Fitting and Smoothing Data Using Algebraic Hyperbolic Cubic Hermite Spline Interpolation

Mohammed Oraiche, Abdellah Lamnii, Mohamed Louzar and Mhamed Madark

Abstract—**In this paper, we provide a new Hermite-type cubic spline interpolating approach that is based on algebraic hyperbolic functions and has an optimal property. The error analysis is included, as well as numerical results illustrating the effectiveness of this technique. Fitting data from second-order systems is made easier using this strategy.**

Index Terms—**algebraic hyperbolic splines, Hermite interpolation, data fitting, second order system.**

I. Introduction

In several sectors of science, engineering, and technology, as well as computer-aided geometric design, data fitting has played an essential role. Interpolation and approximation are the two most often used techniques of the fitting. The data points are interpolated to produce an interpolant curve. Approximation, on the other hand, develops an approximation curve that passes close to the data points, reducing the curves' departure from the data points.

It's challenging to estimate the form of the underlying function of measurement data using a single polynomial when the shape is intricate. It is better to employ a piecewise function in this instance; this specific function is known as a spline. Because of its ease of construction and its ability to approximate complicated shapes via curve fitting, a spline is one of the most useful interpolating and approximating functions.

The usage of quadratic splines[20], [23], cubic splines [6], piecewise polynomial functions of various degrees [7], [8], rational splines [10], a class of polynomial spline curve with free parameters is established in [11] and additional approaches are developed in [12], [13], [14], [17], [21], [22]. Reference [2] proposes an interpolation and fitting approach for smooth curves. It is based on functions with junction point slopes that are specified locally about a geometrical requirement.
In least-squares data fitting issues, Hermite-type cubic splines are used to generate a sub-optimal approach, as detailed in [16]. In [5] an optimum property for Hermite-type cubic interpolation splines is built and applied for several data fitting situations.

In this paper, an optimum Hermite cubic interpolation technique with no polynomial functions is presented. This strategy is centered on optimizing the derivatives at the nodes that define the spline space to design a new interpolant utilizing algebraic hyperbolic functions.

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The following is a breakdown of the paper's structure. The development of a Hermite AH interpolant, whose derivatives at the knots are derived by solving an appropriate minimization problem, is treated in section 2.
A discussion on the study of error is also included. Two numerical examples are given in section 3 to demonstrate the effectiveness of the suggested strategy. Finally, an application of $\mathcal{C}^1$ Hermite AH splines in second order system data fitting problems is established in section 4.

II. Algebraic hyperbolic Hermite splines

Consider the uniform partition $\Delta := \{x_i\}_{1\leq i\leq n}$ of the inteval $I=[a,b]$, with step length $h:=\frac{b-a}{n-1}$ and knots $x_i := a+(i-1)\,h$. Given values y_i and m_i, $1\leq i\leq n$, for all $i\in\{1,\ldots,n-1\}$ there exists a unique AH Hermite interpolant $S_i\in\text{span}\{1,x,\cosh x,\sinh x\}$ such that $S_i(x_{i+j})=y_{i+j}$ and $S_i'(x_{i+j})=m_{i+j}$, $j=0,1$. The function S defined on I from the local interpolants S_i is a $\mathcal{C}^1$-AH spline on Δ that interpolates the data y_i and m_i.

Using once again the notation S_i for the AH cubic Hermite interpolant to the data $\{y_i,m_i\}$ and $\{y_{i+1},m_{i+1}\}$, it is straightforward to prove that the cubic spline based on the Hermite's AH two-point interpolation formula is

$$S_{|[x_i,\ x_{i+1}]}(x) := \sum_{k=0}^{1}\sum_{j=0}^{1} f_{i+k}^{j}\,\Psi_{i+k}^{j}(x), \qquad (1)$$

where

$$\Psi_i^0(x) := \frac{(x_{i+1}-x)\cosh\left(\frac{h}{2}\right)+2\sinh\left(\frac{x-x_{i+1}}{2}\right)\cosh\left(\frac{x-x_i}{2}\right)}{h\cosh\frac{h}{2}-2\sinh\frac{h}{2}},$$

