Analytical and Numerical Solutions of the Modified Coupled Burgers' Equations

Bubpha Jitsom, Surattana Sungnul and Ekkachai Kunnawuttipreechachan

***Abstract*—In this work, we study the coupled Burgers' equations and the modified coupled Burgers' equations, which including nonlinear viscous terms. The Hopf-Cole transformation method is applied to obtain the analytical solutions. The explicit finite difference schemes are used to evaluate the numerical solutions. In addition, we compare the numerical solutions with the analytical solutions for the selected initial and boundary conditions. Moreover, the behaviors of the modified coupled Burger' equations with varied viscosity coefficients are also investigated.**

***Index Terms*—modified Burgers' equations, Hopf-Cole transformation, finite difference method, explicit scheme.**

I. Introduction

BURGERS' equation is a nonlinear partial differential equation that is widely used as a model for convection-diffusion processes. It is closely related to the Navier–Stokes equations. In general, the Burgers' equation plays a vital role to analyze fluid turbulences, gas dynamics, and fluid mechanics problems in many different fields of science and engineering. It is also used to model for traffic flow and computer network problems. As the result, many researchers have been interested in developing methods for finding numerical solutions of the Burgers' equation, which are finite difference methods, finite element methods, and finite volume methods.

The complete nonlinear one-dimensional Burgers' equation is given by [1] as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\mu_0\frac{\partial^2 u}{\partial x^2},\quad (x,t)\in D, \tag{1}$$

where u is fluid velocity, μ_0 is viscosity coefficient, x is position, t is time and D is a continuous space-time domain.

Equation (1) is a parabolic PDE, which can be served as a model equation for boundary–layer problems. For a steady-state boundary–layer and the "parabolized" Navier–Stokes equation, the independent variables t and x can be replaced by x and y, respectively to give

$$\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=\mu_0\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

Manuscript received September 20, 2021; revised April 20, 2022. This research was financially supported by the Faculty of Applied Science, King Mongkut's University of Technology North Bangkok (Contract no. 6242107).

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where $u(x,y)$ gives a marching direction and $(x,y)\in D$. According to [1], the two-dimensional coupled Burgers' equations can be written as

$$u_t+uu_x+vu_y = \mu_0\left(u_{xx}+u_{yy}\right), \tag{3}$$
$$v_t+uv_x+vv_y = \mu_0\left(v_{xx}+v_{yy}\right). \tag{4}$$

In this problem, we assume that the initial conditions are

$$\left.\begin{array}{l} u(x,y,0)=a_1(x,y) \\ v(x,y,0)=a_2(x,y)\end{array}\right\},\quad (x,y)\in\Omega, \tag{5}$$

and the boundary conditions are

$$\left.\begin{array}{l} u(x,y,t)=b_1(x,y,t) \\ v(x,y,t)=b_2(x,y,t)\end{array}\right\},\quad (x,y)\in\partial\Omega,\quad t>0, \tag{6}$$

where $\Omega=\{(x,y),a\le x\le b,c\le y\le d\}$ is the computational domain, $\partial\Omega$ is the boundary of the domain, u and v are the velocity components, and μ_0 is the viscosity coefficient. In the initial and boundary conditions, a_1,a_2,b_1 and b_2 are assumed to be known functions. The terms u_t and v_t are the unsteady terms. In addition, uu_x+vu_y and uv_x+vv_y are the non–linear convection term; $u_{xx}+u_{yy}$ and $v_{xx}+v_{yy}$ are the diffusion terms.

In 2010, Zhua *et al.* [2] applied the discrete Adomian decomposition method to evaluate the numerical solutions of two-dimensional Burgers' nonlinear differential equations. They showed that the numerical results are in a good agreement with the exact solutions. Moreover, Huang and Abduwali [3] applied a modified local Crank-Nicolson method to solve one- and two-dimensional Burgers' equations. The new explicit finite difference schemes with unconditionally stable were obtained.

In 2014, Shukla *et al.* [4] used a modified cubic B-spline differential quadrature method to obtain numerical solutions of the two-dimensional nonlinear coupled viscous Burgers' equation with appropriate initial and boundary conditions.

In 2015, Gulkac [5] illustrated the locally one-dimensional method (LOD) for solving the two-dimensional coupled Burgers' equations. The Fourier method of the LOD method is also investigated to analyze stability of the solutions. The method used in this work can be easily implemented for solving nonlinear problems evolving in several branches of engineering and science.

In 2016, Rotich *et al.* [6] used three methods to solve the coupled Burgers' equations, which are the alternative direction implicit method, the variation of iteration method, and the locally one-dimensional finite difference method. In this work the Hopf-Cole transformation and separation of variables were used to generate appropriate initial and boundary conditions. The result shows that the numerical schemes are unconditionally stable.

In 2018, Sungnul *et al.* [7] investigated the behavior of the modified Burgers' equation of the form

$$u_t + (c + bu)u_x = (\mu_0 + \mu_1 u)u_{xx}$$

where c, b, μ_0 and μ_1 are arbitrary parameters. The numerical solutions of this problem was obtained by the finite difference method with the forward-time central-space (FTCS) implicit scheme. The results obtained by advantages of mathematical software are compared between the numerical solutions and the exact solutions for some examples of initial and boundary conditions.

In 2019, Mukundan *et al.* [8] applied the two-dimensional Hopf-Cole transformation to convert the nonlinear coupled Burgers' equations into a two-dimensional linear diffusion equation with Neumann boundary conditions. Multistep methods based on backward differentiation formulas of order one, two and three were employed to solve the related differential systems. In addition, Mohamed [9] introduced new fully implicit numerical schemes for solving one-dimensional and two-dimensional unsteady Burgers' equations. The nonlinear Burgers' equation was discretized in the spatial direction by using a second-order finite difference method to convert the Burgers' equation to a system of nonlinear ODEs.

In 2021, Camcoon and Pochai [10] introduced a governing equation of a one-dimensional shoreline evolution model, when a couple of groins is added. The introduced model is a transient one-line model. The traditional forward-time centered-space method and the unconditionally stable Saulyev finite difference methods are employed to approximate the incremental model in each year.

In this work, we aims to investigate the behavior of the modified coupled Burgers' equations of the form

$$\begin{aligned} u_t + uu_x + vu_y &= (\mu_0 + \mu_1 u)(u_{xx} + u_{yy}), \quad (7)\\ v_t + uv_x + vv_y &= (\mu_0 + \mu_1 v)(v_{xx} + v_{yy}), \quad (8)\end{aligned}$$

with the initial conditions

$$\left.\begin{aligned} u(x,y,0) = a_1(x,y)\\ v(x,y,0) = a_2(x,y)\end{aligned}\right\}, \quad (x,y) \in \Omega, \tag{9}$$

and the boundary conditions

$$\left.\begin{aligned} u(x,y,t) = b_1(x,y,t)\\ v(x,y,t) = b_2(x,y,t)\end{aligned}\right\}, \quad (x,y) \in \partial\Omega, t > 0, \tag{10}$$

where $\Omega = \{(x,y), a \le x \le b, c \le y \le d\}$ is the computational domain and $\partial\Omega$ is the domain's boundary, u and v are the velocity components to be determined. Here, the parameter $\mu_0 = \dfrac{1}{Re}$, where Re is the Reynolds number, μ_1 are vicosity coefficients. From (9)-(10), functions a_1, a_2, b_1 and b_2 are known functions.

In this work, we also focus on the numerical solutions of (7)-(8) based on the finite difference method. We also apply the Hopf-Cole transformation to find analytical solutions for the case $\mu_1 = 0$, and then compare the obtained analytical solutions with the numerical solutions to investigate the efficiency of the numerical schemes and the behavior of the modified Burgers' equations.

II. Methods

In this section, we present two methods to solve (7)-(10). The first method is the Hopf-Cole transformation method, which is used to find analytical solutions. The second method is the finite difference method to find the related numerical solutions.

A. The Hopf-Cole Transformation Method

The Hopf-Cole transformation was named after Eberhard Hopf and Julian D. Cole [1]. Consider the Burgers' equation system

$$\begin{aligned} u_t + uu_x + vu_y &= \mu_0 (u_{xx} + u_{yy}), \quad (11)\\ v_t + uv_x + vv_y &= \mu_0 (v_{xx} + v_{yy}). \quad (12)\end{aligned}$$

In this problems, we assume that $\mu_0 = \dfrac{1}{Re}$, where Re is the Reynolds number.

The analytical solutions of the Burgers' equation system (11)-(12) with various sets of the initial and boundary conditions (9)-(10) using the Hopf-Cole transformation are followings.

1. Linearization of the Burgers' equations by replacing function $\phi(x,y,t)$ to $u(x,y,t)$ and $v(x,y,t)$ in the following way:

$$u = -2\mu_0 \frac{\phi_x}{\phi}, \tag{13}$$

$$v = -2\mu_0 \frac{\phi_y}{\phi}. \tag{14}$$

For simplicity in calculation, let

$$u = f_1(\phi), \tag{15}$$

$$v = f_2(\phi). \tag{16}$$

2. The derivatives of u and v with respect to t, x, and y are found and substituted back into (11) and (12) to obtain that

$$\begin{aligned} &f_1'(\phi)\phi_t + f_1(\phi)f_1'(\phi)\phi_x + f_2(\phi)f_1'(\phi)\phi_y\\ &= \mu_0 \left(f_1''(\phi)\phi_x^2 + f_1'(\phi)\phi_{xx} + f_1''(\phi)\phi_y^2 + f_1'(\phi)\phi_{yy}\right) \end{aligned} \tag{17}$$

and

$$\begin{aligned} &f_2'(\phi)\phi_t + f_1(\phi)f_2'(\phi)\phi_x + f_2(\phi)f_2'(\phi)\phi_y\\ &= \mu_0 \left(f_2''(\phi)\phi_x^2 + f_2'(\phi)\phi_{xx} + f_2''(\phi)\phi_y^2 + f_2'(\phi)\phi_{yy}\right). \end{aligned} \tag{18}$$

We assume that ϕ is bounded, and therefore it implies that $f_1'(\phi)$ and $f_2'(\phi)$ are all nonzero functions. We can see that equations (17) and (18) can be reduced to the heat equation (19) by dividing $f_1'(\phi)$ and $f_2'(\phi)$, respectively.

$$\phi_t = \mu_0 (\phi_{xx} + \phi_{yy}). \tag{19}$$

3. Equation (19) is linear and it can be solved by separation of variables, which the solution ϕ is transformed back to the original solutions of u and v using (13) and (14), respectively. We seek a general solution of (19) in the form [11],

$$\phi(x,y,t) = a + bx + cy + dxy + X(x)Y(y)T(t), \tag{20}$$

which is the sum of the bilinear solution $a + bx + cy + dxy$ and the separable solution $X(x)Y(y)T(t)$.

