On Relatively Prime Edge Labeling of Graphs

R. Janani, T. Ramachandran

Abstract — **The main purpose of the current work is to create new directions for gaining knowledge in the arena of graph labeling. Graph labeling is helpful to assign values (labels), with some conditions to the vertex set or edge set or both. A new graph labeling technique, namely relatively prime edge labeling, is presented in this study. In prime labeling, adjacent vertices receive relatively prime labels, whereas in relatively prime edge labeling edges are labeled in such a manner that adjacent edges receive relatively prime labels. A graph is a relatively prime edge-labeled graph if it admits relatively prime edge labeling. Also, the relatively prime edge labeling is discussed for graphs with vertex or edge duplications. Relatively prime edge labeling for some trees like centipede tree, X-tree, and Y-tree are determined using algorithms in this work. Finally, a relatively prime index number is defined for the graph which fails to be a relatively prime edge-labeled graph.**

Index Terms— **Centipede tree, Duplication of graph elements, Prime graph, Relatively prime edge labeling, Relatively prime index.**

I. Introduction

The graph $G = (V, E)$ considered here is simple, finite, and undirected. Here, V and E represent the vertex and edge sets, respectively. In the current work, C_n and P_n symbolizes, a cycle and a path having n number of vertices.
Graph Labeling acts as a significant tool to simplify human work in many areas. Labeled graphs serve as a useful model that can be used in various areas of applications. A graph labeling technique assigns labels to the vertices and edges of a graph. In the mid-1960s, the graph labeling approach was initially introduced. Over the last 50 years, more than 2500 research articles use various graph labeling techniques [1].

Cahit [2] introduced the 3-equitable labeling approach in 1990. In addition, [3], [4] discuss the 3-equitable labeling of some particular graphs. Jingwen Li et al., on the other hand, provided an algorithmic approach to tackle the problem of super edge-magic total graph labeling [5].

Entringer proposed the idea of prime labeling, which was examined by Tout et al. [6]. In recent days, many researchers have introduced a variety of prime graphs such as edge vertex prime graph by R. Jagadesh and J. Baskar Babujee [7], K-Prime graph by S.K. Vaidya and U.M. Prajapati [8], the edge-prime graph was developed by Wai-Chee - Shiu et al. [9], SD-Prime graph notion was examined by G.C. Lau, W.C. Shiu [10], and so on.

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R. Janani is an Assistant professor of the Department of Mathematics, SSM Institute of Engineering and Technology, Tamil Nadu, India. (Corresponding author phone: 9994606901; e-mail: raseja3@gmail.com)

T. Ramachandran is a Head and Assistant Professor of Mathematics Department, M.V.M Government Arts College for Women, Tamilnadu, India (e-mail: yasrams@gmail.com).

Motivated by the above work, a new labeling technique named as relatively prime edge labeling for a simple graph is introduced. Also, the relatively prime edge labeling for some trees like centipede tree, X-tree, and Y-tree are determined using algorithms.

II. Preliminaries

The basic definitions needed for further understanding are presented here.

Let $G = (V, E)$ be a graph. A bijection $f : V \rightarrow \{1,2,3 \ldots \ldots, p\}$ is called prime labeling if for each edge $e = uv \in E$, we have $GCD\ (f(u), f(v)) = 1$. A graph that admits prime labeling is called a prime graph [11]. In other words, a graph having n vertices is said to be a prime graph if its vertices are labeled with the first n natural numbers, under the condition that the adjacent vertices take relatively prime labels. Paths, stars, caterpillars, complete binary trees, spiders, palm trees, fans, flowers, and many more graph families are found to admit prime labeling [1], [12]. Entringer inferred in his paper, that any tree can be prime labeled; yet, this conjecture remains open in general. [1].

Now, we extend our study of prime labeling to the relatively prime edge labeling. The significance of relatively prime edge labeling is that, every pair of adjacent edges receives relatively prime labels.

Definition 2.1
The $n -$centipede tree is a graph formed from P_n by attaching pendent edges to each vertex of path P_n [13].

Definition 2.2
One point union of path graph is a tree of t paths with exactly t vertices of degree one, one vertex of degree t and other vertices of degree two. In other words, P_n^t is a tree with t copies of path P_n and it is denoted by P_n^t with nt edges. Also, it is perceived that Y - tree and X - tree are special cases of one-point union of path graph [14].

The generalized Y-tree is a tree of three paths with exactly three vertices of degree one, one vertex of degree three and other vertices of degree two. It is denoted by P_n^3 with $3n$ edges. The generalized X -tree is a tree of four paths with exactly four vertices of degree one, one vertex of degree four and other vertices of degree two. It is denoted by P_n^4 with $4n$ edges.

III. Main Results

A. Relatively Prime Edge Labeling

A prime graph G is a bijection $f : V \rightarrow \{1,2,3 \ldots., p\}$ such that, for each edge $e = uv \in E$, we have $GCD\ (f(u), f(v)) = 1$. By extending the above definition for edges, a relatively prime edge labeled graph is proposed and it is as follows:

Definition 3.1
Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a graph. A bijection $f : E \rightarrow \{1,2,3 \ldots \ldots q\}$ is called relatively prime edge labeling if, for each vertex $\mathrm{v} \in \mathrm{V(G)}$, the labels of the edges incident on v are pairwise relatively prime. A graph that admits a relatively prime edge labeling is called a relatively prime edge labeled graph.

