Existence of Generalized Augmented Lagrange Multipliers for Cone Constrained Optimization Problems

Liping Geng, Jinchuan Zhou*, Yue Wang, Jingyong Tang

***Abstract*—The concept of generalized augmented Lagrange multiplier was introduced originally for nonlinear programming problems. In this paper we further study this concept in the framework of convex cone constrained optimization problems. In particular we establish the relation of optimal solution sets between the primal problem and augmented Lagrange minimization problem. Some discussions on related concepts such as saddle points and exact penalty are given as well.**

***Index Terms*—Cone constraints, augmented Lagrange multipliers, exact penalty function, localization principle.**

I. INTRODUCTION

IN this paper we manly consider the following convex cone constrained optimization problem

$$\begin{array}{ll}\min\limits_{x\in Q} & f(x)\\ \text{s.t.} & G(x)\in\mathcal{K},\end{array}$$

where $Q\subseteq X$ is a nonempty closed set, $\mathcal{K}\subseteq Y$ is a closed convex cone, $f:X\to\mathbb{R}$, $G:X\to Y$, and X,Y are Banach spaces, respectively. This model includes the standard nonlinear programming, second-order cone programming, semidefinite programming, etc.

For a fixed $y\in Y$, the perturbation problem of (P) is defined as

$$\begin{array}{ll}\min\limits_{x\in Q} & f(x)\\ \text{s.t.} & G(x)+y\in\mathcal{K}.\end{array}$$

Denote by $v(y)$ the optimal value of (P_y), i.e.,

$$v(y):=\inf_{x\in Q}\{f(x)+\delta_{\mathcal{K}}(G(x)+y)\}.$$

Clearly, $v(0)$ reduces to the optimal value of primal problem (P), i.e., $v(0)=val(P)$.

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The concept of augmented Lagrange multiplier was introduced in [1]. More precisely, a vector $\lambda\in Y^*$ is said to be an augmented Lagrange multiplier of (P) if there exists $r\geq 0$ such that

$$v_r(y)\geq v_r(0)+\langle\lambda,y\rangle,\quad\forall y\in Y,$$

where $v_r(y):=v(y)+r\sigma(y)$. This means that the perturbation function has a linear support at zero. In other words, λ^* belongs to the subdifferential of v_r at zero provided that v_τ is convex. The existence of augmented Lagrange multiplier has been studied in different circumvents, such as semi-infinite programming [1, 2], second-order cone programming [3], matrix programming [4], convex cone programming [5–7]. However, a linear support may be failing to hold in general for nonconvex programming problems. This difficulty is remedied by considering the possibility of some types of nonlinear support at zero. It naturally leads to the concept of generalized augmented Lagrange multiplier; see [8]. In [8], the related theoretical results are mainly focused on the standard nonlinear programming, i.e, $\mathcal{K}:=\{0\}^l\times\mathbb{R}^m_-$, a convex polyhedral set. In this paper, we further extend these results to more general framework, i.e., $\mathcal{K}$ is a non-polyhedral convex cone, which includes second-order cone, positive semi-definite matrix cone, homogeneous cone, as special cases.

The existence of augmented Lagrange multiplier is closely related to the important concept of saddle points. In particular, (x^*,λ^*) is a global saddle point of (P) if and only if x^*,λ^* are the optimal solutions of the primal and dual problems respectively, and meanwhile the zero dual gap property holds. Some related works can be found in [9–15]. However, in some practical case, it is impossible to know the real solution in advance. In addition, some types of approximating problems have to be solved in order to deal with the constraints. The usual approaches include penalty function methods and augmented Lagrange multiplier methods; see [16–25] for more information. To overcome the numerical difficulty caused as penalty parameter is too large, it makes sense to study the exact penalty and exact augmented Lagrange function. For example, we hope $\inf_x L(x,\lambda^*,r)=val(P)$ for some λ^* and $r>0$. This is equivalent to saying that λ^* is an augmented Lagrange multiplier. If $\lambda^*=0$, then it reduces to the exact penalty function. In this paper, we further establish the relation of optimal solution set between primal problems and augmented Lagrangian relaxed problems. Finally, the existence of generalized augmented Lagrange multiplier is discussed by using exact penalty and the analysis technique called localization principle which was introduced in [26].