The bilinear solution is added as a stabilizer while the separable solution is defined as

$$X(x)Y(y)T(t) = W(x,y)T(t). \tag{21}$$

So, (20) becomes

$$\phi(x,y,t) = a + bx + cy + dxy + W(x,y)T(t). \tag{22}$$

Substitute (22) into (19), then we obtain that

$$WT' = \mu_0 \left(W_{xx}T + W_{yy}T \right). \tag{23}$$

For simplicity, (23) can also be written as

$$\frac{1}{\mu_0} \left(WT' \right) = \left(\Delta W \right) T. \tag{24}$$

Equation (24) can be rearranged as

$$\frac{1}{\mu_0} \frac{T'}{T} = \frac{\Delta W}{W} = -\alpha^2, \tag{25}$$

where α^2 is a separation constant. The negative sign is used because a decaying function of time is anticipated. Therefore, (25) can be splited into two equations as

$$T' + \alpha^2 \mu_0 T = 0, \tag{26}$$

and

$$\Delta W + \alpha^2 W = 0, \tag{27}$$

where Δ is Laplacian operator. By solving (26), we have

$$T(t) = Ae^{-\alpha^2 \mu_0 t}. \tag{28}$$

Equation (27) is solved by the separation of variable method. Let $W(x, y) = X(x)Y(y)$, then (27) becomes

$$X''Y + XY'' + \alpha^2 XY = 0. \tag{29}$$

After rearrangement, we have

$$\frac{X''}{X} = -\frac{Y''}{Y} - \alpha^2 = -\beta^2, \tag{30}$$

where β^2 is a separation constant. By splitting the expression (30), two equations are obtained in the form

$$X'' + \beta^2 X = 0, \tag{31}$$

$$Y'' + \left(\alpha^2 - \beta^2 \right) Y = 0. \tag{32}$$

The general solutions of (31) and (32) are given by

$$X(x) = B\sin(\beta x) + C\cos(\beta x), \tag{33}$$

and

$$Y(y) = D\sin(\gamma y) + E\cos(\gamma y), \tag{34}$$

respectively. Here $\gamma^2 = \left(\alpha^2 - \beta^2 \right)$. Substitute the solutions into the general solution (20), it yields that

$$\begin{aligned} \phi(x, y, t) &= a + bx + cy + dxy \\ &+ (B \sin \beta x + C \cos \beta x)(D \sin \gamma y + E \cos \gamma y) Ae^{-\alpha^2 \mu_0 t}. \end{aligned} \tag{35}$$

Then we have the partial derivative of ϕ with respect to x and y as

$$\begin{aligned} \phi_x &= b + dy \\ &+ \beta(B \cos \beta x - C \sin \beta x)(D \sin \gamma y + E \cos \gamma y) Ae^{-\alpha^2 \mu_0 t}, \end{aligned} \tag{36}$$

$$\begin{aligned} \phi_y &= c + dx \\ &+ \gamma(B \sin \beta x + C \cos \beta x)(D \cos \gamma y - E \sin \gamma y) Ae^{-\alpha^2 \mu_0 t}. \end{aligned} \tag{37}$$

Substitute (35), (36) and (37) into the original solutions $u(x, y, t)$ and $v(x, y, t)$ in (13) and (14), then we obtain the analytical solutions of the coupled Burgers' equations.

B. The Finite Difference Method

The finite difference methods can be used to convert ordinary differential equations (ODEs) or partial differential equations (PDEs) into a system of algebraic equations that can be solved by matrix algebra methods. For example, in the forward-time central-space scheme (FTCS), the time derivatives are discretized in time by the forward Euler scheme, and the space by the second-order central-difference scheme. For the time domain $[0, T]$, the discrete time points are given by $\{0, t_1, t_2, \ldots, T\}$, with $t_{n+1} = t_n + \tau$, where τ is a time-step. Similarly, for each space dimension $[a, b] \times [c, d]$, the discrete mesh points are given by $\{a = x_1, x_2, x_3, \ldots, x_M = b\}$, $\{c = y_1, y_2, y_3, \ldots, y_n = d\}$, with $x_{i+1} = x_i + h$, $y_{j+1} = y_j + h$, where h is the space-step. Let $u_{i,j}^n = u(x_i, y_j, t_n)$ and $v_{i,j}^n = v(x_i, y_j, t_n)$, the modified coupled Burgers' equations in FTCS scheme are as follows:

$$\begin{aligned} \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\tau} = & -u_{i,j}^n \left(\frac{u_{i+1,j}^n - u_{i-1,j}^n}{2h} \right) \\ & -v_{i,j}^n \left(\frac{u_{i,j+1}^n - u_{i,j-1}^n}{2h} \right) \\ & +(\mu_0 + \mu_1 u_{i,j}^n) \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} \right) \\ & +(\mu_0 + \mu_1 u_{i,j}^n) \left(\frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2} \right), \end{aligned} \tag{38}$$

$$\begin{aligned} \frac{v_{i,j}^{n+1} - v_{i,j}^n}{\tau} = & -u_{i,j}^n \left(\frac{v_{i+1,j}^n - v_{i-1,j}^n}{2h} \right) \\ & -v_{i,j}^n \left(\frac{v_{i,j+1}^n - v_{i,j-1}^n}{2h} \right) \\ & +(\mu_0 + \mu_1 v_{i,j}^n) \left(\frac{v_{i+1,j}^n - 2v_{i,j}^n + v_{i-1,j}^n}{h^2} \right) \\ & +(\mu_0 + \mu_1 v_{i,j}^n) \left(\frac{v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n}{h^2} \right). \end{aligned} \tag{39}$$

Equation (38)-(39) can be rearranged as

$$\begin{aligned} u_{i,j}^{n+1} &= u_{i,j}^n - \tau \left(u_{i,j}^n \cdot \text{dudx} + v_{i,j}^n \cdot \text{dudy} \right) \\ &+ \tau(\mu_0 + \mu_1 u_{i,j}^n) \left(\text{d2udx} + \text{d2udy} \right) \end{aligned} \tag{40}$$

and

$$\begin{aligned} v_{i,j}^{n+1} &= v_{i,j}^n - \tau \left(v_{i,j}^n \cdot \text{dvdx} + v_{i,j}^n \cdot \text{dvdy} \right) \\ &+ \tau(\mu_0 + \mu_1 v_{i,j}^n) \left(\text{d2vdx} + \text{d2vdy} \right), \end{aligned} \tag{41}$$

where

$$\begin{aligned} \text{dudx} &= \frac{u_{i+1,j}^n - u_{i-1,j}^n}{2h}, \\ \text{dudy} &= \frac{u_{i,j+1}^n - u_{i,j-1}^n}{2h}, \\ \text{dvdx} &= \frac{v_{i+1,j}^n - v_{i-1,j}^n}{2h}, \\ \text{dvdy} &= \frac{v_{i,j+1}^n - v_{i,j-1}^n}{2h}, \\ \text{d2udx} &= \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2}, \\ \text{d2udy} &= \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2}, \\ \text{d2vdx} &= \frac{v_{i+1,j}^n - 2v_{i,j}^n + v_{i-1,j}^n}{h^2}, \\ \text{d2vdy} &= \frac{v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n}{h^2}. \end{aligned}$$

In the case that $\mu_1 = 0$, the scheme (38) - (39) can be proved to be conditionally stable for $\left(\frac{\tau}{h}\right)^2 \leq 2\left(\frac{\tau}{\mu_0 h^2}\right) \leq 1$, see [1] for more details.

The computational domain for the finite difference scheme is staggered grid as shown in Fig. 1.

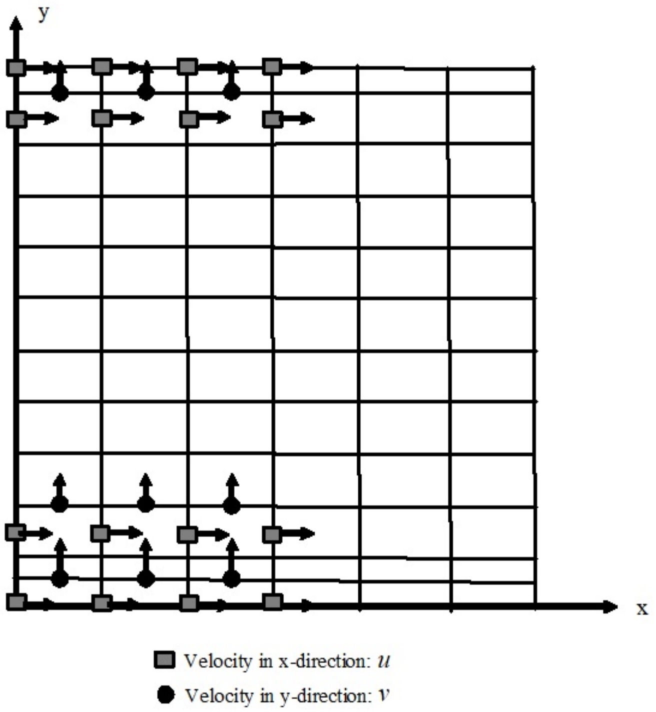


Fig. 1: Computational domain

III. Results

The analytical solutions and the numerical solutions of the modified coupled Burgers' equation using the Hopf-Cole transformation and the finite difference method are presented in this section. We consider the solutions of the modified coupled Burgers' equation in two cases as follows:

$$\text{Case } 1: \quad \mu_0 = \frac{1}{Re}, \quad \mu_1 = 0,$$

and

$$\text{Case } 2: \quad \mu_0 = \frac{1}{Re}, \quad 5 \leq \mu_1 \leq 30.$$

Case 1: $\mu_0 = \dfrac{1}{Re}, \mu_1 = 0$ with $Re = 500, 10000, 50000$.