In other words, a graph with p vertices and q edges is said to be a relatively prime edge labeled graph, if the edges are labeled with the first q natural numbers with the condition that any two adjacent edges have relatively prime labels.

Theorem 3.1
Every cycle admits relatively prime edge labeling.
Proof
Let $\mathrm{C_n}$ be the cycle having n number of vertices and n number of edges. It's clear, that any two consecutive numbers are relatively prime, thus by labeling the edges of the cycle consecutively forms a relatively prime edge labeling. Hence every cycle admits relatively prime edge labeling.

Corollary 3.2
Every path admits relatively prime edge labeling.

Illustration
As an illustration of Theorem 3.1, relatively prime edge labeling of $\mathrm{C_6}$ and $\mathrm{P_7}$ is given in Fig 1, Fig 2.

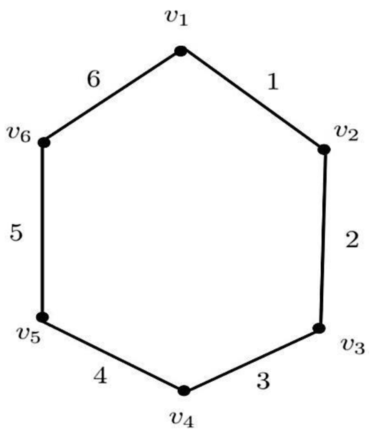

Fig 1: Cycle - C_6

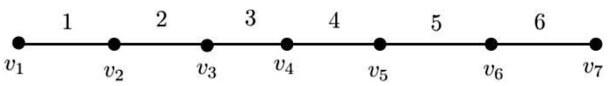

Fig 2: Path - P_7

The next theorem explains about the graph which is not relatively prime edge labeled.

Theorem 3.3
If $\mathrm{G} = (\mathrm{p}, \mathrm{p}-1)$ is a connected acyclic graph with a vertex of degree $p-1$, $p > 4$, then G is not a relatively prime edge labeled graph.
Proof
Suppose G is a relatively prime edge labeled graph for $p > 4$, then for each vertex in G, the labels of edges incident on v are pairwise relatively primes. Let w be the vertex with degree $p-1$. Therefore, the labels of edges incident on w are pairwise relatively prime, which is a contradiction. Since, label of edges incident on w will be $1,2,3 \ldots \ldots p-1$.

As a continuation, relatively prime edge labeling of some graphs namely, n- centipede tree, X – tree, Y- tree are discussed and illustrated with an example.

Theorem 3.4
$n-$ Centipede tree is a relatively prime edge labeled graph.
Proof
Let $\mathrm{G} = (\mathrm{V(G)}, \mathrm{E(G)})$ be n-centipede tree, where $V(G) = \{u_i, v_i \ / \ 1 \leq i \leq n\}$, $E(G) = \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{v_i u_i / 1 \leq i \leq n\}$, where, $v_i's$ the vertex set of the path P_n & $u_i's$ are the pendent vertices.
Also, $|V(G)| = 2n$ and $|E(G)| = 2n - 1$.
A bijection $f : E(G) \rightarrow \{1, 2, \ldots, q\}$ is defined as follows:
For $i = n, f(v_i u_i) = 1$ and
$f(v_i u_i) = 2i$, for $1 \leq i \leq n-1$,
$f(v_i v_{i+1}) = 2i + 1, 1 \leq i \leq n-1$
Now $v_i v_{i+1}, v_{i-1} v_i, v_i u_i$ are the edges incidents with the vertex v_i, $for\ 1 < i < n$
Also, $v_i v_{i+1},\ v_i u_i\ and\ \ v_{i-1} v_i, v_i u_i$ are the edges incidents with v_i for $\mathrm{i} = 1, \mathrm{n}$ respectively.
Thus from the above-mentioned bijection, edge labels incident on v_i are $2i-1, 2i, 2i+1, for\ i \neq 1, n$, are successive integers. Hence they are pairwise relatively prime.
For $\mathrm{i} = 1, \mathrm{n}$, the label of the edges incident on v_1, v_n are $2, 3$ and $1, 2\mathrm{n}-1$, which are relatively prime. Hence n-centipede tree is a relatively prime edge labeled graph.

Illustration
As an illustration of Theorem 3.4, relatively prime edge labeling of 5- centipede tree is given in Fig 3.