II. Preliminaries

Definition 1. *Let $\sigma : Y \to \mathbb{R}_+ := [0, +\infty)$. We say that σ has a valley at zero provided that σ is continuous at 0 with $\sigma(0) = 0$ and $\inf\{\sigma(y)|\ \|y\| \geq \varepsilon\} > 0$ whenever $\varepsilon > 0$.*

Definition 2. *A vector $\lambda \in Y^*$ is said to be a generalized augmented Lagrange multiplier of (P), if there exists $r \geq 0$ such that*

$$v_r(y) \geq v_r(0) + \phi(\lambda, y), \quad \forall y \in Y,$$

where $\phi : Y^* \times Y \to \mathbb{R}$ has the following properties:
(B_1) ϕ is continuous with $\phi(\cdot, 0) = 0$;
(B_2) If $x \in Q$ satisfies $G(x) \notin \mathcal{K}$, then there exists a vector $u_0 \in Y^*$ such that

$$\sup_{\xi\in\mathcal{K}} \phi(\tau u_0, \xi - G(x)) \to -\infty \quad \text{as} \quad \tau \to +\infty.$$

Notice that $\mathcal{K}$ is a closed convex set. For any $x \in Q$ satisfying $G(x) \notin \mathcal{K}$ and $\xi \in \mathcal{K}$, by the Strong Separation Theorem of convex sets in [27], there exist a nonzero vector u_0 and a scalar $\varepsilon > 0$ such that

$$\langle u_0, \xi - G(x)\rangle < -\varepsilon, \quad \forall \xi \in \mathcal{K}. \tag{1}$$

Next let us introduce two simple examples satisfying the assumptions (B_1) and (B_2) required in Definition 2.

(a) Let

$$\phi(x, y) := \langle Ax, y\rangle,$$

where A is a positive definite matrix. Obviously (B_1) is valid due to $\phi(x, 0) = \langle Ax, 0\rangle = 0$. Pick $u_0' := A^{-1}u_0$. It then follows from (1) that

$$\begin{aligned}\phi(\tau u_0', \xi - G(x)) &= \langle \tau A u_0', \xi - G(x)\rangle \\ &= \tau\langle u_0, \xi - G(x)\rangle \\ &< -\tau\varepsilon.\end{aligned}$$

Hence

$$\sup_{\xi\in\mathcal{K}} \phi(\tau u_0', \xi - G(x)) \to -\infty \quad \text{as} \quad \tau \to +\infty.$$

(b) Let $\theta(\cdot)$ be a continuous and strict increasing function on $\mathbb{R}$ with $\theta(0) = 0$. Define

$$\phi(x, y) := \|x\|\theta(x^T y).$$

Taking into account the monotonicity of function θ and the fact $u_0^T(\xi - G(x)) < -\varepsilon$ appeared in (1), we obtain

$$\begin{aligned}\phi(\tau u_0, \xi - G(x)) &= \|\tau u_0\|\theta\left(\tau u_0^T\left(\xi - G(x)\right)\right) \\ &= \tau\|u_0\|\theta\left(\tau u_0^T\left(\xi - G(x)\right)\right) \\ &\leq \tau\|u_0\|\theta(-\tau\varepsilon) \\ &\leq \tau\|u_0\|\theta(-\varepsilon),\end{aligned}$$

whenever $\tau \geq 1$. Note that $\theta(-\varepsilon) < 0$ by the strict monotonicity of θ. Hence

$$\sup_{\xi\in\mathcal{K}} \phi(\tau u_0, \xi - G(x)) \to -\infty \quad \text{as} \quad \tau \to +\infty.$$

For $(x, \lambda, r) \in X \times Y^* \times \mathbb{R}_+$, the augmented Lagrange function of (P) is defined as

$$\begin{aligned}&L(x, \lambda, r) \\ &:= \inf_{x\in Q, y\in Y} \left\{f(x) + \delta_{\mathcal{K}}(G(x) + y) - \phi(\lambda, y) + r\sigma(y)\right\}.\end{aligned} \tag{2}$$

Definition 3. *A point $(\tilde{x}, \tilde{\lambda}) \in Q \times Y^*$ is said to be a global saddle point of the generalized augmented Lagrangian L, if there exists $r \geq 0$ such that*

$$L(\tilde{x}, \lambda, r) \leq L(\tilde{x}, \tilde{\lambda}, r) \leq L(x, \tilde{\lambda}, r), \quad \forall x \in Q, \lambda \in Y^*.$$

Furthermore, if the above inequalities hold for all $x \in \mathbb{B}_X(\tilde{x}, \delta) \cap Q$, then $(\tilde{x}, \tilde{\lambda})$ is said to be a local saddle point of L, where $\mathbb{B}_X(\tilde{x}, \delta)$ denotes a ball centered at $\tilde{x}$ with radius $\delta > 0$.

