko, "Branched continued fractions for double power series," *Journal of Computational and Applied Mathematics*, vol. 6, pp. 121-125, 1980.

[15] W. Siemaszko, "Thiele-type branched continued fractions for two-variable functions," *Journal of Computational and Applied Mathematics* , vol. 9, pp. 137-153, 1983.

[16] A. Cuyt and B, Verdonk, "Multivariate reciprocal differences for branched Thiele continued fraction expansions," *Journal of Computational and Applied Mathematics*, vol. 21, pp. 145-160, 1988.

[17] P. Wynn, "Continued fractions whose coefficients obey a non-commutative law of multiplication," *Archive for Rational Mechanics and Analysis volume*, vol. 12, pp. 273-312, 1963.

[18] H. Zhao, G. Zhu, P. Xiao, "A backward three-term recurrence relation for vector valued continued fractions and its applications," *Journal of Computational and Applied Mathematics*, vol. 142, pp. 389-400, 2002.

[19] C. Gu, Z. Chen, "Matrix valued rational interpolants and its error formula," *Mathematics Numerical Sinica*, vol. 17, pp. 73-77, 1995.

[20] C. Gu, "Matrix Padé type approximant and directional matrix Padé approximat in the inner product space," *Journal of Computational and Applied Mathematics*, vol. 164-165, pp. 365-385, 2004.

[21] C. Gu, "Thiele-type and largrange-type generalized inverse rational interpolation for rectangular complex matrices," *Linear Algebra and its Applications*, vol. 295, pp. 7-30, 1999.

[22] C. Gu, "A Practical Two-Dimensional Thiele-Type Matrix Padé Approximation," *IEEE Transactions on Automatic Control*, vol. 48, pp. 2259-2263, 2003.

[23] C. Gu, "Bivariate Thiele-type matrix valued rational interpolants,"*Journal of Computational and Applied Mathematics*, vol. 80, pp. 71-82, 1997.

[24] X. Qin, Q. Xu, "A New C2 Piecewise Bivariate Rational Interpolation Scheme with Bi-quadratic Denominator," *IAENG International Journal of Applied Mathematics*, vol. 47, no. 4, pp. 454-458, 2017.

[25] X. Qin, Y. Zhu, "A Class of Bivariate Rational Interpolation Surfaces with C2 Continuity," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 3, pp. 601-608, 2020.

[26] R. Cui, C. Gu, "Bivariate generalized inverse Newton-Thiele type matrix Padé approximation", *Applied Mathematics and Computation*, vol. 236, pp. 202-214. 2014.

[27] J. Wang, C. Gu, "Three-Dimensional Generalized Inverse Matrix Rational Interpolation," *Journal of Shanghai University (English Edition)*, vol. 5, pp. 276-281, 2001.