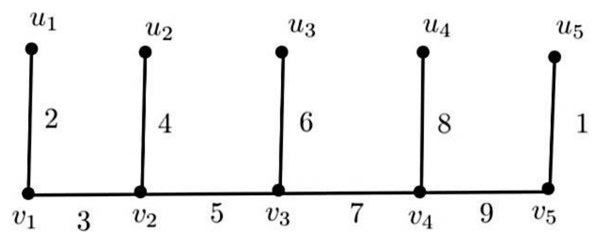

Fig 3: 5- Centipede Tree

Theorem 3.5
Every Y - tree P_n^3 is relatively prime edge labeled for even n.
Proof
The vertices and edges of P_n^3 are denoted as,
$V(P_n^3) = \{u_i \ / \ 0 \leq i \leq n\} \cup \{v_i \ /1 \leq i \leq n\} \cup \{w_i \ / \ 1 \leq i \leq n\}$ and
$E(P_n^3) = \{u_i u_{i+1} \ / \ 0 \leq i \leq n-1\} \cup \{v_i v_{i+1} \ /1 \leq i \leq n-1\} \cup \{w_i w_{i+1} \ / \ 1 \leq i \leq n-1\} \cup \ u_0 v_1 \cup u_0 w_1$.
Also P_n^3 have $3n$ edges.
For even n, $\mathrm{f} : \mathrm{E}(\mathrm{P_n^3}) \rightarrow \{1, 2, \ldots, 3\mathrm{n}\}$ is as follows:
$f(u_i u_{i+1}) = i + 1, for\ i = 0,1, \ldots n-1$
$f(v_i v_{i+1}) = n + i + 1, for\ \ i = 1,2, \ldots, n-1$
$f(w_i w_{i+1}) = 2n + i + 1, for\ i = 1,2, \ldots., n-1$
and $f(u_0 v_1) = n + 1, f(u_0 w_1) = 2n + 1$
Now, for the vertices $u_i, v_i, w_i, 1 \leq i \leq n-1$, the edges incident on u_i are $u_i u_{i+1}, u_{i-1} u_i$ with labels $i, i+1$, which are relatively prime.
Similarly, for the vertices $v_i, w_i, 2 \leq i \leq n-1$, the edges incident on v_i, w_i are $v_{i-1} v_i,\ v_i v_{i+1}$ and

$w_{i-1}w_i$, w_iw_{i+1} with labels $n+i, n+i+1$ and $2n+i, 2n+i+1$ which are relatively prime.
And for the vertex, v_1, w_1, the edges incident on, v_1, w_1 are u_0v_1, v_1v_2 and u_0w_1, w_1w_2 with labels $n+i, n+i+1$ and $2n+i, 2n+i+1$ which are relatively prime. The edges u_0u_1, u_0v_1, u_0w_1 are also incident on the vertex u_0 with labels $1, n+1, 2n+1$. Since n is even, $1, n+1, 2n+1$ are pairwise relatively prime. As a result, the labels of edges incident on each vertex are pairwise relatively prime. Thus, every Y - tree P_n^3 admits relatively prime edge labeling for even n.

Illustration:
As an illustration of Theorem 3.5, relatively prime edge labeling of P_4^3 is given in Fig 4.

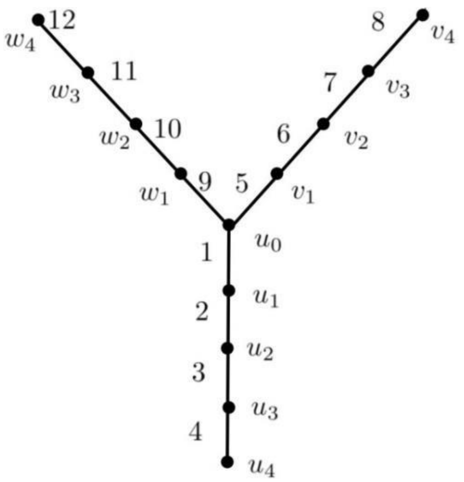

Fig 4: Y - Tree P_4^3

Theorem 3.6
Every X - tree P_n^4 is a relatively prime edge labeled, for even n.
Proof
The vertices and edges of P_n^4 are denoted as,
$V(P_n^4) = \{u_i / 0 \le i \le n\} \cup \{v_i / 1 \le i \le n\} \cup \{x_i / 1 \le i \le n\} \cup \{y_i / 1 \le i \le n\}$ and
$E(P_n^4) = \{u_iu_{i+1} / 0 \le i \le n-1\} \cup \{v_iv_{i+1} / 1 \le i \le n-1\} \cup \{x_ix_{i+1} / 1 \le i \le n-1\} \cup \{y_iy_{i+1} / 1 \le i \le n-1\} \cup u_0v_1 \cup u_0x_1 \cup u_0y_1$.
Also, the number of edges in P_n^4 are $4n$. We now define a labeling function f such that,
For even n, f : $E(P_n^4) \rightarrow \{1, 2, \ldots, 4n\}$ is,
$f(u_iu_{i+1}) = i+1, for\ i = 0,1, \ldots n-1$
$f(v_iv_{i+1}) = n+i+1, for\ i = 1,2, \ldots, n-1$
$f(x_ix_{i+1}) = 2n+i+1, for\ i = 1,2, \ldots, n-1$
$f(y_iy_{i+1}) = 3n+i+1, for\ i = 1,2, \ldots, n-1$
and $f(u_0v_1) = n+1, f(u_0x_1) = 2n+1$,
$f(u_0y_1) = 3n+1$
For the vertices $u_i, 1 \le i \le n-1$, the edges incident on, u_i are $u_iu_{i+1}, u_{i-1}u_i$ with labels $i, i+1$, which are relatively prime.
Similarly, for the vertices v_i, x_i, y_i, $2 \le i \le n-1$, the edges incident on v_i, x_i, y_i, are $v_{i-1}v_i$, v_iv_{i+1}, $x_{i-1}x_i$, x_ix_{i+1} and $y_{i-1}y_i$, y_iy_{i+1} with labels $\mathrm{n+i, n+i+1}$, $\mathrm{2n+i, 2n+i+1}$ and $\mathrm{3n+i, 3n+i+1}$ which are relatively prime. And for the vertex, v_1, x_1, y_1 the edges incident on, v_1, x_1, y_1 are u_0v_1, v_1v_2, u_0x_1, x_1x_2 and u_0y_1, y_1y_2 with labels $n+1, n+2, 2n+1, 2n+2$ and $3n+1, 3n+2$ which are relatively prime.
Also, the edges u_0u_1, u_0v_1, u_0x_1, u_0y_1 are incident into the vertex u_0 with labels $1, n+1, 2n+1$ and $3n+1$. Since n is even, $1, n+1, 2n+1, 3n+1$ are pairwise relatively prime. Therefore, for each vertex, the labels of edges incident on each vertex are pairwise relatively prime. Thus, every Y - tree P_n^4 is a relatively prime edge labeled, for even n.