Consider the modified coupled Burgers' equations (7) and (8), where $\mu_0 = \frac{1}{Re}$ and $\mu_1 = 0$. We obtain the two-dimentional coupled Burgers' equations of the form

$$u_t + uu_x + vu_y = \frac{1}{Re}\left(u_{xx} + u_{yy}\right), \quad (42)$$

$$v_t + uv_x + vv_y = \frac{1}{Re}\left(v_{xx} + v_{yy}\right), \quad (43)$$

where $\Omega = \{(x, y), 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

(1.1) The analytical solutions and the numerical solutions of the coupled Burgers' equations (42) and (43) for $Re = 500$ and the parameters [11] $a = 100, b = 0, c = 0, d = 1, A = 1, B = 1, C = 1, D = 1, E = 0, \beta = \pi,\ \gamma = \pi$, with the initial conditions (44) and the boundary conditions (45)-(46) are investigated.

The initial conditions for $0 \leq x \leq 1$ and $0 \leq y \leq 1$ are

$$\begin{aligned} u(x, y, 0) &= -\frac{y + (\pi\cos(\pi x) - \pi\sin(\pi x))\sin(\pi y)}{25000 + 250xy + (250(\sin(\pi x) + \cos(\pi x))\sin(\pi y)}, \\ v(x, y, 0) &= -\frac{x + (\sin(\pi x) + \cos(\pi x))\pi\cos(\pi y))}{250xy + 25000 + (250(\sin(\pi x) + \cos(\pi x))\sin(\pi y)}. \end{aligned} \quad (44)$$

The boundary conditions for $0 \leq y \leq 1$ where $t > 0$ are

$$\begin{aligned} u(0, y, t) &= \frac{-\pi\sin(\pi y)e^{\frac{-\pi^2 t}{250}} - y}{(25000 + 250\sin(\pi y)e^{\frac{-\pi^2 t}{250}}}, \\ v(0, y, t) &= \frac{-\pi\cos(\pi y)e^{\frac{-\pi^2 t}{250}}}{25000 + 250\sin(\pi y)e^{\frac{-\pi^2 t}{250}}}, \\ u(1, y, t) &= \frac{-\pi\sin(\pi y)(\cos(\pi) - \sin(\pi))e^{\frac{-\pi^2 t}{250}} - y}{25000 + 250y + (250(\sin(\pi) + \cos(\pi)))\sin(\pi y)e^{\frac{-\pi^2 t}{250}}}, \\ v(1, y, t) &= \frac{\pi\cos(\pi y)e^{\frac{-\pi^2 t}{250}} - 1}{250y + 25000 - 250\sin(\pi y)e^{\frac{-\pi^2 t}{250}}}. \end{aligned} \quad (45)$$

The boundary conditions for $0 \leq x \leq 1$ where $t > 0$ are

$$\begin{aligned} u(x, 0, t) &= 0, \\ v(x, 0, t) &= -\frac{x}{25000} - \frac{\sin(\pi x) + \cos(\pi x)\pi e^{\frac{-\pi^2 t}{250}}}{25000}, \\ u(x, 1, t) &= \frac{-1}{(25000 + 250x)}, \\ v(x, 1, t) &= \frac{(-x + (\sin(\pi x) + \cos(\pi x))\pi e^{\frac{-\pi^2 t}{250}}}{25000 + 250x}. \end{aligned} \quad (46)$$

The analytical solution from the Hopf-Cole transformation method for case 1 (1.1) is

$$u = \left(-\frac{1}{250}\right)\left(\frac{y + (\cos\pi x - \sin\pi x)\sin\pi y e^{\left(\frac{-\pi^2}{250}\right)t}}{100 + xy + (\sin\pi x + \cos\pi x)\sin\pi y e^{\left(\frac{-\pi^2}{250}\right)t}}\right), \quad (47)$$

$$v = \left(-\frac{1}{250}\right)\left(\frac{x + \pi(\cos\pi x + \sin\pi x)\cos\pi y e^{\left(\frac{-\pi^2}{250}\right)t}}{100 + xy + (\sin\pi x + \cos\pi x)\sin\pi y e^{\left(\frac{-\pi^2}{250}\right)t}}\right). \quad (48)$$

Fig. 2a-3a show the analytical solutions for $Re = 500$ at $t = 1$, with 10×10 grids size. The graphs of the numerical solutions by the finite difference method with the FTCS scheme of u and v in Eqs. (40) and (41), where $\Delta x = \Delta y = 0.1$, and $\Delta t = 0.001$ at $t = 1$ are shown in Fig. 2b-3b. The plots of absolute error between the analytical and the numerical solutions for this case are shown in Fig. 2c-3c. We can see that the maximum of absolute error of u and v are 10^{-8} and 10^{-5}, respectively.

(1.2) The analytical solutions and the numerical solutions of the coupled Burgers' equation (42) and (43) for $Re = 10000$ and the parameters [11] $a = 0, b = 5, c = 10, d = 0, A = 1, B = 0, C = 1, D = 0, E = 1, \beta = 0, \gamma = 2\pi$ with the initial conditions (49), and boundary conditions (50)-(51) are considered.

The initial conditions for $0 \leq x \leq 1$ and $0 \leq y \leq 1$ are

$$u(x,y,0) = -\frac{1}{1000(5x+10y+\cos(2\pi y))},$$
$$v(x,y,0) = \frac{\pi\sin(2\pi y)-5}{12500x+25000y+2500\cos(2\pi y)}. \quad (49)$$

The boundary conditions for $0 \leq y \leq 1$ where $t > 0$ are

$$\begin{aligned} u(0,y,t) &= -\frac{1}{10000y+1000\cos(2\pi y)e^{-(1/2500)\pi^2 t}}, \\ v(0,y,t) &= -\frac{10-2\pi\sin(2\pi y)^{e}-(1/2500)\pi^2}{50000y+5000\cos(2\pi y)e^{-(1/2500)\pi^2 t}}, \\ u(1,y,t) &= -\frac{1}{5000+10000y+1000\cos(2\pi y)e^{-(1/2500)\pi^2 t}}, \\ v(1,y,t) &= -\frac{10-2\pi\sin(2\pi y)e^{(-(1/2500)\pi^2 t)}}{25000+50000y+5000\cos(2\pi y)e^{-(1/2500)\pi^2 t}}. \quad (50) \end{aligned}$$

The boundary conditions for $0 \leq x \leq 1$ where $t > 0$ are

$$\begin{aligned} u(x,0,t) &= -\frac{1}{5000x+1000e^{-(1/2500)\pi^2 t}}, \\ v(x,0,t) &= -\frac{1}{2500x+500e^{-(1/2500)\pi^2}}, \\ u(x,1,t) &= -\frac{1}{5000x+10000+1000\cos(2\pi)e^{-(1/2500)\pi^2 t}}, \\ v(x,1,t) &= -\frac{1}{500(5x+10+e^{-(1/2500)\pi^2 t}}. \quad (51) \end{aligned}$$

The analytical solution obtained by the Hopf-Cole transformation method for case 1 (1.2) is

$$u = \left(-\frac{1}{5000}\right)\left(\frac{1}{x+2y+\frac{1}{5}\cos 2\pi y e^{\left(\frac{-\pi^2}{2500}\right)t}}\right) \quad (52)$$

$$v = \left(-\frac{1}{5000}\right)\left(\frac{10-2\pi\sin 2\pi y e^{\left(\frac{-\pi^2}{2500}\right)t}}{5x+10y+\cos 2\pi y\cos 2\pi y e^{\left(\frac{-\pi^2}{2500}\right)t}}\right) \quad (53)$$

The graphs of the analytical solutions for case 1 (1.2), with $t = 1$ and 100×100 grids size, are shown in Fig. 4a-5a. The numerical solutions by finite-differnce method with the FTCS scheme of u and v in Eqs. (40) and (41), where $\Delta x = \Delta y = 0.01$ and $\Delta t = 0.0001$ are shown in Fig. 4b-5b. Moreover, Fig. 4c-5c show the graphs of absolute errors between the analytical and the numerical solutions for case 1 (1.2). We can see that the absolute error of u and v are 10^{-9} and 10^{-4}, respectively.

(1.3) The analytical solutions and the numerical solutions of the coupled Burgers' equations (42) and (43) for $Re = 50000$ and the parameters [11] $a = 10, b = 50, c = 0, d = 0, A = 1, B = 0, C = 1, D = 1, E = 0, \beta = 2\pi, \gamma = 2\pi$ with the initial conditions (54) and the boundary conditions (55)-(56) are presented.