$$\begin{aligned}\Psi_i^1(x) := &-\frac{csch^2\frac{h}{2}}{2(h\coth\frac{h}{2}+2)}(\sinh(x-x_i)-\sinh(x-x_{i+1})\\ &-\sinh h - x\cosh h + x_i(\cosh(x-x_{i+1})-1)\\ &+x_{i+1}(\cosh h-\cosh(x-x_{i+1}))+x),\end{aligned}$$

$$\Psi_{i+1}^0(x) := \frac{(x_i-x)\cosh(\frac{h}{2})+2\sinh(\frac{x-x_i}{2})\cosh(\frac{x-x_{i+1}}{2})}{-h\cosh\frac{h}{2}+2\sinh\frac{h}{2}},$$

$$\begin{aligned}\Psi_{i+1}^1(x) := &-\frac{csch^2\frac{h}{2}}{2(h\coth\frac{h}{2}-2)}(\sinh(x-x_i)-\sinh(x-x_{i+1})\\ &-\sinh h + x\cosh h + x_i(\cosh(x-x_i)\\ &-\cosh(h)) - x_{i+1}(\cosh(x-x_i)-1)-x).\end{aligned}$$

In data-fitting problems the given values are x_i, y_i $i=1,..,n$, while the first derivatives of S $(m_1,...,m_n)$ remain to be determined. Akima proposes in [6] a method for estimating these derivatives based on geometric local procedures. In [7], the Hermite type cubic splines are applied in order to obtain a suboptimal algorithm in least squares data-fitting problems. In [4], [15], I_1 is minimized in order to determine the slopes

m_i yielding the spline S having minimal mean derivative oscillation to L'.

Therefore, the proposed methods does not provide satisfactory results when such some remarkable curves such as the hyperbola and the catenary are interpolated. For this purpose in the present work we propose to explore the performance of AH spline spaces with respect to the capacity to reproduce such curves and the minimization of the mean oscillation of the derivative. We determine the values $m_1, ..., m_n$ in order to minimize the quadratic oscillation in average of S' to L', in other words minimizing the functionals :

$$\begin{aligned} I_1\left(m_1, ..., m_n\right) &:= \int_a^b \left(S'(x) - L'(x)\right)^2 dx \\ &= \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \left(S_i'\left(x\right) - L_i'\left(x\right)\right)^2 dx, \quad (2) \end{aligned}$$

where L is the linear interpolating spline with pieces $L_i\left(x\right) := \frac{x_{i+1}-x}{x_{i+1}-x_i} y_i + \frac{x-x_i}{x_{i+1}-x_i} y_{i+1}$. It is the simplest interpolant, but not enough regular.

From these local AH Hermite interpolants a $\mathcal{C}^1$ AH Hermite spline interpolant is produced, and the values m_i could be chosen by minimizing the functional I_1 defined in (2). We obtain the following tridiagonal system of normal equations:

$$\begin{pmatrix} \alpha & 2\beta & & & & \\ \beta & \alpha & \beta & & & \\ & \beta & \alpha & \beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & \beta & \alpha & \beta \\ & & & & 2\beta & \alpha \end{pmatrix} \begin{pmatrix} m_1 \\ \\ \vdots \\ \\ m_n \end{pmatrix} = \lambda \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_{n-1} \\ \Delta_n \end{pmatrix}, \quad (3)$$

with

$$\beta := \frac{1}{4(-h\coth(\frac{h}{2})+2)^2} \times h^4(\text{csch}^4(\frac{h}{2})((-3h^2+2)\sinh(h) + (h^2+6)h\cosh(-h) - 6h - \sinh(2h)))$$

$$\alpha := \frac{1}{4(-h\coth(\frac{h}{2})+2)^2} h^4(\text{csch}^4(\frac{h}{2})((h^2+2)\sinh(2h) + 4(h^2-1)\sinh(h) + 2(-h)(h^2+2\cosh(2h) - 2\cosh(h))))$$

$$\lambda := \frac{2e^h h^4(h^2 + h\sinh(h) - 4\cosh(h) + 4)}{(e^h(h-2)+h+2)^2}$$

$\Delta_1 := 2(f2 - f_1),\ \Delta_i := f_{i+1} - f_{i-1},\ i = 2, \cdots, n-1$

and $\Delta_n := 2(f_n - f_{n-1})$.