Illustration:
As an illustration of Theorem 3.6, relatively prime edge labeling of P_4^4 is given in Fig 5.

B. Algorithm

An algorithm for relatively prime edge labeling of Y –tree and X – tree are discussed below.

TABLE 1
Graph theoretical notations

Symbol	Description
$\vert \boldsymbol{V} \vert$	Vertex cardinality of G
$\vert \boldsymbol{E} \vert$	Edge cardinality of G
$\boldsymbol{\mathcal{L}}$	Set of unlabeled edges in G

Algorithm for relatively prime edge labeling of Y–tree

INPUT: Vertices and Edges of P_n^3
$V(P_n^3) \leftarrow \{u_0, u_1, \cdots\cdots, u_n, v_1, \cdots\cdots, v_n, w_1, \cdots\cdots, w_n\}$
$E(P_n^3) \leftarrow \{u_0u_1, u_1u_2 \cdots\cdots, u_{n-1}u_n,$
$u_0v_1, v_1v_2, \cdots\cdots, v_{n-1}v_n, u_0w_1, w_1w_2, \cdots\cdots, w_{n-1}w_n\}$

OUTPUT: Graph with relatively prime edge labeling.
Step 1: For, $n \equiv 0(\,mod\ 2)$
Step 2: if $(e \in \mathcal{L})$ do Step 3;
Else do Step 4;
Step 3: if (i < n and i ≠ 0) then
Label $u_iu_{i+1} \rightarrow i+1$
Label $v_iv_{i+1} \rightarrow n+i+1$
Label $w_iw_{i+1} \rightarrow 2n+i+1$
Else if $(i = 0)$ then
Label $u_0u_1 \rightarrow 1$
Label $u_0v_1 \rightarrow n+1$
Label $u_0w_1 \rightarrow 2n+1$
End if
Step 4: increment e;
if ($|\mathcal{L}|$ = 0) do step 5;
Else do Step 2;
Step 5: Stop.

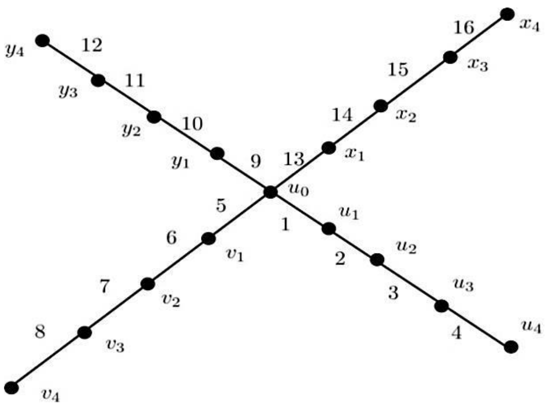

Fig 5: X - Tree P_4^4

Algorithm for relatively prime edge labeling of X –tree

INPUT: Vertices and Edges of P_n^4
$V(P_n^3) \leftarrow \{u_0, u_1, \cdots\cdots, u_n, v_1, \cdots\cdots, v_n, x_1, \cdots\cdots, x_n, y_1, \cdots\cdots, y_n\}$
$E(P_n^3) \leftarrow \{u_0u_1, u_1u_2 \cdots\cdots, u_{n-1}u_n, u_0v_1, v_1v_2, \cdots\cdots, v_{n-1}v_n, u_0x_1, x_1x_2, \cdots\cdots, x_{n-1}x_n, u_0y_1, y_1y_2, \cdots\cdots, y_{n-1}y_n\}$

OUTPUT: Graph with relatively prime edge labeling.

Step 1: For, $n \equiv 0(\,mod\ 2)$
Step 2: if $(e \in \mathcal{L})$ do Step 3;
Else do Step 4;
Step 3: if (i < n and i $\neq$ 0) then
Label $u_iu_{i+1} \rightarrow i+1$
Label $v_iv_{i+1} \rightarrow n+i+1$
Label $x_ix_{i+1} \rightarrow 2n+i+1$
Label $y_iy_{i+1} \rightarrow 3n+i+1$
Else if $(i = 0)$ then
Label $u_0u_1 \rightarrow 1$
Label $u_0v_1 \rightarrow n+1$
Label $u_0x_1 \rightarrow 2n+1$
Label $u_0y_1 \rightarrow 3n+1$
End if
Step 4: increment e;
if ($|\mathcal{L}|$ = 0) do step 5;
Else do Step 2;
Step 5: Stop.