The initial conditions for $0 \leq x \leq 1$ and $0 \leq y \leq 1$ are

$$u(x,y,0) = \frac{\pi\sin(2\pi x)sin(2\pi y)-25}{12500\cos(2\pi x)\sin(2\pi y)+625000x+125000},$$
$$v(x,y,0) = -\frac{\cos(2\pi x)\pi\cos(2\pi y)}{12500(50x+10+\cos(2\pi x)\sin(2\pi y))}. \quad (54)$$

The boundary conditions for $0 \leq y \leq 1$ where $t > 0$ are

$$\begin{aligned} u(0,y,t) &= -\frac{1}{500\sin(2\pi y)e^{-(1/6250)\pi^2 t}+5000}, \\ v(0,y,t) &= -\frac{\pi\cos(2\pi y)e^{-(1/6250)\pi^2 t}}{125000+12500\sin(2\pi y)e^{-(1/6250)\pi^2 t}}, \\ u(1,y,t) &= \frac{1}{500\sin(2\pi y))e^{-(1/6250)\pi^2 t}+30000}, \\ v(1,y,t) &= -\frac{\pi\cos(2\pi y)e^{-(1/6250)\pi^2}}{750000+12500\sin(2\pi y)e^{-(1/6250)\pi^2 t}}. \quad (55) \end{aligned}$$

The boundary conditions for $0 \leq x \leq 1$ where $t > 0$ are

$$\begin{aligned} u(x,0,t) &= -\frac{1}{25000x+5000}, \\ v(x,0,t) &= -\frac{\cos(2\pi x)\pi e^{-(1/6250)\pi^2 t}}{125000(5x+1)}, \\ u(x,1,t) &= -\frac{1}{25000x+5000}, \\ v(x,1,t) &= -\frac{\cos(2\pi x)\pi e^{-(1/6250)\pi^2 t}}{125000(5x+1)}. \quad (56) \end{aligned}$$

The analytical solution obtained by the Hopf-Cole transformation method for case 1 (1.3) is

$$u = \left(-\frac{1}{25000}\right)\left(\frac{50-2\pi\sin 2\pi x\sin 2\pi x y e^{\left(\frac{-\pi^2}{6250}\right)t}}{10+50x+\cos 2\pi x\sin 2\pi y e^{\left(\frac{-\pi^2}{6250}\right)t}}\right), \quad (57)$$

$$v = \left(-\frac{1}{25000}\right)\left(\frac{2\pi\sin 2\pi x\cos 2\pi y e^{\left(\frac{-\pi^2}{6250}\right)t}}{10+50x+\cos 2\pi x\sin 2\pi y e^{\left(\frac{-\pi^2}{6250}\right)t}}\right). \quad (58)$$

Then, we obtain analytical solutions for case 1 (1.3) with $Re = 50000$ and 100×100 grids size. The graphs of the analytical solutions are shown in Fig. 6a-7a. The numerical solutions by the finite difference method with the FTCS scheme of u and v in Eqs. (40) and (41) where $\Delta x = \Delta y = 0.01$ and $\Delta t = 0.00001$ are presented in Fig. 6b-7b. The plots of absolute errors between the analytical and the numerical solutions for case 1 (1.3) are shown in Fig. 6c-7c. We can see that the absolute error of u and v are 10^{-11} and 10^{-7}, respectively.

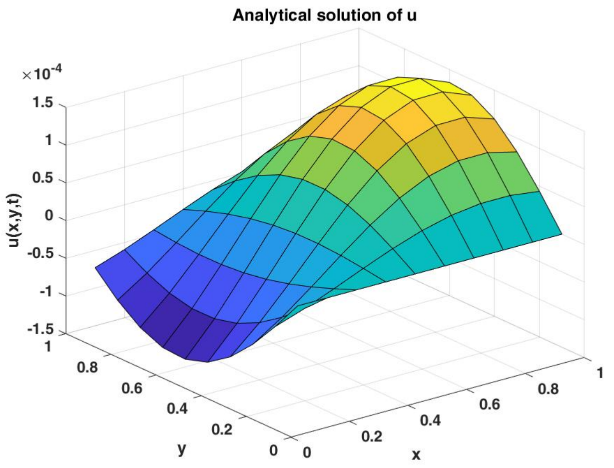
In addition, numerical solutions of u and v at $x = 0.5, y = 0.5$ for $0 \leq t \leq 1$ are computed. The results are shown in Fig. 8 with $Re = 500, 10000$ and 50000.

For $Re = 500$ it can be seen that the value of u increases rapidly at the beginning and then decreases slowly and it is converges to 5.7176×10^{-5}. The value of v fastly decrease at the beginning and then slightly increases, and it converges to -3.9163×10^{-5}. For $Re = 10000$ and 50000, we obtained that the values of u and v decrease rapidly during the first 0.1 seconds, and then the values of u and v remains contant.

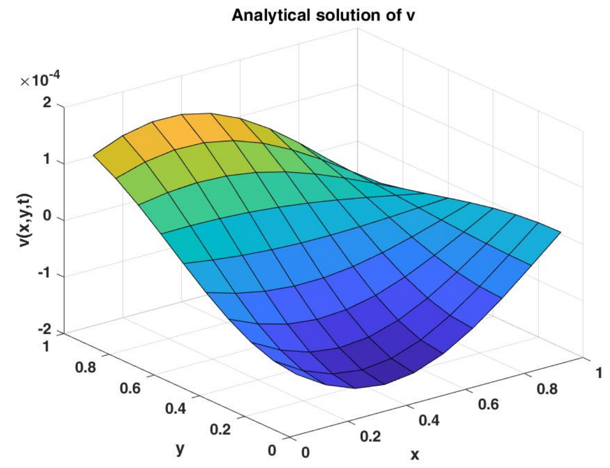
Case 2: $\mu_0 = \frac{1}{Re}$, $5 \leq \mu_1 \leq 30$

The modified coupled Burgers' equations (7) and (8) with fixed $\mu_0 = \frac{1}{Re}$ and various $5 \leq \mu_1 \leq 30$ are considered which present in equations (59) and (60).

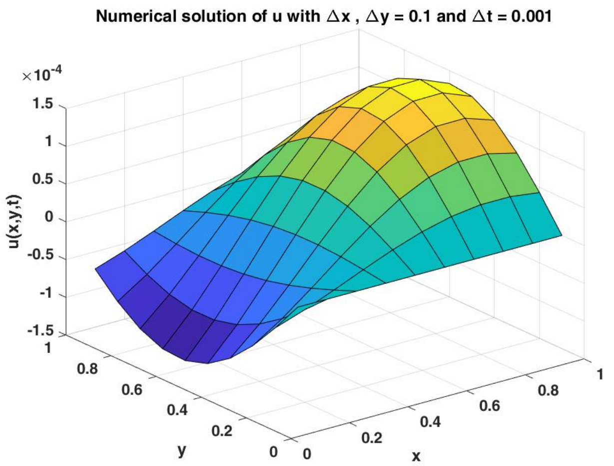
$$u_t + uu_x + vu_y = \left(\frac{1}{500} + \mu_1 u\right)(u_{xx} + u_{yy}), \quad (59)$$
$$v_t + uv_x + vv_y = \left(\frac{1}{500} + \mu_1 v\right)(v_{xx} + v_{yy}). \quad (60)$$