Theorem 1: For $h > 0$ the matrix system in (3) is a diagonally dominant matrix.

Proof: Note That

$$\begin{aligned} \alpha - 2\beta &= \frac{h^4 \text{csch}^4}{4(h\coth(\frac{h}{2})-2)^2}(4(\sinh(h)-h) \\ &\quad (h\cosh(\frac{h}{2}) - 2\sinh(\frac{h}{2}))^2) \\ &= (h^4(\sinh(h)-h)\text{csch}^2(\frac{h}{2}) \end{aligned}$$

Since $sinh(h) - h > 0$ for all $h > 0$ then $\alpha - 2\beta > 0$, which completes the proof. ■

Let $A := (a_{ij})_{1\leq i,j\leq n}$ be the coefficient matrix of system (3). The index on the diagonally dominant property of matrix A is given by

$$D_h\left(A\right) := \max_{i=1,\ldots,n} \frac{1}{|a_{ii}|} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{i,j}| = -2 \max_{h>0} \frac{\alpha\left(h\right)}{\beta\left(h\right)}$$

Its limit when h goes to zero is equal to $\frac{1}{4}$. Therefore the stability of LU factorization method is satisfactory for solving (3) for enough small h.

To analyze the interpolation error, we need the following lemma to establish an error bound for our operator.

Lemma 2: Let $f \in \mathcal{C}^3\left([a, b]\right)$, then the local truncation errors t_i, $i = 1, 2, \ldots, n$, associated with the scheme (3) are given by the expressions

$$t_i = \left(\alpha h^2 - \lambda \frac{h^3}{3}\right) f_i^{(3)} + o\left(h^2\right).$$

Proof: Since f is of class $\mathcal{C}^3\left([a, b]\right)$, then

$$\begin{aligned} f\left(x_i + h\right) &= f\left(x_i\right) + f'\left(x_i\right) h + f''\left(x_i\right) \frac{h^2}{2} + f^{(3)}\left(x_i\right) \frac{h^3}{6} \\ &\quad + o\left(h^3\right), \\ f\left(x_i - h\right) &= f\left(x_i\right) - f'\left(x_i\right) h + f''\left(x_i\right) \frac{h^2}{2} - f^{(3)}\left(x_i\right) \frac{h^3}{6} \\ &\quad + o\left(h^3\right), \\ f'\left(x_i + h\right) &= f'\left(x_i\right) + f''\left(x_i\right) h + f^{(3)}\left(x_i\right) \frac{h^2}{2} + o\left(h^2\right), \\ f'\left(x_i - h\right) &= f'\left(x_i\right) - f''\left(x_i\right) h + f^{(3)}\left(x_i\right) \frac{h^2}{2} + o\left(h^2\right). \end{aligned}$$

From equation (3), it follows that

$$\beta f'\left(x_{i-1}\right) + \alpha f'\left(x_i\right) + \beta f'\left(x_{i+1}\right) = \lambda\left(f\left(x_{i+1}\right) - f\left(x_{i-1}\right)\right)$$

Replacing $f'(x_{i-1})$, $f'(x_{i+1})$ and $f(x_{i+1}) - f(x_{i-1})$ by their Taylor expansions, we get

$$\begin{aligned} t_i &= \left(\left(2\beta + \alpha\right) f'\left(x_i\right) + \beta h^2 f^{(3)}\left(x_i\right) + o\left(h^2\right)\right) - \\ &\quad \left(2\lambda h f'\left(x_i\right) + \lambda \frac{h^3}{3} f^{(3)}\left(x_i\right) + o\left(h^3\right)\right) \\ &= \left(2\beta + \alpha - 2\lambda h\right) f'\left(x_i\right) + \left(\beta h^2 - \lambda \frac{h^3}{3}\right) f^{(3)}\left(x_i\right) \\ &\quad + o\left(h^2\right). \end{aligned}$$