The dual function and dual problem of (P) are defined respectively as $\varphi(\lambda, r) := \inf_{x\in Q} L(x, \lambda, r)$ and

$$\sup_{(\lambda,r)\in Y^*\times\mathbb{R}_+} \varphi(\lambda, r). \tag{3}$$

Note that

$$\sup_{(\lambda,r)\in Y^*\times\mathbb{R}_+} \varphi(\lambda, r) = \sup_{r\in\mathbb{R}_+} \sup_{\lambda\in Y^*} \varphi(\lambda, r).$$

Hence it is natural to introduce the following r-dual problem by fixing the parameter r in advance, i.e.,

$$\sup_{\lambda\in Y^*} \varphi(\lambda, r^*) = \sup_{\lambda\in Y^*} \inf_{x\in Q} (x, \lambda, r^*). \tag{4}$$

Denote by $val(D)$ and $val(D_r)$ as the optimal value of the above dual problems (3) and (4) respectively.

III. Dual theory

Proposition 1. *If x is feasible for (P) and $(\lambda, r) \in Y^* \times \mathbb{R}_+$, then*

$$\varphi(\lambda, r) \leq val(D_r) \leq val(D) \leq val(P) \leq f(x). \tag{5}$$

Furthermore, $\lambda \in Y^*$ is a generalized augmented Lagrangian multiplier of (P) if and only if there exists some $r \geq 0$ such that

$$\varphi(\lambda, r) = \inf_{x\in Q} L(x, \lambda, r) = val(P). \tag{6}$$

Proof: Since x is feasible, then $0 \in \mathcal{K} - G(x)$. Hence it follows from (2) that

$$\begin{aligned}L(x, \lambda, r) &= \inf_{y\in\mathcal{K}-G(x)} \{f(x) - \phi(\lambda, y) + r\sigma(y)\} \\ &\leq f(x),\end{aligned}$$

where the inequality is due to $\phi(\cdot, 0) = 0$ and $\sigma(0) = 0$. Taking the infimum over $x \in Q$ yields

$$\varphi(\lambda, r) = \inf_{x\in Q} L(x, \lambda, r) \leq \inf_{\substack{x\in Q \\ G(x)\in\mathcal{K}}} f(x) = val(P) \leq f(x).$$

The formula (5) can be obtained by further taking the supremum over $\lambda \in Y^*$ and $(\lambda, r) \in Y^* \times \mathbb{R}_+$ respectively.

It is known from Definition 2 that λ is a generalized augmented Lagrange multiplier of (P) if and only if there exists $r \geq 0$ such that

$$v(y) \geq v(0) + \phi(\lambda, y) - r\sigma(y), \quad \forall y \in Y.$$

This is equivalently written as

$$\begin{aligned} v(0) &= \inf_{y\in Y} \{v(y) - \phi(x,y) + r\sigma(y)\} \\ &= \inf_{y\in Y} \inf_{x\in Q} \{f(x) + \delta_{\mathcal{K}}(G(x)+y) - \phi(\lambda,y) + r\sigma(y)\} \\ &= \inf_{x\in Q} \inf_{y\in Y} \{f(x) + \delta_{\mathcal{K}}(G(x)+y) - \phi(\lambda,y) + r\sigma(y)\} \\ &= \inf_{x\in Q} L(x,\lambda,r) = \varphi(\lambda,r). \end{aligned} \tag{7}$$

The proof is complete. ■

Lemma 1. *For $r \geq 0$, one has*

$$\sup_{\lambda\in Y^*} L(x,\lambda,r) = \begin{cases} f(x), & \text{if } x\in Q,\ G(x)\in\mathcal{K}; \\ +\infty, & \text{otherwise.} \end{cases} \tag{8}$$

Proof: This is discussed in two cases of $x \in \mathcal{F}$ or $x \notin \mathcal{F}$, where $\mathcal{F}$ denotes the feasible region of (P).