IV. RELATIVELY PRIME EDGE LABELING FOR GRAPH DUPLICATION IN P_N

Some basic definitions necessary for the current work is presented below.

Definition 4.1
Let G be a graph with n vertices. Then the new graph G′ with $n+1$ vertices is obtained by duplicating a vertex $v \in G$ by a new vertex v', with $N(v') = N(v)$. In other words a vertex v' is a duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G′ [15].

Definition 4.2
A new graph G' obtained from G by duplicating a vertex $v_k \in G$ by a new edge $e = v_k' v_k''$ with $N(v_k') = \{v_k, v_k''\}$ and $N(v_k'') = \{v_k, v_k'\}$ [16].

Definition 4.3
Duplicating an edge $e = uv$ by a new vertex w in a graph G forms a new graph G′ such that $N(w) = \{u, v\}$ [16].

Definition 4.4
Duplicating an edge $e = uv$ of a graph G produces a new graph G′ by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v'\} - \{v\}$, $N(v') = N(v) \cup \{u'\} - \{u\}$ [17].

Theorem 4.1
Let P_n be a path with n vertices, then duplicating a pendant vertex in P_n forms a relatively prime edge labeled graph.

Proof
Let the vertices of the path P_n is $\{v_1, v_2, \ldots, v_n\}$ and G′ is attained by duplicating a pendant vertex v_i of P_n, by a new vertex v_i', $i = 1, n$ having n edges.
Case 1: Taking, the pendant vertex v_n, as a duplication vertex.
Then, the labeling function $f: E(P_n) \rightarrow \{1, 2, \ldots, n\}$ is
$f(v_n'v_{n-1}) = 1$, and
$f(v_iv_{i+1}) = n - i + 1, for\ i = 1,2, \ldots n-1$

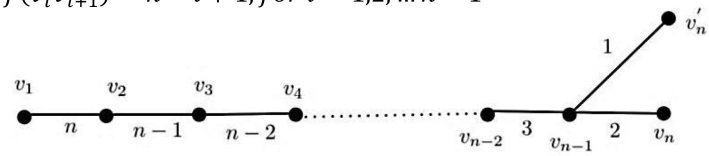

Fig 6: Duplication of pendant vertex v_n

Case 2: Taking, the pendant vertex v_1, as a duplication vertex
Then the labeling function $f: E(P_n) \rightarrow \{1, 2, \ldots, n\}$ is defined as follows:
$f(v_1'v_2) = 1$, and $f(v_iv_{i+1}) = i + 1, for\ i = 1, 2, \ldots n-1$

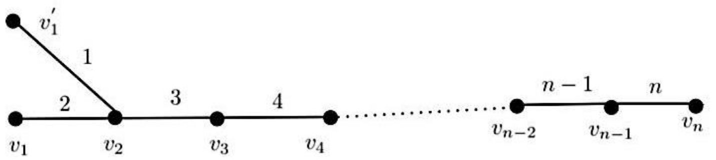

Fig 7: Duplication of pendant vertex v_1

In both the cases, for each vertex v_i, $3 < i < n-2$, the edges incident on v_i are $v_iv_{i+1}, v_{i-1}v_i$ with labels $i+1, i$ and $n-i+1, n-i+2$, which are relatively prime. The edges incident on v_2, v_{n-1} are $v_2v_1, v_2v_3, v_2v_1{}'$ and $v_{n-1}v_n$, $v_{n-1}v_{n-2}$, $v_{n-1}v_n{}'$ with labels $1, 2, 3$, which are relatively prime.

Theorem 4.2
Let P_n be the path having n vertices, then duplicating a pendant vertex of P_n with an edge forms a relatively prime edge labeled graph.
Proof
Let the vertex set of P_n is $\{v_1, v_2, \ldots, v_n\}$ and G′ is acquired by duplicating a pendent vertex by an edge $v_i'v_i''$, for $i = 1, n$.
It's clear that, the number of edges in G′ is $n + 2$.

Case 1: Duplicating pendant vertex v_n, by an edge $v_n' \, v_n''$
Define $f: E(P_n) \to \{1, 2, \ldots, n+2\}$ as follows:
$f(v_n' \, v_n'') = 2, \quad f(v_n' v_n) = 1, \quad f(v_n \, v_n'') = 3$
$f(v_i v_{i+1}) = n - i + 3, for\ i = 1,2, \ldots n-1$

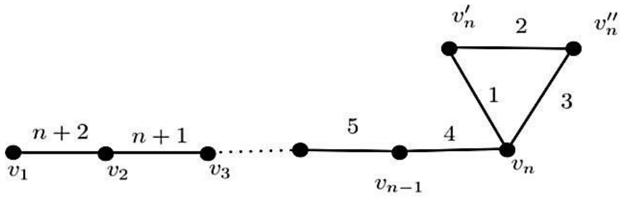

Fig 8: Duplication of pendant vertex by an edge

Case 2: Duplicating pendent vertex v_1, by an edge $v_1' \, v_1''$
Define $f: E(P_n) \to \{1, 2, \ldots, n+2\}$ as follows:
$f(v_1' v_1'') = 2,\ f(v_1' v_1) = 1,\ f(v_1 \, v_1'') = 3$ & $f(v_i v_{i+1}) = i + 3$, for $i = 1,2, \ldots n-1$
In both cases, for each vertices v_i , $1 < i < n$, the edges incident on v_i are $v_i v_{i+1}, v_{i-1} v_i$ with labels $i+2,\ i+3$ and $n-i+3,\ n-i+4$, which are relatively prime. And for the vertices v_1 , v_n , the edges incident on v_1, v_n are $v_1 v_2, v_1 v_1', v_1 v_1''$ and $v_{n-1} v_n,\ v_n v_n', v_n v_n''$ with labels 1,3,4 , which are relatively prime.