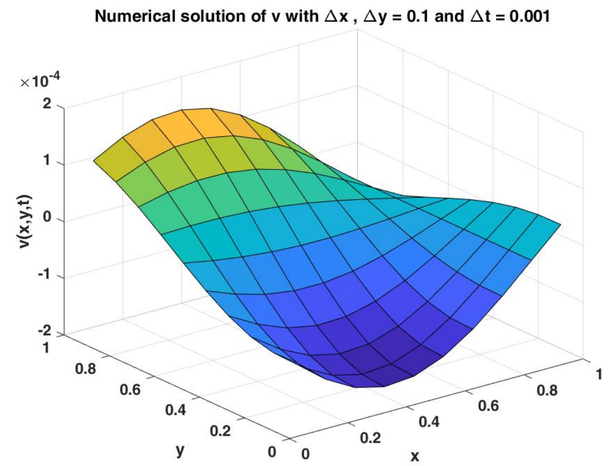
(a) Analytical solution of u



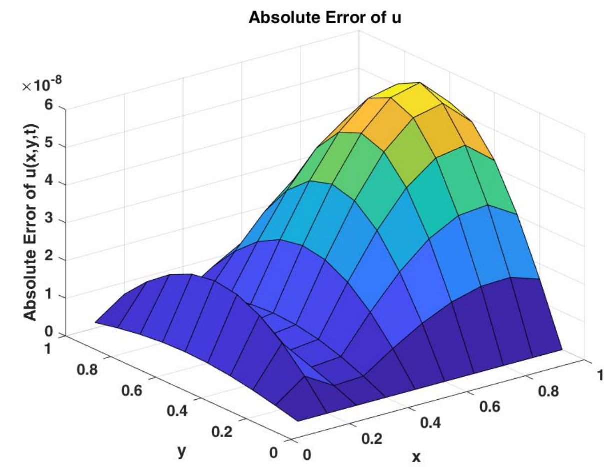
(a) Analytical solution of v



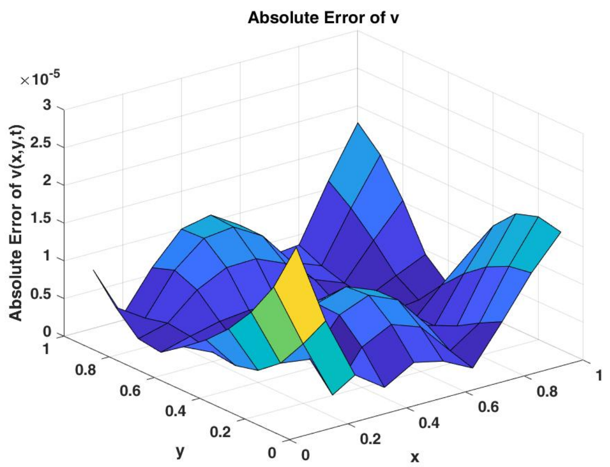
(b) Numerical solution of u



(b) Numerical solution of v



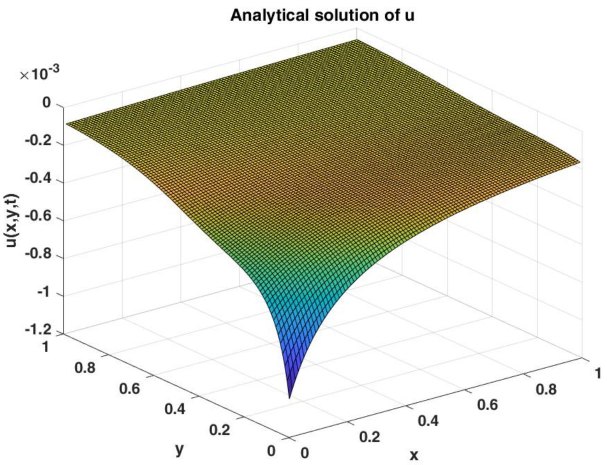
(c) Absolute error of u



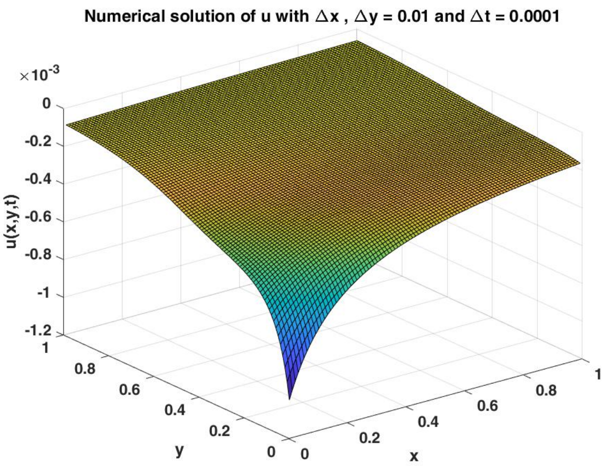
(c) Absolute error of v

Fig. 2: Graph of analytical and numerical solutions with absolute error of u at $t = 1$ for case 1 (1.1)

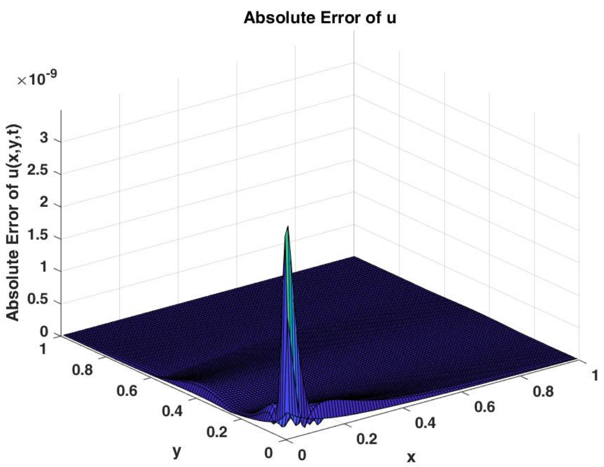
Fig. 3: Graph of analytical and numerical solutions with absolute error of v at $t = 1$ for case 1 (1.1)



(a) Analytical solution of u

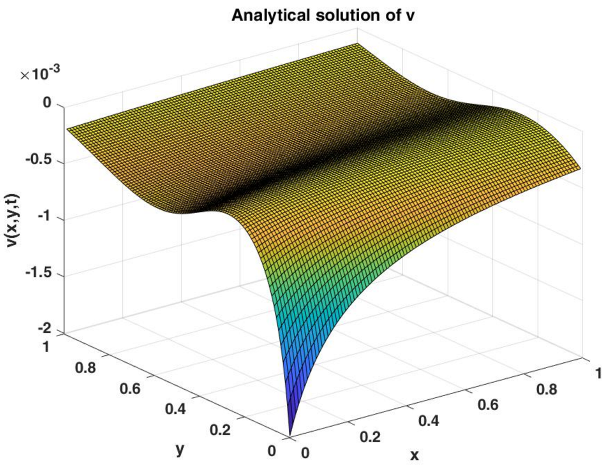


(b) Numerical solution of u

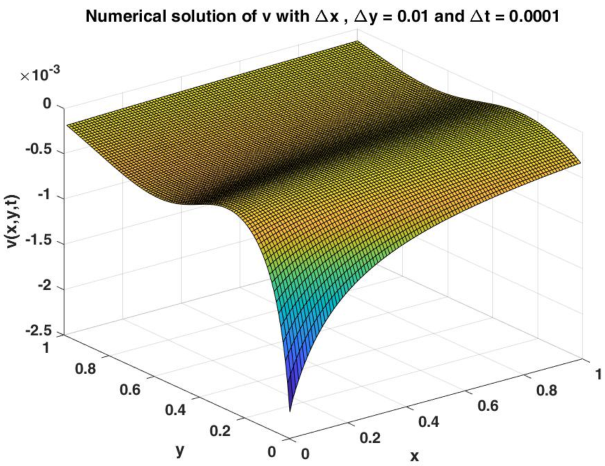


(c) Absolute error of u

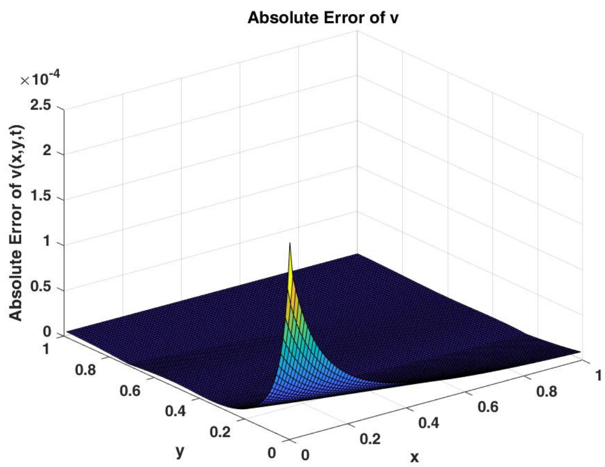
Fig. 4: Graph of analytical and numerical solutions with absolute error of u at $t = 1$ for case 1 (1.2)



(a) Analytical solution of v

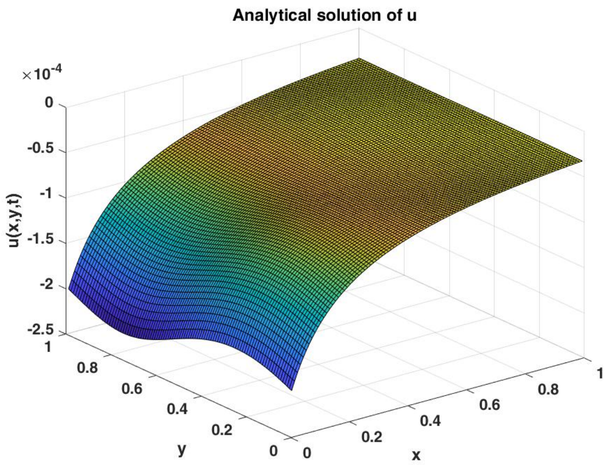


(b) Numerical solution of v

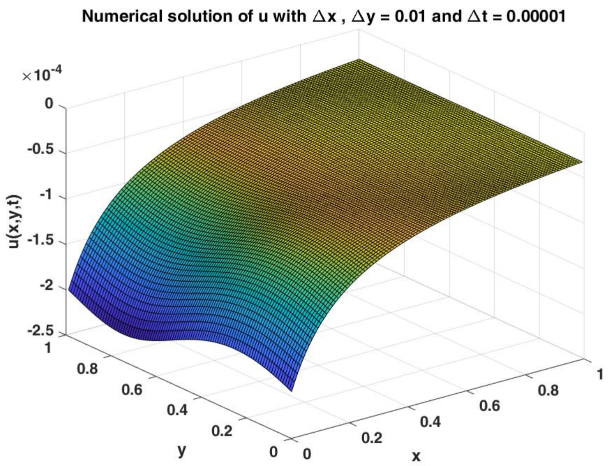


(c) Absolute error of v

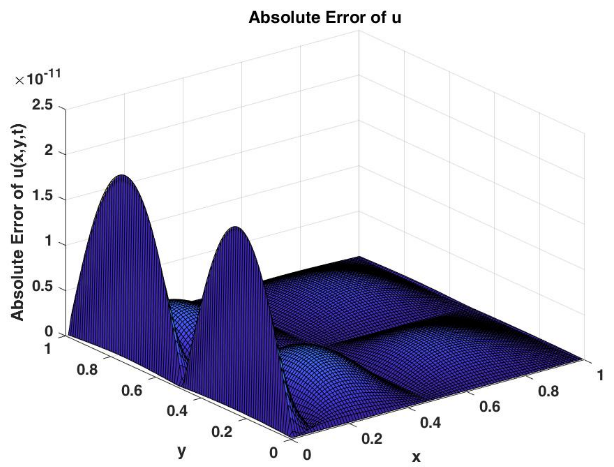
Fig. 5: Graph of analytical and numerical solutions with absolute error of v at $t = 1$ for case 1 (1.2)