Case 1. $x \in \mathcal{F}$, i.e., $x \in Q$, $G(x) \in \mathcal{K}$. It then follows from (7) that $L(x,\lambda,r) \leq f(x)$ for all $\lambda \in Y^*$. On the other hand

$$\sup_{\lambda\in Y^*} L(x,\lambda,r) \geq L(x,0,r) \geq f(x), \tag{9}$$

where the last inequality comes from the nonnegativity of σ. Combining (7) and (9) yields

$$\sup_{\lambda\in Y^*} L(x,\lambda,r) = f(x),\ \forall x\in\mathcal{F}.$$

Case 2. $x \notin \mathcal{F}$, i.e., $x \notin Q$ or $x \in Q$ but $G(x) \notin \mathcal{K}$. If $x \notin Q$, it is clear from (2) that

$$L(x,\lambda,r) = +\infty, \quad \forall\lambda\in Y^*. \tag{10}$$

If $x \in Q$ but $G(x) \notin \mathcal{K}$, it follows from (B_2) that there exists a vector $u_0 \in Y^*$ such that

$$\sup_{\xi\in\mathcal{K}} \phi(\tau u_0, \xi - G(x)) \to -\infty \quad \text{as} \quad \tau\to+\infty.$$

Hence

$$\begin{aligned} &\sup_{\lambda\in Y^*} L(x,\lambda,\tilde{r}) \\ &\geq L(x,\tau u_0, r) \\ &= \inf_{y\in\mathcal{K}-G(x)} \{f(x) - \phi(\tau u_0, y) + r\sigma(y)\} \\ &\geq f(x) - \sup_{\xi\in\mathcal{K}} \phi(\tau u_0, \xi - G(x)) \\ &\to +\infty, \quad \text{as} \quad \tau\to+\infty. \end{aligned} \tag{11}$$

Putting (10) and (11) together yields

$$\sup_{\lambda\in Y^*} L(x,\lambda,\tilde{r}) = +\infty, \quad \forall x\notin\mathcal{F}.$$

■

According to Lemma 1 we can obtain the following result.

Corollary 1.

$$\begin{aligned} v(0) &= \min_{x\in Q} \sup_{\lambda\in Y^*} L(x,\lambda,r) \\ &= \min_{x\in Q} \sup_{\substack{\lambda\in Y^* \\ r\in\mathbb{R}_+}} L(x,\lambda,r), \quad \forall r\in\mathbb{R}_+. \end{aligned}$$

Theorem 1. *Let $\tilde{r} \geq 0$ be given. The following statements are equivalent.*

(i) *$(\tilde{x},\tilde{\lambda})$ is a global saddle point of $L(x,\lambda,\tilde{r})$;*
(ii) *$\tilde{x} \in Q$ and $\tilde{\lambda} \in Y^*$ are optional solutions of (P) and $(D_{\tilde{r}})$ respectively, and $val(P) = val(D_{\tilde{r}})$;*
(iii) *$\tilde{x} \in Q$ and $(\tilde{\lambda},\tilde{r}) \in Y^* \times \mathbb{R}_+$ are optional solutions of (P) and (D) respectively, and $val(P) = val(D)$.*

Proof: Note that $(\tilde{x},\tilde{\lambda})$ is a global saddle point of $L(\cdot,\cdot,\tilde{r})$ if and only if

$$\begin{aligned} \varphi(\tilde{\lambda},\tilde{r}) &= \min_{x\in Q} L(x,\tilde{\lambda},\tilde{r}) = L(\tilde{x},\tilde{\lambda},\tilde{r}) \\ &= \max_{\lambda\in Y^*} L(\tilde{x},\lambda,\tilde{r}) = f(\tilde{x}) \end{aligned} \tag{12}$$

and

$$\begin{aligned} L(\tilde{x},\tilde{\lambda},\tilde{r}) &= \max_{\lambda\in Y^*} \min_{x\in Q} L(x,\lambda,\tilde{r}) \\ &= \min_{x\in Q} \max_{\lambda\in Y^*} L(x,\lambda,\tilde{r}) = v(0), \end{aligned} \tag{13}$$

where the last equalities in (12) and (13) come from Lemma 1 and Corollary 1 respectively. Combining (12) and (13) ensures that $\tilde{x}$ and $\tilde{\lambda}$ are optimal solution of (P) and $(D_{\tilde{r}})$ respectively and $val(P) = val(D_{\tilde{r}})$.