Theorem 4.3
Let $G = P_n$ be the path and if $n+1$ is prime, then G' be a graph formed from P_n by duplicating any edge ($e = uv$) with new vertex w is a relatively prime edge labeled graph.
Proof
Let the vertex set of P_n is $\{v_1, v_2, \ldots, v_n\}$ and G' is constructed from $G = P_n$ by duplicating any edge ($e = v_k v_{k+1}$) with a new vertex w and the number of edges in G' is $n+1$.
A labeling function, $f: E \to \{1,2,3 \ldots, q\}$ is defined as follows,

$$f(v_i v_{i+1}) = i, \text{for } i = 1,2,3 \ldots, k-1 \text{ and } f(v_i v_{i+1}) = i+1, \text{for } i = k+1, k+2, \ldots, n$$
$$f(v_k v_{k+1}) = n+1,\ f(v_k w) = k, f(w\, v_{k+1}) = k+1$$

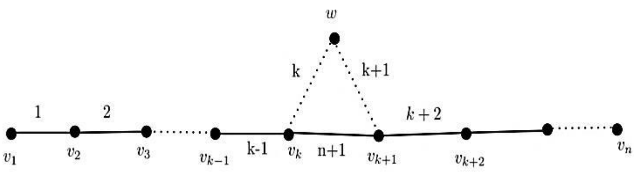

Fig 9: Duplication of any edge with a vertex

Since $n+1$ is a prime number. Then for the vertex v_k, the edges incident on v_k are $v_{k-1} v_k, v_k w, v_k v_{k+1}$ with labels $k-1, k, n+1$ and for the vertex v_{k+1}, the edges incident on v_{k+1} are $v_k v_{k+1}, w\, v_{k+1}, v_{k+1} v_{k+2}$ with labels $n+1, k+1, k+2$.
In both the cases, $n+1$ are relatively prime to $k-1, k$ and $k+1, k+2$. Also, $k-1, k$ and $k+1, k+2$ are successive integers. Therefore, for the vertices v_k, v_{k+1}, the edge labels incident on them are relatively prime. Also, label of the edges incident on the remaining vertices are consecutive integers which are relatively prime. Hence the proof.

Theorem 4.4
Let P_n be the path with n vertices, then the duplication of pendant edge of P_n forms a relatively prime edge labeled graph.
Proof
Let the vertex set of P_n is $\{v_1, v_2, \ldots, v_n\}$ and G' is the graph found by duplicating a pendant edge $e(= v_1 v_2\ or\ v_{n-1} v_n)$ with an edge $v_1' v_2'\ or\ v_{n-1}' v_n'$.
From the definition, duplication of any edge with an edge, the number of edges in G' is $q (= n+1)$.
Case 1: Duplication of a pendant edge $v_1 v_2$, by an edge $v_1' v_2'$
A labeling function $f: E \to \{1, 2, \ldots, q\}$ is defined as follows: $f(v_1' v_2') = 2,\ f(v_2' v_3) = 1$, and $f(v_i v_{i+1}) = i + 2, for\ i = 1,2, \ldots n-1$

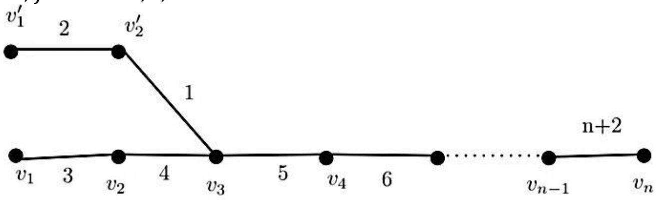

Fig 10: Duplication of a pendant edge

Case 2: Duplication of a pendant edge $v_{n-1} v_n$, by an edge $v_{n-1}' v_n'$
A labeling function $f: E \to \{1, 2, \ldots, q\}$ is,
$f(v_{n-1}' v_n') = 2, f(v_{n-1}' v_{n-2}) = 1$, and $f(v_i v_{i+1}) = n - i + 2$, for $i = 1, 2, \ldots, n-1$
In both cases, for the vertices v_3, v_{n-2} , the edges incident on $v_3\ and\ v_{n-2}$ are $v_2 v_3, v_2' v_3, v_3 v_4$ and $v_{n-2} v_{n-3}, v_{n-1}' v_{n-2}, v_{n-2} v_{n-1}$ with labels 4, 1, 5 and 5, 1, 4 are relatively prime. Also, the label of the edges incident on the remaining vertices are consecutive integers which are relatively prime.

V. DUPLICATION OF GRAPH ELEMENTS IN C_N

In this section, relatively prime edge labeling of duplication of graph elements in C_n is discussed.