(a) Analytical solution of u

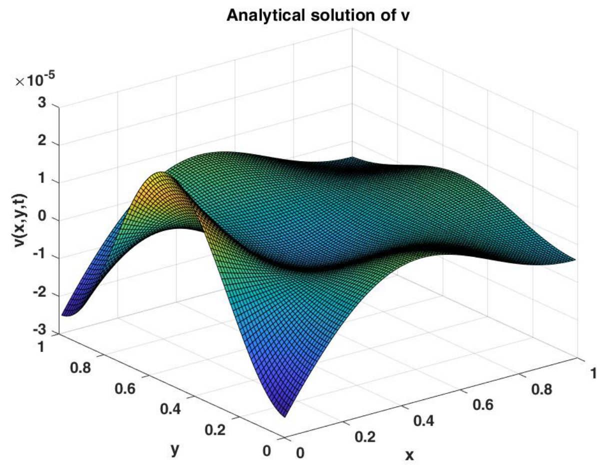


(b) Numerical solution of u

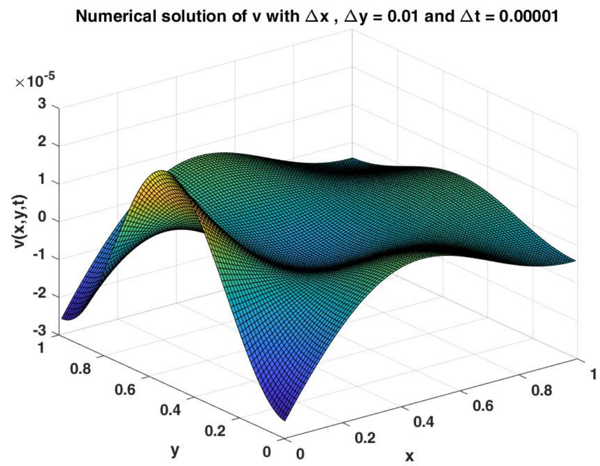


(c) Absolute error of u

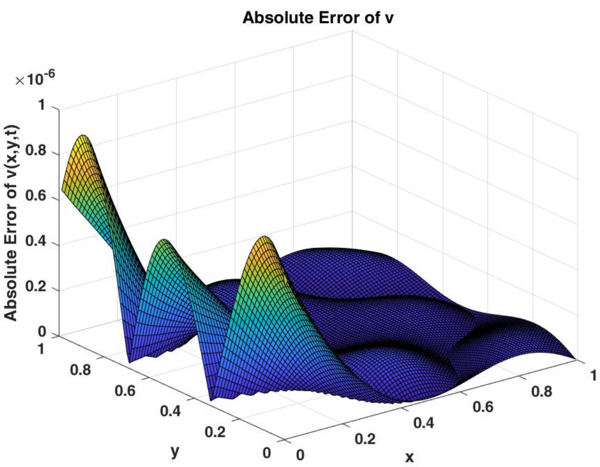
Fig. 6: Graph of analytical and numerical solutions with absolute error of u at $t = 1$ for case 1 (1.3)



(a) Analytical solution of v

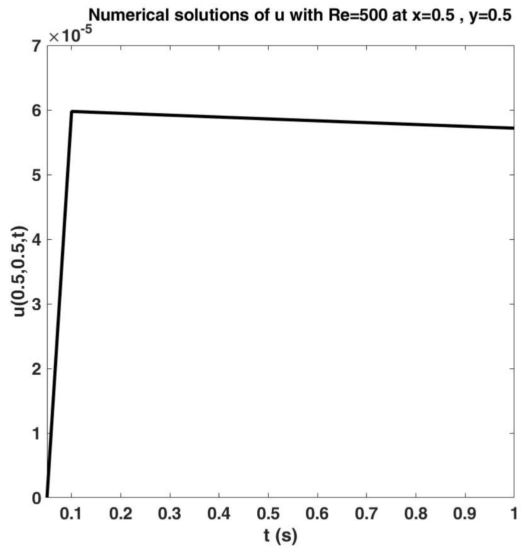


(b) Numerical solution of v

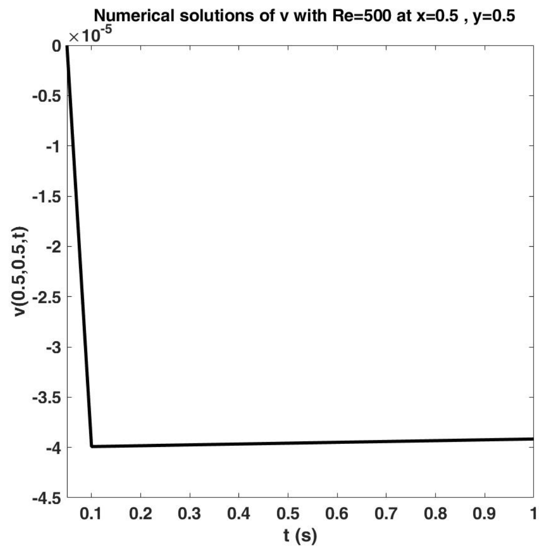


(c) Absolute error of v

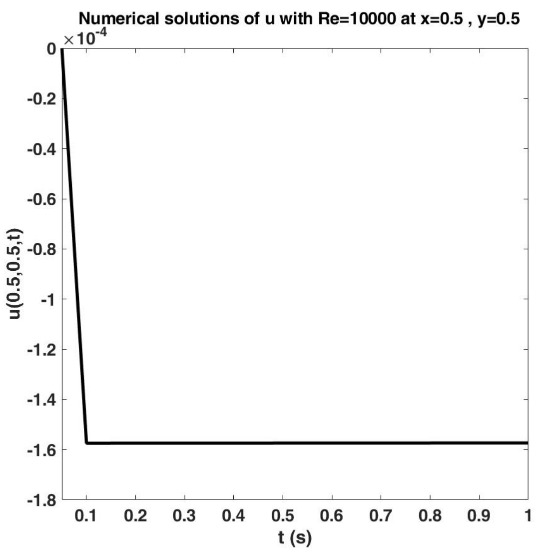
Fig. 7: Graph of analytical and numerical solutions with absolute error of v at $t = 1$ for case 1 (1.3)



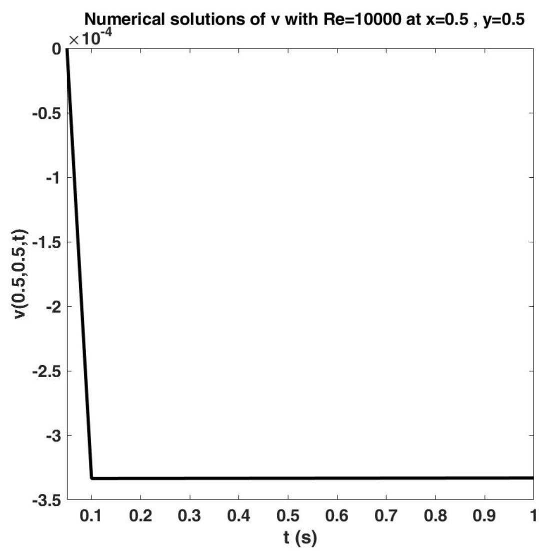
(a) $Re = 500:\ u(0.5, 0.5, t)$ when $0 \leq t \leq 1$



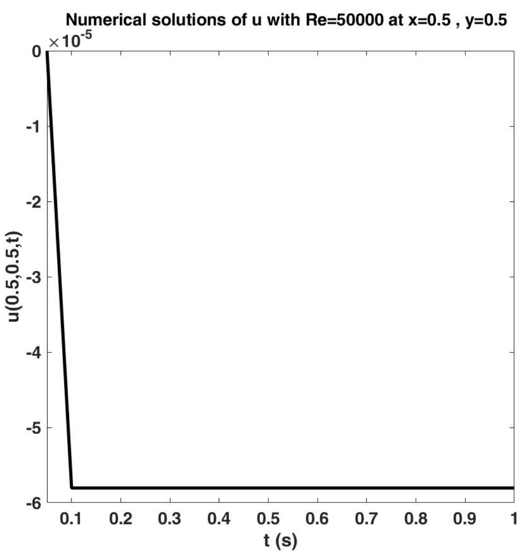
(b) $Re = 500:\ v(0.5, 0.5, t)$ when $0 \leq t \leq 1$



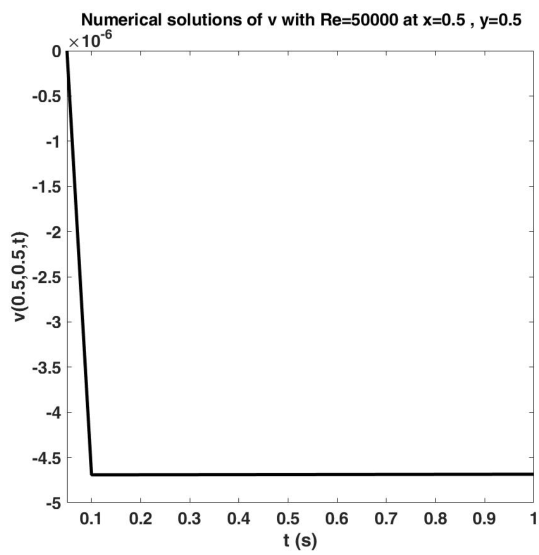
(c) $Re = 10000:\ u(0.5, 0.5, t)$ when $0 \leq t \leq 1$



(d) $Re = 10000:\ v(0.5, 0.5, t)$ when $0 \leq t \leq 1$



(e) $Re = 50000:\ u(0.5, 0.5, t)$ when $0 \leq t \leq 1$



(f) $Re = 50000:\ v(0.5, 0.5, t)$ when $0 \leq t \leq 1$

Fig. 8: Graph of numerical solutions u and v at $x = y = 0.5$ for case 1 (1.1-1.3)

The initial conditions in this case are

$$u(x,y,0) = -\frac{y + (\pi\cos(\pi x) - \pi\sin(\pi x))\sin(\pi y)}{25000 + 250xy + (250(\sin(\pi x) + \cos(\pi x))\sin(\pi y)},$$
$$v(x,y,0) = -\frac{x + (\sin(\pi x) + \cos(\pi x))\pi\cos(\pi y))}{250xy + 25000 + (250(\sin(\pi x) + \cos(\pi x))\sin(\pi y)}.$$
$$\text{for } 0 \le x \le 1, 0 \le y \le 1 \tag{61}$$