Conversely, if $\tilde{x}$ and $\tilde{\lambda}$ are optimal solution of (P) and $(D_{\tilde{r}})$ respectively, and one has $val(P) = val(D_{\tilde{r}})$, then $\varphi(\tilde{\lambda},\tilde{r}) = f(\tilde{x})$. Note that

$$\begin{aligned} L(\tilde{x},\tilde{\lambda},\tilde{r}) &\leq \max_{\lambda} L(\tilde{x},\lambda,\tilde{r}) = f(\tilde{x}) \\ &= \varphi(\tilde{\lambda},\tilde{r}) = \min_{x\in Q} L(x,\tilde{\lambda},\tilde{r}) \end{aligned} \tag{14}$$

and

$$\begin{aligned} L(\tilde{x},\tilde{\lambda},\tilde{r}) &\geq \min_{x\in Q} L(x,\tilde{\lambda},\tilde{r}) = \varphi(\tilde{\lambda},\tilde{r}) \\ &= f(\tilde{x}) = \max_{\lambda} L(\tilde{x},\lambda,\tilde{r}), \end{aligned} \tag{15}$$

where the first equation in (14) and the third equation in (15) follow from Lemma 1. The formula (14) and (15) show that $(\tilde{x},\tilde{\lambda})$ is a global saddle point of $L(\cdot,\cdot,\tilde{r})$.

The similar argument is applicable to the relation between items (i) and (iii). ■

Theorem 2. *Let $\tilde{r} \geq 0$ be given. Then the following statements hold.*

(i) *$\tilde{\lambda}$ is a generalized augmented Lagrangian multiplier of (P) with $\tilde{r}$;*
(ii) *$(\tilde{\lambda},\tilde{r})$ is an optional solution of (D) and $val(P) = val(D)$;*
(iii) *$\tilde{\lambda}$ is an optional solution of $(D_{\tilde{r}})$ and $val(P) = val(D_{\tilde{r}})$.*

Proof: (i)⇒(ii) According to Definition 2, $\tilde{\lambda}$ is a generalized augmented Lagrangian multiplier of (P) with $\tilde{r}$ if and only if

$$v_{\tilde{r}}(0) = \inf_{y\in Y} \left\{ v_{\tilde{r}}(y) - \phi(\tilde{\lambda},y) \right\} = \varphi(\tilde{\lambda},\tilde{r}),$$

where the last step is due to (7). Furthermore, according to the weak duality theorem between (P) and (D) we know that $(\tilde{\lambda},\tilde{r})$ is an optional solution of (D) and the dual gap is zero, i.e., $val(P) = val(D)$.

(ii)⇒(i) If $(\tilde{\lambda},\tilde{r})$ is an optional solution of (D) and zero duality gap property holds between (P) and (D), then

$$\begin{aligned} \varphi(\tilde{\lambda},\tilde{r}) &= \sup_{\lambda,r\geq 0} \varphi(\lambda,r) = val(D) \\ &= val(P) = v(0) = v_{\tilde{r}}(0). \end{aligned} \tag{16}$$

Recall from (7) that

$$\varphi(\tilde{\lambda},\tilde{r}) = \inf_{y\in Y} \left\{ v_{\tilde{r}}(y) - \phi(\tilde{\lambda},y) \right\}.$$

This together with (16) implies

$$v_{\tilde{r}}(y) \leq v_{\tilde{r}}(0) + \phi(\tilde{\lambda}, y), \quad \forall y \in Y.$$

Thus $\tilde{\lambda}$ is a generalized augmented Lagrangian multiplier of (P) by definition.

The similar argument is applicable to the relation between items (i) and (iii). ■

The conditions involved in the concept of generalized augmented Lagrange multiplier can be further weaken to some neighborhood of the origin provided that the augmented Lagrangian function is bounded from below.