Theorem 5.1
Let C_n denotes a cycle and if n is odd, then by duplicating a vertex in C_n results a relatively prime edge labeled graph.

Proof
The vertex set of C_n is $\{v_1, v_2, \ldots, v_n\}$ and G' is obtained by duplicating any vertex v_i by a new vertex v_i'. Thus, G' is the union of the cycle C_4 and the path P_{n-1} .

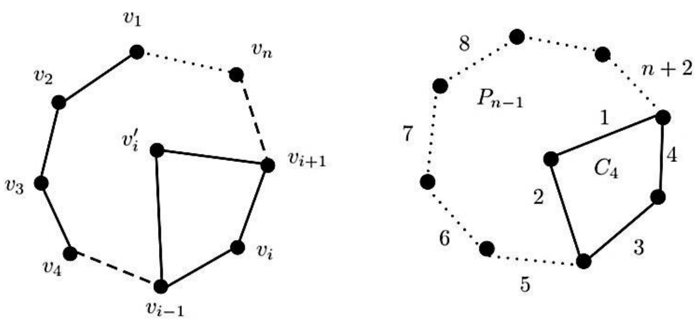

Fig 11: Duplication of a vertex

Label the edges of C_4 with 1,2,3,4 and label the path P_{n-1} with $5, 6, \ldots . n+2$, in such a way that, the edges incident on three degree vertices of C_4 will be labeled as 2, 3, 5 and 1, 4, n + 2. The above-defined labeling results in a relatively prime edge labeled graph.

Theorem 5.2
If $n+3$ is prime, then duplicating any vertex in C_n with an edge results in a relatively prime edge labeled graph
Proof
Let the vertex set of C_n is $\{v_1, v_2, \ldots, v_n\}$. Duplicating any vertex in C_n with an edge results in a graph G' with $n+3$ edges.

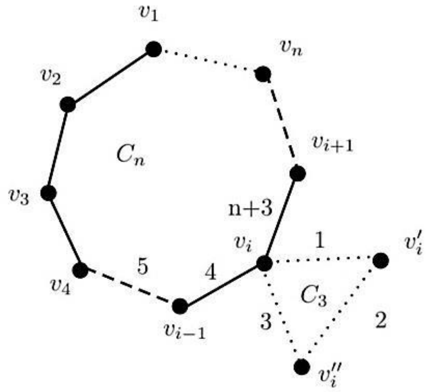

Fig 12: Duplication of a vertex by an edge

Then the resultant graph G' is the union of C_3 and C_n with a vertex in common. From the definition of duplication of a vertex with an edge, the number of edges in G' is $n+3$. Now, the edges of C_3 are labeled with the consecutive integers 1,2,3 and C_n are labeled with $4, 5, \ldots . n+3$, in such a way that, the edges incident on the common vertex will be labeled as $1, 3, 4, n+3$, which is relatively prime. Since $n+3$ is a prime number. Also, the label of the edges incident on the remaining vertices are consecutive integers which are relatively prime.

Theorem 5.3
For a graph G', which is obtained from $G = C_n$ by duplicating an edge with a vertex w is a relatively prime edge labeled graph if $n+2$ is prime.
Proof
Let $\{v_1, v_2, \ldots, v_n\}$ be the consecutive vertices of C_n and G' be the graph obtained by duplication of an edge with a vertex. Then the resultant graph G' is the union of C_3 and C_n with an edge in common. From the definition of duplication of an edge with a vertex, the number of edges in G' is $n+2$.

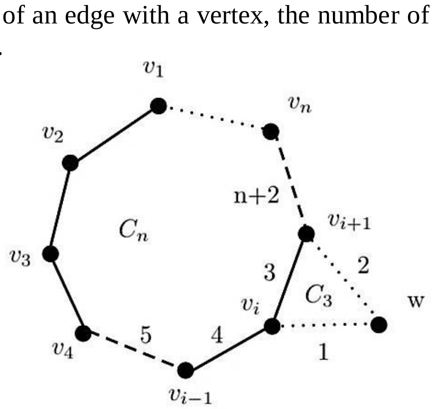

Fig 13: Duplication of an edge by a vertex

Then label the edges of C_3 with the consecutive integers 1,2,3 and C_n with $3, 4, 5, \ldots . n+2$. In addition to that, the common edge is labeled with 3. Also, the edge labels incident on three degree vertices of C_3 are 1, 3, 4 and 2, 3, n + 2. Since n+2 is prime. The edge labels incident on three degree vertices of G' are relatively prime. Moreover, the label of the edges incident on the remaining vertices are consecutive integers which are relatively prime. Hence, the above-defined labeling results in a relatively prime edge labeled graph.

Theorem 5.4
For a graph G', which is obtained from $G = C_n$ by duplicating an edge is a relatively prime edge labeled graph if $n+3$ is prime.
Proof
Let the vertex set of C_n is $\{v_1, v_2, \ldots, v_n\}$ and G' is found by duplicating an edge. Then the resultant graph G' is the union of C_6 and P_{n-2} having two common vertices. From the definition of duplication of an edge, the number of edges in G' is $n+3$ which is a prime number.