The boundary conditions for $0 \le y \le 1$ where $t > 0$ are

$$u(0,y,t) = \frac{-\pi\sin(\pi y)e^{\frac{-\pi^2 t}{250}} - y}{(25000 + 250\sin(\pi y)e^{\frac{-\pi^2 t}{250}}},$$
$$v(0,y,t) = \frac{-\pi\cos(\pi y)e^{\frac{-\pi^2 t}{250}}}{25000 + 250\sin(\pi y)e^{\frac{-\pi^2 t}{250}}},$$
$$u(1,y,t) = \frac{-\pi\sin(\pi y)(\cos(\pi) - \sin(\pi))e^{\frac{-\pi^2 t}{250}} - y}{25000 + 250y + (250(\sin(\pi) + \cos(\pi)))\sin(\pi y)e^{\frac{-\pi^2 t}{250}}},$$
$$v(1,y,t) = \frac{\pi\cos(\pi y)e^{\frac{-\pi^2 t}{250}} - 1}{250y + 25000 - 250\sin(\pi y)e^{\frac{-\pi^2 t}{250}}}. \tag{62}$$

The boundary conditions for $0 \le x \le 1$ where $t > 0$ are

$$u(x,0,t) = 0,$$
$$v(x,0,t) = -\frac{x}{25000} - \frac{\sin(\pi x) + \cos(\pi x)\pi e^{\frac{-\pi^2 t}{250}}}{25000},$$
$$u(x,1,t) = \frac{-1}{(25000 + 250x)},$$
$$v(x,1,t) = \frac{(-x + (\sin(\pi x) + \cos(\pi x))\pi e^{\frac{-\pi^2 t}{250}}}{25000 + 250x}. \tag{63}$$

The numerical simulation of the nonlinear Burgers' equations (59) and (60) with fixed $\mu_0 = 1/Re = 1/500$ and various $5 \le \mu_1 \le 30$ are computed using the finite difference method in the FTCS scheme. The graph of the numerical solutions of u and v at $t = 1$ for $\mu_0 = 1/500$ and various $5 \le \mu_1 \le 30$ are shown in Fig. 9a-9d and Fig. 10a-10d respectively. In addition, we compare the numerical solutions u and v in case that $x = 0.5$, $\mu_0 = 1/500$ and $5 \le \mu_1 \le 30$ at $t = 1$. The two-dimensional plots of this case are shown in Fig. 11a-11b. We can see that the numerical solutions u are slightly decreasing when μ_1 increased, which occur around the middle of the domain of y, while the numerical solutions v have a little different at the initial part of the domain of y. The two-dimensional plots for $y = 0.5$, where $\mu_0 = 1/500$ and $5 \le \mu_1 \le 30$ at $t = 1$ are presented in Fig. 12a-12b. It is shown that the numerical solutions u are slightly decreasing when μ_1 is increasing, which occur at the end of domain x about $0.5 \le x \le 0.9$ while the behavior of numerical solutions v have almost no difference when μ_1 is changed.

In addition, we show a comparison of the numerical solutions u and v vs. time at the center of domain $x = 0.5, y = 0.5$ for the four different values of μ_1 in Fig. 13a-13b. It can be seen that the values of u and v are linearly decrease with time while μ_1 is increasing.

IV. Conclusion

The modified coupled Burgers' equations which include a nonlinear viscous term are investigated. In the first case $\mu_0 = 1/Re$ and $\mu_1 = 0$, the numerical solutions are solved by using the finite difference method in the FTCS scheme. The analytical solutions are obtained by the Hopf-Cole transformation in three different values of Reynolds numbers, $Re = 500, 10000$ and 50000. We found that the numerical solutions of u and v converge to the analytical solutions in all cases are presented in Table I. In the second case, $\mu_0 = 1/500$ and various $5 \le \mu_1 \le 30$, the numerical solutions are obtained by using the finite difference method in explicit FTCS scheme. The two-dimensional and three-dimensional graphs of this case are presented in Fig. 9-12. We found that the behavior of numerical solutions u are slightly decreasing when μ_1 increased in both cases for fixed $x = 0.5$ and $y = 0.5$, while the behavior of numerical solutions v have a little different when μ_1 is changed.

TABLE I: The maximum of absolute errors for Case 1

Re	The maximum of absolute error	
	velocity u	velocity v
500	10^{-8}	10^{-5}
10000	10^{-9}	10^{-4}
50000	10^{-11}	10^{-7}

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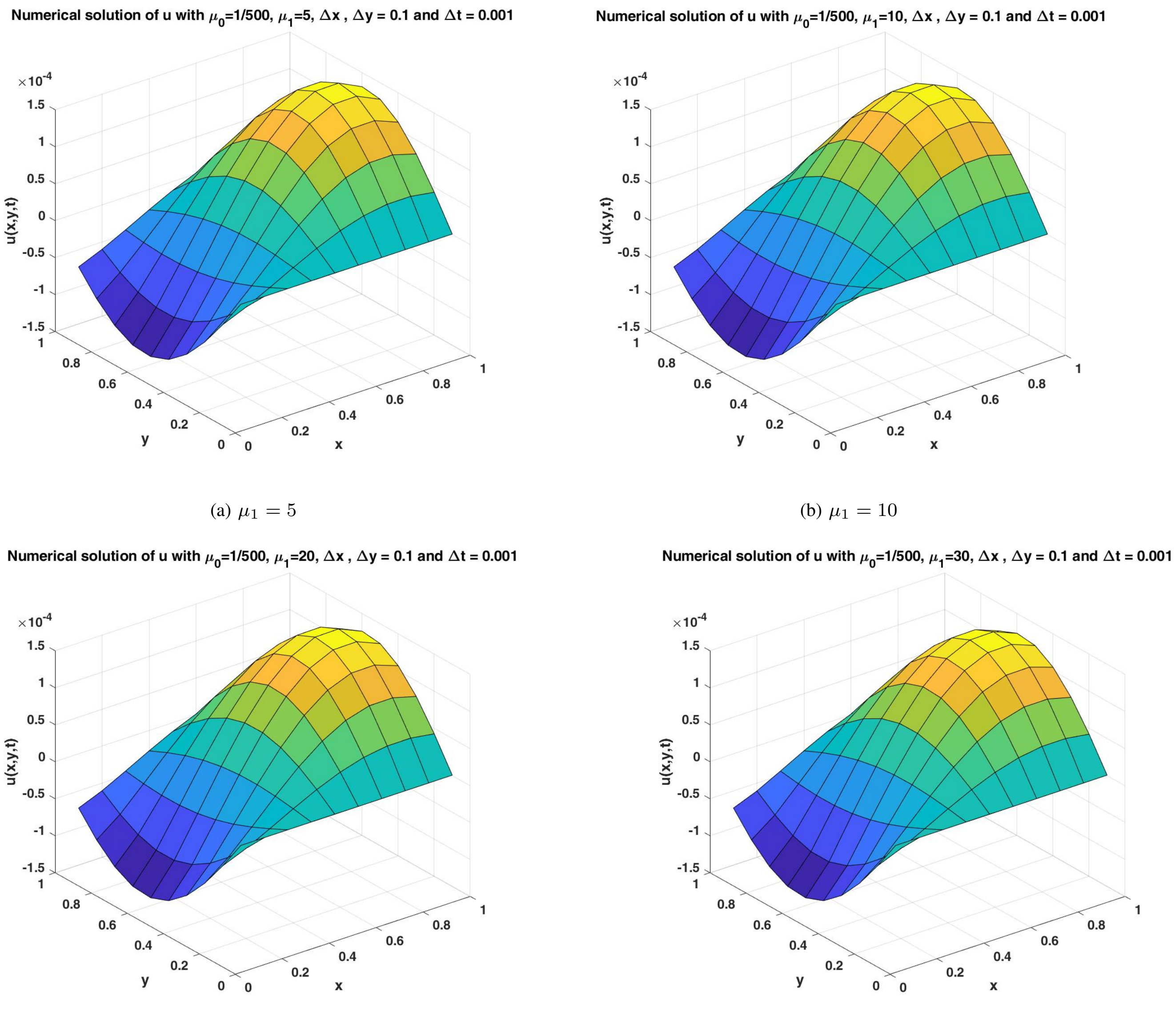


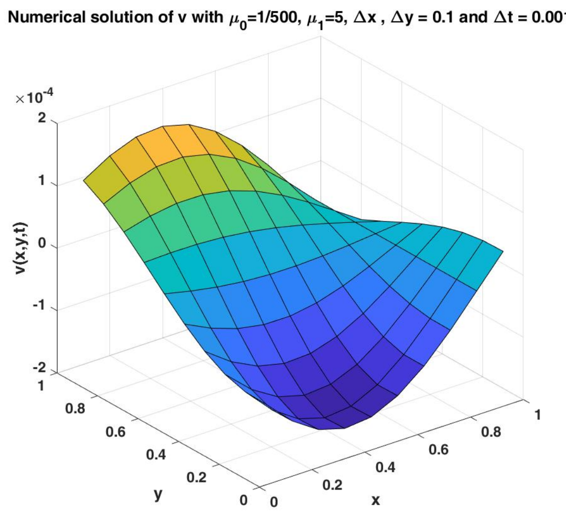
(a) $\mu_1 = 5$

(b) $\mu_1 = 10$

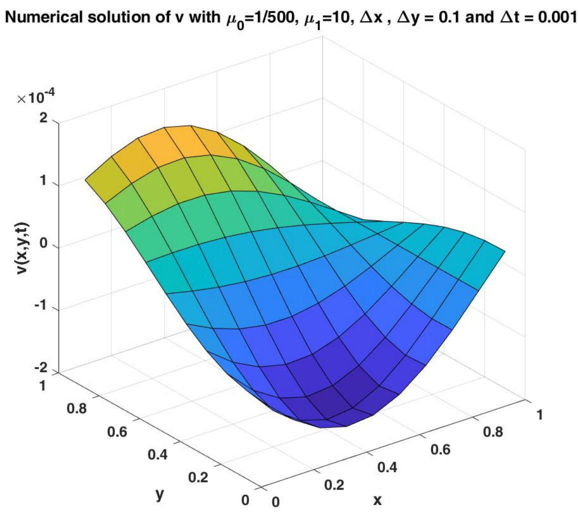
(c) $\mu_1 = 20$

(d) $\mu_1 = 30$

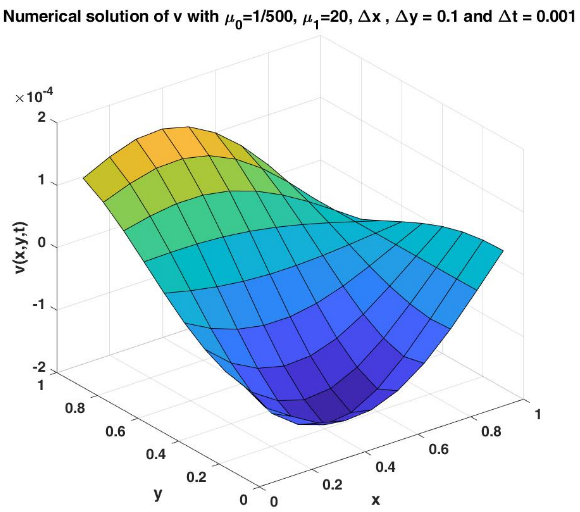
Fig. 9: Graph of numerical solutions of u at $t = 1$ for $\mu_0 = 1/500$ and $5 \leq \mu_1 \leq 30$