Theorem 3. *A vector $\tilde{\lambda}$ is a generalized augmented Lagrange multiplier of (P) if and only if there exist $\tilde{r} \in \mathbb{R}_+$ and a scalar $\tau > 0$ such that*

$$v(y) \geq v(0) + \phi(\tilde{\lambda}, y) - \tilde{r}\sigma(y), \quad \forall y \in \tau\mathbb{B}, \tag{17}$$

and the function $L(\cdot, \tilde{\lambda}, \tilde{r})$ is bounded from below in Q.

Proof: The necessity is followed by Definition 2 and (6) in Proposition 1.

We now prove the sufficiency. Since $L(\cdot, \tilde{\lambda}, \tilde{r})$ is bounded from below on Q, then there exists $\gamma \in \mathbb{R}$ such that

$$\begin{aligned} f(x) + \delta_{\mathcal{K}}\left(G(x) + y\right) - \phi(\tilde{\lambda}, y) + \tilde{r}\sigma(y) \\ \geq L(x, \tilde{\lambda}, \tilde{r}) \geq \gamma, \quad \forall (x, y) \in Q \times Y. \end{aligned}$$

Taking the infimum of the above formula over $x \in Q$ yields

$$v(y) - \phi(\tilde{\lambda}, y) + \tilde{r}\sigma(y) \geq \gamma, \quad \forall y \in Y.$$

Using the fact that σ has valley at zero, for the above τ in (17) there exists $\beta > 0$ such that

$$\sigma(y) \geq \beta, \quad \forall y \notin \tau\mathbb{B}.$$

Pick

$$\overline{r} := 1 + \tilde{r} + (v(0) - \gamma)/\beta.$$

Then for each $y \notin \tau\mathbb{B}$ we have

$$\begin{aligned} & v(y) - \phi(\tilde{\lambda}, y) + \overline{r}\sigma(y) \\ & \quad = v(y) - \phi(\tilde{\lambda}, y) + \tilde{r}\sigma(y) + (\overline{r} - \tilde{r})\sigma(y) \\ & \quad \geq v(y) - \phi(\tilde{\lambda}, y) + \tilde{r}\sigma(y) + (\overline{r} - \tilde{r})\beta \\ & \quad \geq \gamma + (\overline{r} - \tilde{r})\beta \geq v(0), \quad y \notin \tau\mathbb{B}. \end{aligned}$$

This together with (17) means that $\tilde{\lambda}$ is a generalized augmented Lagrangian multiplier. ■

The results obtained in this section are similar to those given in [26] by replacing the inner produce by more general function ϕ.

Definition 4. *For any $\rho > 0$, we say that $v(y)$ has a growth condition with σ at zero, if there exists $a, b \in \mathbb{R}$ such that*

$$v(y) \geq b - a\sigma(y), \quad \forall y \in Y \backslash \rho\mathbb{B}_Y.$$

By utilizing the growth condition, we can obtain a sufficient condition for the existence of generalized augmented Lagrangian multipliers.

Theorem 4. *Assume that $v(y)$ satisfies the growth condition with σ at zero. If there exist a vector $\widetilde{\lambda}$ and scalars $\rho, \tilde{r}, \tau > 0$ such that*

$$v(y) \geq v(0) + \phi(\tilde{\lambda}, y) - \tilde{r}\sigma(y), \quad \forall y \in \tau\mathbb{B}, \tag{18}$$

and

$$\rho\sigma(y) - \phi(\widetilde{\lambda}, y) \geq 0, \quad \forall y \in Y \backslash \tau\mathbb{B}_Y, \tag{19}$$

then the vector $\tilde{\lambda}$ is a generalized augmented Lagrange multiplier of (P).