We have

$$\begin{aligned} 2\beta + \alpha - 2h\lambda &= \frac{h^5\left(h^2 + h\sinh(h) - 4\cosh(h) + 4\right)}{\left(h\cosh\left(\frac{h}{2}\right) - 2\sinh\left(\frac{h}{2}\right)\right)^2} - \\ &\quad \frac{2h\left(2e^h h^4\left(h^2+h\sinh(h)-4\cosh(h)+4\right)\right)}{\left(e^h(h-2)+h+2\right)^2} \\ &= 0 \end{aligned}$$

and $\beta h^2 - \lambda\frac{h^3}{3} \neq 0$, which completes the proof. ■

Theorem 3: Let $f \in \mathcal{C}^3([a,b])$, then

$$|m_i - f'(x_i)| = o(h^2).$$

Proof: For simplicity, we write $f_i = f(x_i)$ and $f_i' = f'(x_i)$.
Let $F = (f_i)$, $M = (m_i)$, $D = \lambda(\Delta_i)$, $T = (t_i)$ for $i = 1, 2, ..., n$, and $E = F - M$ all be n-dimensional columns vectors.
With these notations, the system (3) can be written as

$$AM = D \text{ and } AF = D + T.$$

We get

$$AE = T$$

then

$$\|E\|_\infty = \left\|A^{-1}\right\|_\infty \|T\|_\infty.$$

We deduce for $i = 1, 2, ..., n$ the inequalities

$$|m_i - f'(x_i)| \leqslant C_1 \|T\|_\infty$$

where $C_1 := \left\|A^{-1}\right\|_\infty$
Using the fact that $f \in \mathcal{C}^3([a,b])$, $\alpha = \frac{8h^5}{15} + o(h^7)$ and $\lambda = \frac{h^6}{350} + \frac{h^4}{5} + o(h^7)$, a simple calculation show that

$$|t_i| \leqslant C_2 \frac{2h^7}{15} + o(h^2)$$

where C_2 is a constant such that $\left|f^{(3)}(x)\right| \leqslant C_2$ for all $x \in [a,b]$.
Finally, it holds

$$|m_i - f_i'| \leqslant C_1 C_2 \frac{2h^7}{15} + o(h^2)$$

and the proof is complete. ■

Theorem 4: Let $f \in \mathcal{C}^4([a,b])$ and S be the interpolation operator defined by (**??**). For a uniform step size h, we have

$$\|f(x) - S(x)\|_{\infty,[a,b]} = o(h^2).$$

Proof: The proof of this theorem is similar to the proof of Theorem 3 in [4]. ■

III. NUMERICAL RESULTS

This section performs some numerical tests to demonstrate the cubic Hermite AH spline's performance. First, we approximate two functions with the S-operator, where the functional values of the two functions are given. Still, the derivatives must be calculated using the minimization method described in section 2.
The first example is to consider the data taken from the function :

$$G(x)=2\sinh(x) - \frac{\cosh(x)}{5}, \ x \in [0;3] \quad (4)$$

we consider the abscissae $x_i = i\,h$, with $h = 3/n$ and $n = 3$.
The second example is to consider the data taken from the function : The first example is to consider the data taken from the function :

$$F(x)=\exp(x+2) - \exp(1-x), \ x \in [0;5] \quad (5)$$

For F we consider the abscissae $x_i = i\,h$, with $h = 5/n$ and $n = 4$.
The third example illustrates how to smooth a piecewise function H that has no derivative at some knots. For reasons of simplicity, we suppose that our function H is defined as in [18] by :

$$H(x) = \begin{cases} \log\left(\left(x - \frac{1}{2}\right)^2 + 1\right), & if\ x \in [0, \frac{1}{2}[\\ \sin(2\pi x), & if\ x \in]\frac{1}{2}, 1[\\ (1-x)\exp\left(\frac{1}{2}x(-\log(2))\right), & if\ x \in [1, 2[\\ (3-x)\left(x - \frac{5}{2}\right), & if\ x \in]2, 3] \end{cases} \quad (6)$$

It is obvious that H is only continuous on $[0,3]$. To make it of class $\mathcal{C}^1$ we apply the S-operator and we adopt a uniform subdivision of $[0,3]$.
The figures 1, 2 and 3 shows the curves of the functions G, F and H respectively.