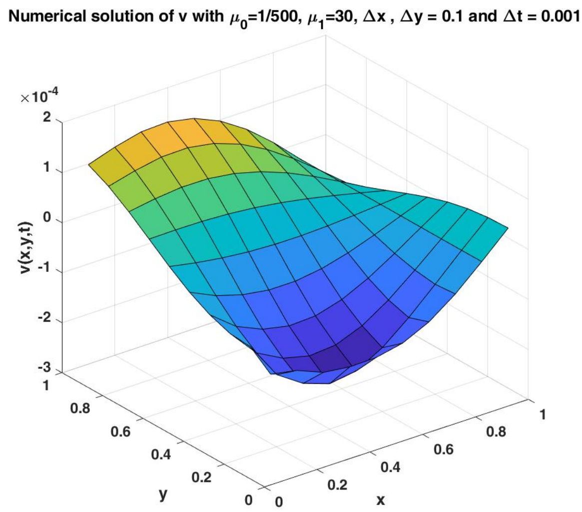
(a) $\mu_1 = 5$



(b) $\mu_1 = 10$

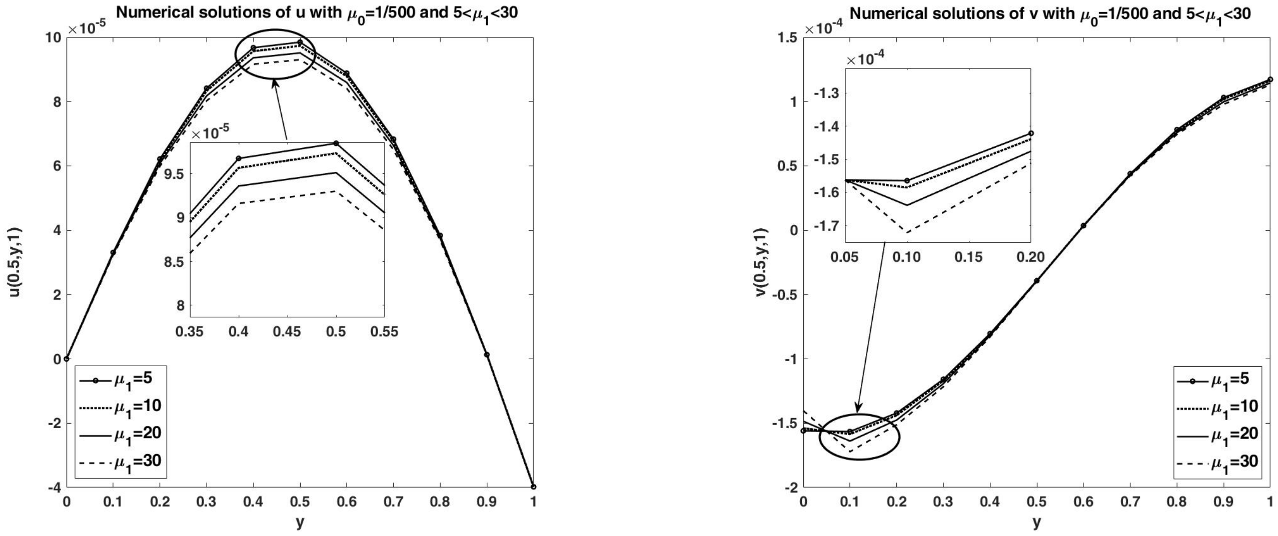


(c) $\mu_1 = 20$



(d) $\mu_1 = 30$

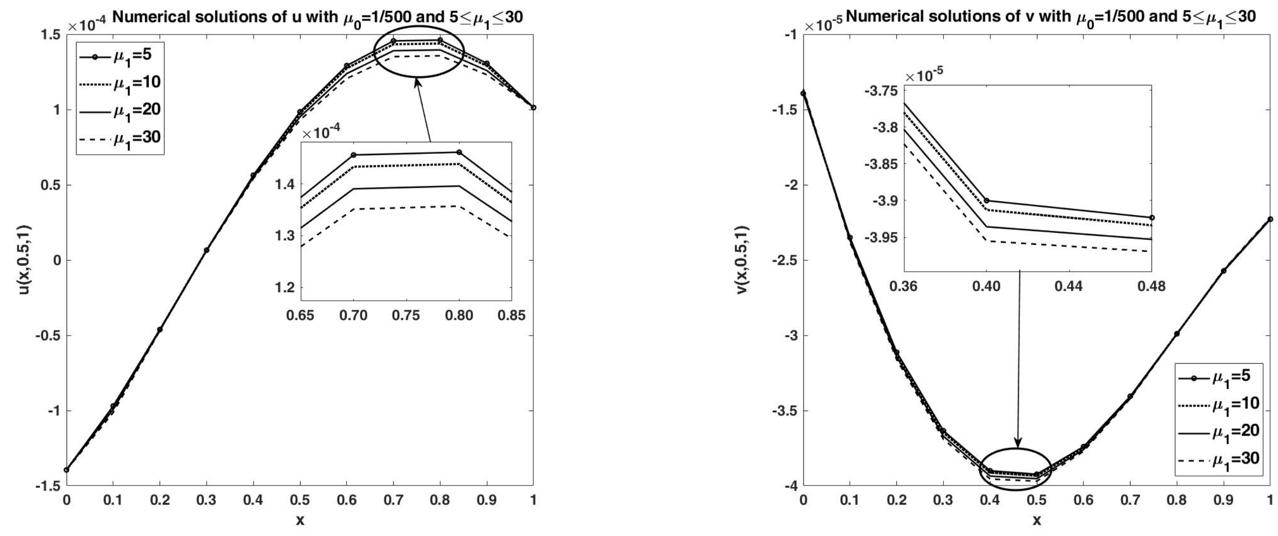
Fig. 10: Graph of numerical solutions of v at $t = 1$ for $\mu_0 = 1/500$ and $5 \leq \mu_1 \leq 30$



(a) velocity in x direction : u

(b) velocity in y direction : v

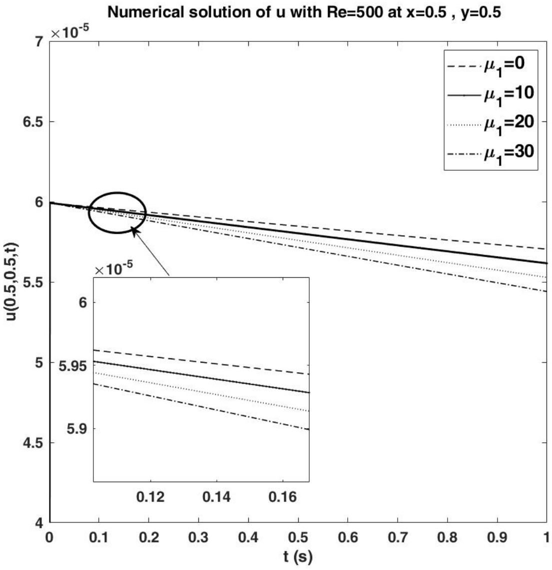
Fig. 11: Comparison the numerical solutions of u and v at $t = 1$ for fixed $x = 0.5$ with $\mu_0 = 1/500$ and $5 \leq \mu_1 \leq 30$



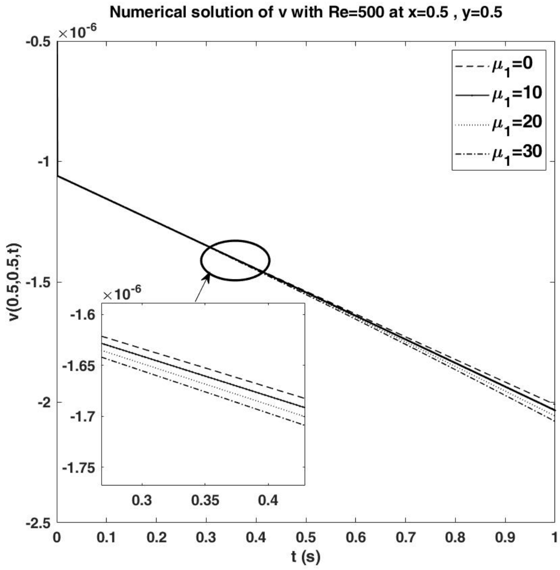
(a) velocity in x direction : u

(b) velocity in y direction : v

Fig. 12: Comparison the numerical solutions of u and v at $t = 1$ for fixed $y = 0.5$ with $\mu_0 = 1/500$ and $5 \leq \mu_1 \leq 30$



(a) $u(0.5, 0.5, t)$



(b) $v(0.5, 0.5, t)$

Fig. 13: Comparison of the numerical solutions u and v at fixed $x = y = 0.5$ with $0 \leq t \leq 1$ and $0 \leq \mu_1 \leq 30$

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