Proof: Since $v(y)$ satisfies the growth condition, then by definition for the above $\tau > 0$ in (18), there exist $a, b \in \mathbb{R}$ such that

$$v(y) \geq b - a\sigma(y), \quad \forall y \in Y \backslash \tau\mathbb{B}_Y.$$

The valley at zero property of σ guarantees the existence of $\beta > 0$ such that

$$\sigma(y) \geq \beta, \quad \forall y \notin \tau\mathbb{B}. \tag{20}$$

Pick

$$\overline{r} := (v(0) - b)/\beta + a + \rho + 1.$$

Hence for each $y \notin \tau\mathbb{B}$ it follows from (19) and (20) that

$$\begin{aligned} & v(y) - \phi(\tilde{\lambda}, y) + \overline{r}\sigma(y) \\ & \quad \geq b - a\sigma(y) - \phi(\tilde{\lambda}, y) + \overline{r}\sigma(y) \\ & \quad = b + (\overline{r} - a - \rho)\sigma(y) + \rho\sigma(y) - \phi(\tilde{\lambda}, y) \\ & \quad \geq b + (\overline{r} - a - \rho)\beta \geq v(0). \end{aligned}$$

This together with (18) ensures that $\tilde{\lambda}$ is a generalized augmented Lagrangian multiplier. ■

IV. Optimal solutions

Given $\lambda \in Y^*$, let us define

$$r(\lambda) := \inf\left\{r \geq 0 \middle| \, v(y) \geq v(0) + \phi(\lambda, y) - r\sigma(y)\right\}.$$

It is clear to see that λ^* is a generalized augmented Lagrange multiplier if and only if $r(\lambda^*) < \infty$.

Theorem 5. *If $\lambda \in Y^*$ is a generalized Lagrange multiplier of (P), then for any $r > r(\lambda)$,*

$$\begin{aligned} \arg \min_{(x,y) \in Q \times Y} \left\{f(x) + \delta_{\mathcal{K}}(G(x) + y) - \phi(\lambda, y) + r\sigma(y)\right\} \\ = (S, 0). \end{aligned}$$

Proof: It follows from [28, Proposition 1.35] that

$$(\bar{x}, \bar{y}) \in \arg \min_{x \in Q, G(x) + y \in \mathcal{K}} \{f(x) - \phi(\lambda, y) + r\sigma(y)\}$$

if and only if $\bar{y}$ belongs to

$$\arg \min_{y \in Y} \Big\{ \inf_{x \in Q, G(x) + y \in \mathcal{K}} \left\{f(x) - \phi(\lambda, y) + r\sigma(y)\right\} \Big\} \tag{21}$$

and $\bar{x}$ lies in

$$\arg \min_{x \in Q} \left\{f(x) + \delta_{\mathcal{K}}(G(x) + \bar{y}) - \phi(\lambda, \bar{y}) + r\sigma(\bar{y})\right\}. \tag{22}$$

Now we claim that $\bar{y} = 0$, i.e.,

$$\begin{aligned} & \arg \min_{y \in Y} \Big\{ \inf_{x \in Q, G(x) + y \in \mathcal{K}} \{f(x) - \phi(\lambda, y) + r\sigma(y)\} \Big\} \\ & = \{0\}. \end{aligned} \tag{23}$$

Since $\lambda \in Y^*$ is a generalized Lagrange multiplier, then

$$\begin{aligned} & v(y) - \phi(\lambda, y) + r\sigma(y) \\ & \quad = \inf_{x \in Q} \{f(x) + \delta_{\mathcal{K}}(G(x) + y) - \phi(\lambda, y) + r\sigma(y)\} \\ & \quad \geq v(0) = \inf_{x \in Q} \left\{f(x) + \delta_{\mathcal{K}}\left(G(x)\right)\right\}, \quad \forall y \in Y. \end{aligned}$$

So

$$0 \in \arg\min_{y\in Y}\Big\{\inf_{x\in Q}\{f(x)+\delta_{\mathcal{K}}(G(x)+y)-\phi(\lambda,y)+r\sigma(y)\}\Big\}. \tag{24}$$

Conversely, pick

$$y^* \in \arg\min_{y\in Y}\Big\{\inf_{x\in Q}\{f(x)+\delta_{\mathcal{K}}(G(x)+y)-\phi(\lambda,y)+r\sigma(y)\}\Big\}.$$