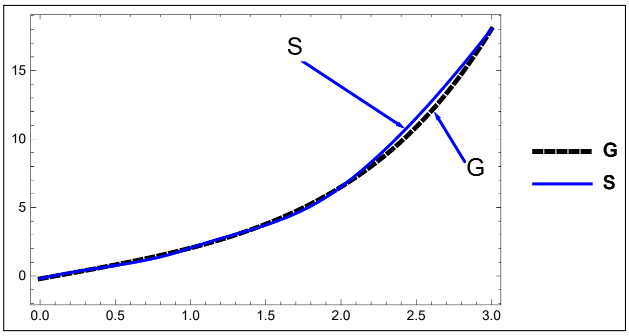

Fig. 1: Interpolating G by cubic AH Hermite splines.

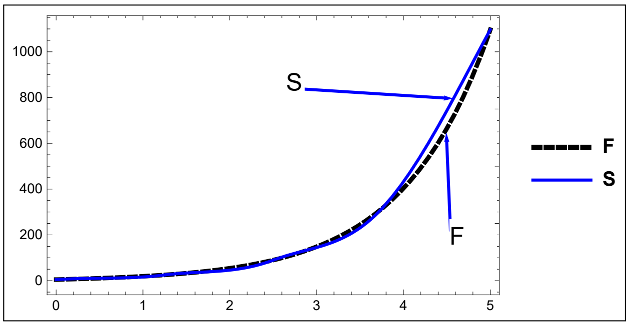

Fig. 2: Interpolating F by cubic AH Hermite splines.

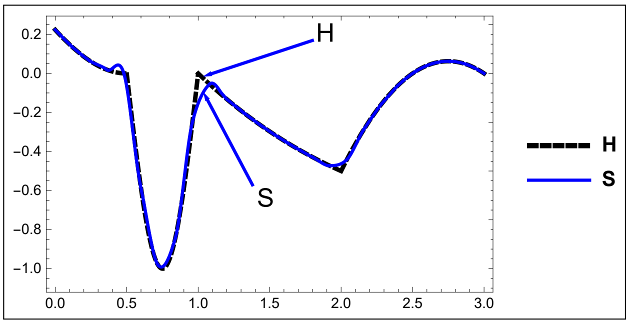

Fig. 3: Interpolating H by cubic AH Hermite splines.

From the present examples, we can see that the AH cubic spline interpolation method proposed here gives a good approximation of the interpolated functions.

IV. Data fitting application

We'll formalize our discussion of second-order responses in this section, as well as define two specifications for analyzing and designing general second-order systems. In addition, we will discuss the underdamped example and provide quantitative specifications particular to this reaction. Natural frequency and damping ratios are the names given to these numbers. The oscillation frequency of a second-order system without damping is called its natural frequency. The damping ratio, on the other hand, compares the envelope's exponential decay frequency to natural frequency. Modeling of second-order system a system is known as a second-order linear invariant if the response $y(t)$ is related to the excitation $e(t)$ by a linear second-order differential equation with constant coefficients:

$$y'' + ay' + by = be \tag{7}$$

Using the Laplace transform and assuming that initial conditions are equal to zero, the transfer function of a second order system is expressed as:

$$H(s) = \frac{Y(s)}{E(s)} = \frac{b}{s^2 + \text{as} + b} \tag{8}$$

The natural frequency is the oscillation frequency without damping. That means $a = 0$: The transfer function becomes:

$$H(s) = \frac{Y(s)}{E(s)} = \frac{b}{s^2 + b} \tag{9}$$

This function has two imaginary poles:

$$s_1 = j\sqrt{b}; s_2 = -j\sqrt{b} \tag{10}$$

Thus

$$\omega_n = \sqrt{b} \tag{11}$$

The poles of an under-damped system $(0 < \xi < 1)$ are given as:

$$s_1 = \frac{a}{2} + j\sqrt{b}; s_2 = \frac{a}{2} - j\sqrt{b} \tag{12}$$

Hence

$$\xi = \frac{Exponential\ decay\ frequency}{natural\ frequency} = \frac{a}{2\omega_n} \tag{13}$$