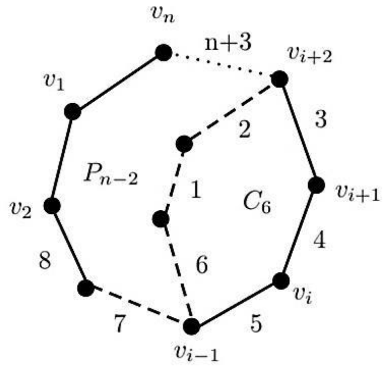

Fig 14: Edge duplication

Now, the edges of C_6 with the consecutive integers $1, 2, \ldots, 6$ and P_{n-2} with $7, 8, \ldots . n+3$, is labeled in such a way that, the edges incident on the common vertices is labeled as $5, 6, 7$ *and* $2, 3, n+3$, which are pairwise relatively prime. Since $n+3$ is a prime number. Also, the label of the edges incident on the remaining vertices are consecutive integers which are relatively prime.

VI. RELATIVELY PRIME INDEX

It has been observed that there are graphs that fail to have relatively prime edge labeling. It can, however, be made into a relatively prime edge labeled graph by removing some edges. The following definition arises by finding the answer to the above question. That is the number of edges that need to be removed to become a relatively prime edge labeled graph.

Definition 6.1
Relatively prime index of G is defined to be the minimum number of edge removal, resulting in a relatively prime edge labeled graph G^*. And it is denoted by, $\varepsilon_r(G)$. In other words,
$\varepsilon_r(G) = \min\{e_i \mid G^* = G - \{e_i\}$ is a relatively prime edge labeled graph$\}$

The next theorem determines the relatively prime index of the complete graph K_n.

Theorem 6.2
For a graph $G = K_n$, the relatively prime index is given by

$$\varepsilon_r(\mathrm{G}) = \begin{cases} \frac{n(n-3)}{2} & , if\ n \equiv 1(mod\ 2) \\ \frac{n(n-3)}{2} - 1 & , if\ n \equiv 0(mod\ 2) \end{cases}$$

Proof
Let the number of vertices and edges in a complete graph $\mathrm{K_n}$ is n and $\frac{n(n-1)}{2}$ respectively.

Case 1: If $n \equiv 1(mod\ 2)$.
It's enough to prove that, the removal of $\frac{n(n-3)}{2}$ edges results in a relatively prime edge labeled graph. Suppose the removal of $\frac{n(n-3)}{2} - 1$ edges in $\mathrm{K_n}$ results in a relatively prime edge labeled graph. That is, remaining $\frac{n(n-1)}{2} - \frac{n(n-3)}{2} + 1 = n + 1$ edges of $\mathrm{K_n}$ can be labeled from 1 to $n + 1$. Hence by removing $\frac{n(n-3)}{2} - 1$ interior edges of $\mathrm{K_n}$, resultant graph will be of the form $\mathrm{C_n}$ with n edges and edge connecting any non-adjacent vertices of $\mathrm{C_n}$. As n is odd, $\mathrm{C_n}$ can be labeled from 1 to n and the remaining edge is labeled with n+1, which is a contradiction. Since the label incident on the vertices of the edge with label $n + 1$ fails to be relatively prime.
Case 2: If $n \equiv 0(mod\ 2)$.
It is enough to prove that, the removal of $\frac{n(n-3)}{2} - 1$ edges results in a relatively prime edge labeled graph. Suppose the removal of $\frac{n(n-3)}{2} - 2$ edges in $\mathrm{K_n}$ results in a relatively prime edge labeled graph. That is, remaining $\frac{n(n-1)}{2} - \frac{n(n-3)}{2} + 2 = n + 2$ edges of $\mathrm{K_n}$ can be labeled from 1 to $n + 2$. Hence by removing $\frac{n(n-3)}{2} - 2$ interior edges of $\mathrm{K_n}$, the resultant graph will be of the form $\mathrm{C_n}$ with n edges and any two edges connecting any two pair of non-adjacent vertices of $\mathrm{C_n}$. As n is even, $\mathrm{C_n}$ can be labeled from 1 to n and the remaining edge is labeled with n+1 and n+2, which is a contradiction. Since the label incident on the vertices of the edge with label n+2 fails to be relatively prime.

Illustration
As an illustration of above theorem, relatively prime index of $\mathrm{K_6}$ is given below.

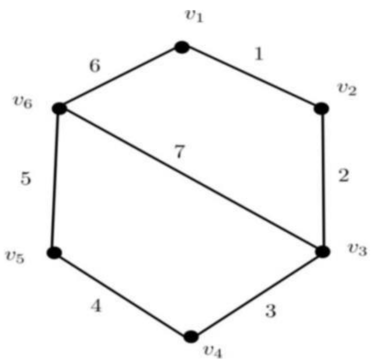

Figure 15: $\varepsilon_r(\mathrm{K_6}) = 8$

VII. CONCLUSION

The work presented here provides some new results in the theory of relatively prime edge labeling of graphs. The main focus of this study is to find the relatively prime edge labeling for some duplication of graph elements of $\mathrm{C_n}$ and $\mathrm{P_n}$. Also, an algorithm is provided to label the edges of Y-tree and X- tree using relatively prime edge labeling technique. Finally, a relatively prime index of complete graph is found. In future, relatively prime index of some class of graphs can be calculated.

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