It is possible to deduce that

$$a = 2\xi\omega_n \tag{14}$$

The transfer function looks like this:

$$H(s) = \frac{Y(s)}{E(s)} = \frac{\omega_n^2}{s^2 + 2\xi s\omega_n + \omega_n^2} \tag{15}$$

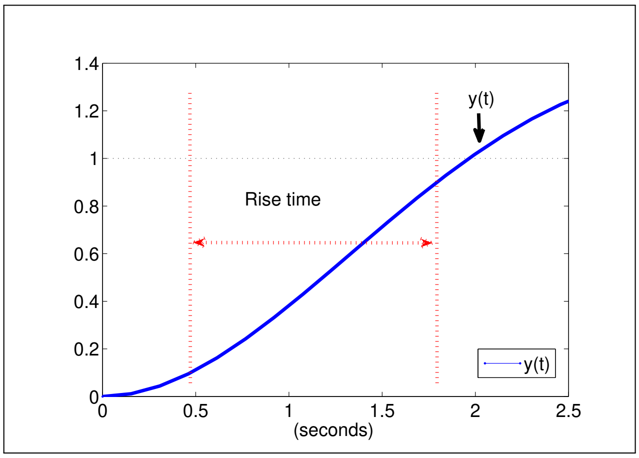

Fig. 4: Step response characteristic of an underdamped second order system.

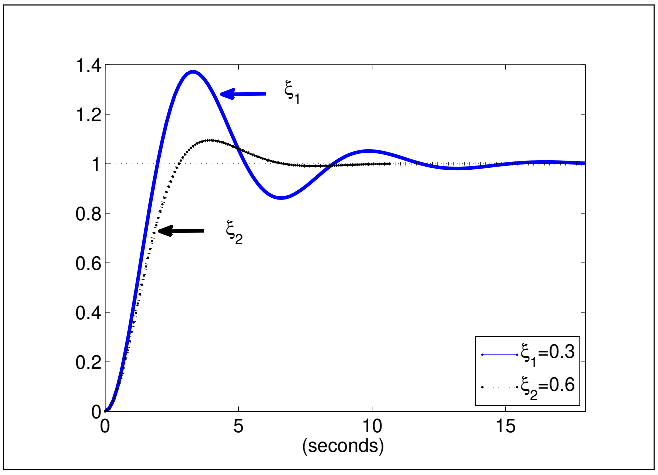

Fig. 5: Second-order underdamped responses characteristics for damping ratio values.

A. Response of an under-damped second-order system

The step response of an under-damped second-order system is its response $y(t)$ to a unit-step excitation $e(t)$ where it is assumed that $(0 < \xi < 1)$. The response is defined using inverse Laplace transform of $Y(s)$ $y(t)$:

$$Y(s) = \frac{E(s)\omega_n^2}{s^2 + 2\xi s\omega_n + \omega_n^2} = \frac{\omega_n^2}{s\left(s^2 + 2\xi s\omega_n + \omega_n^2\right)} \tag{16}$$

with $E(s) = \frac{1}{s}$.

Taking the inverse Laplace transform, the step response of an under-damped second-order system is given by:

$$y(t) = 1 - \exp(\xi t(-\omega_n))(\sin(\sqrt{1-\xi^2}t\omega_n) + \frac{\xi\cos(\sqrt{1-\xi^2}t\omega_n)}{\sqrt{1-\xi^2}}) \tag{17}$$

We have defined quantitative specifications as the natural frequency and the damping ratio. From figure 4, other parameters associated with an underdamped second-order system are rise time and settling time. According to figure 5, the rise time and the settling time depend on the damping ratio.

TABLE I: Error behavior of AH splines and the Polyfit function in Matlab.

ξ_i	Tr_i	Error of AH	Error of Polyfit
0.1	1.121	0	1.95×10^{-2}
0.15	1.178	1.69×10^{-2}	2.5×10^{-2}
0.2	1.205	0	2×10^{-4}
0.25	1.268	4.5×10^{-3}	6.8×10^{-3}
0.3	1.328	0	6.3×10^{-3}
0.35	1.397	6.2×10^{-3}	8.9×10^{-3}
0.4	1.461	0	5.2×10^{-4}
0.45	1.5451	1.5×10^{-3}	1.5×10^{-3}
0.5	1.642	0	6.7×10^{-3}
0.55	1.738	4.7×10^{-3}	1.3×10^{-4}
0.6	1.855	0	1.5×10^{-3}
0.65	1.979	3.2×10^{-3}	3.5×10^{-3}
0.7	2.125	0	1.6×10^{-3}
0.75	2.281	6.5×10^{-3}	6.2×10^{-3}
0.8	2.466	0	5.6×10^{-4}
0.85	2.672	4.7×10^{-3}	9.2×10^{-3}
0.9	2.876	0	4.3×10^{-3}

TABLE II: The absolute mean error $E(n)$

AH splines	Polyfit
3.6×10^{-3}	11.9×10^{-3}

B. Rise time Tr

The time required for the response to rising from 0.1 to 0.9 of its steady value. In theory, there is no method to define the rise time concerning the damping ratio. Therefore, an accurate analytical relationship between rise time and damping ratio, ξ, cannot be determined. However, using a numerical method and the analytical response expression $y(t)$, it is possible to find the relationship between rise time and damping ratio. In this paper, an AH Hermite spline interpolating is proposed to establish a precise relationship between rise time and damping ratio. In this work, a AH Hermite spline interpolating is proposed to establish a precise relationship between rise time and damping ratio.
With the use of the computer, we can resolve the values of $\omega_n t_2$ that result in $y(\omega_n t_2) = 0.9$ and $y(\omega_n t_1) = 0.1$. If we subtract the two values of $\omega_n t_1$ and $\omega_n t_2$, we derive the normalized rise time, $\omega_n Tr = \omega_n t_2 - \omega_n t_1$, for this specific value of ξ.
To show the accuracy of our proposed method, we will present in Table I the error made by our approach and by the Polyfit function in Matlab for the approximation of Rise time and in table II the absolute mean error. We define the error between a value Tr_i and its approximation S_i by :

$$e_i = |Tr_i - S_i|$$

and the absolute mean error $E(n)$ by

$$E(n) = \frac{1}{n}\sum_{i=1}^{n} e_i$$

the figure 6 showed fitting the rise time using AH spline.

V. Conclusion

This paper proposes a new Hermite interpolation method based on algebraic hyperbolic functions. The $\mathcal{C}^1$ Hermite

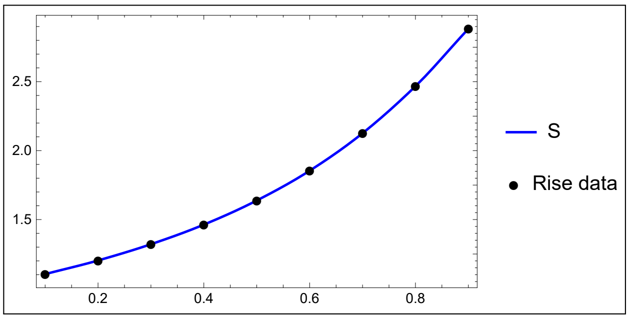

Fig. 6: Fitting the rise time using AH splines.

AH splines interpolation is exact on the space spanned by $1, x, sinh(x), cosh(x)$. The unknowns of the interpolating are determined by minimizing an integral expression measuring the squared value of the difference between the first derivative of the interpolation error. The proposed method has been applied to a second-order sub-damped system to fit the rise time as a function of the damping ratio. From the results presented in section 4 and 5, this approach showed satisfactory performance.

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