Then

$$\begin{aligned}&\inf_{x\in Q}\{f(x)+\delta_{\mathcal{K}}(G(x))\}\\ &\quad\geq \inf_{x\in Q}\{f(x)+\delta_{\mathcal{K}}(G(x)+y^*)-\phi(\lambda,y^*)+r\sigma(y^*)\},\end{aligned}$$

i.e.,

$$v(0)\geq v(y^*)-\phi(\lambda,y^*)+r\sigma(y^*). \tag{25}$$

Take $\epsilon>0$ satisfying $r(\lambda)+\epsilon<r$. Then

$$\begin{aligned}&v(y^*)-\phi(\lambda,y^*)+r\sigma(y^*)\\ &\quad\geq v(y^*)-\phi(\lambda,y^*)+(r(\lambda)+\epsilon)\sigma(y^*)\\ &\quad\geq v(0),\end{aligned} \tag{26}$$

where the last inequality is due to the definition of $r(\lambda)$. Comparing (25) and (26) immediately leads to $(r-r(\lambda))\sigma(y^*)=0$, which in turn implies $\sigma(y^*)=0$. So $y^*=0$ since σ has the valley property. This together with (24) guarantees the validity of (23).

According to (23), we know $\bar{y}=0$ by (21). Hence substituting $\bar{y}=0$ to (22) leads to

$$\begin{aligned}\bar{x}&\in \arg\min_{x\in Q}\{f(x)+\delta_{\mathcal{K}}(G(x))-\phi(\lambda,0)-r\sigma(0)\}\\ &=\arg\min_{x\in Q,G(x)\in\mathcal{K}} f(x)=S.\end{aligned}$$

■

Given $x\in Q$, let

$$\Gamma(x):=\arg\min_{y\in Y}\big\{f(x)+\delta_{\mathcal{K}}(G(x)+y)-\phi(\lambda,y)+r\sigma(y)\big\}.$$

Theorem 6. *If $\lambda\in Y^*$ is a generalized augmented Lagrange multiplier of (P), then for any $r>r(\lambda)$,*

$$S=\{\bar{x}|\,\bar{x}\in\arg\min_{x\in Q}L(x,\lambda,r),\ \Gamma(\bar{x})\neq\emptyset\}.$$

Proof: We first show that

$$S\subseteq\{\bar{x}|\,\bar{x}\in\arg\min_{x\in Q}L(x,\lambda,r),\ \Gamma(\bar{x})\neq\emptyset\}. \tag{27}$$

There is nothing to prove if S is empty. Now take $x^*\in S$. Then $(x^*,0)\in(S,0)$. Note that

$$\begin{aligned}&\arg\min_{(x,y)\in Q\times Y}\{f(x)+\delta_{\mathcal{K}}(G(x)+y)-\phi(\lambda,y)+r\sigma(y)\}\\ &=\left\{(\bar{x},\bar{y})\;\middle|\;\begin{array}{l}\bar{x}\in\arg\min_{x\in Q}\big\{\inf_{y\in Y}f(x)+\delta_{\mathcal{K}}(G(x)+y)\\ \quad-\phi(\lambda,y)+r\sigma(y)\big\},\\ \bar{y}\in\arg\min_{y\in Y}\big\{f(\bar{x})+\delta_{\mathcal{K}}(G(\bar{x})+y)\\ \quad-\phi(\lambda,y)+r\sigma(y)\big\}.\end{array}\right\}\\ &=\left\{(\bar{x},\bar{y})\;\middle|\;\begin{array}{l}\bar{x}\in\arg\min_{x\in Q}L(x,\lambda,r),\\ \bar{y}\in\Gamma(\bar{x}).\end{array}\right\}.\end{aligned} \tag{28}$$

Hence it follows from Lemma 5 that $x^*\in\arg\min_{x\in Q}L(x,\lambda,r)$ and $0\in\Gamma(x^*)$. This shows the validity of (27).

Conversely, pick x^* satisfying $x^*\in\arg\min_{x\in Q}L(x,\lambda,r)$ and $\Gamma(x^*)\neq\emptyset$. Then it follows from (28) and Lemma 5 that $x^*\in S$. ■

V. Sufficient conditions

Definition 5. [26] *Let $C\subseteq Y$. Define*

$$F(x,r,C):=\inf_{y\in C}\{f(x)+\delta_{\mathcal{K}}(G(x)+y)+\delta_Q(x)+r\sigma(y)\},$$

where $x\in X$ and $r\geq 0$. We simplify write $F(x,r,C)$ as $F(x,r)$ if $C=Y$. The penalty function $F(x,r,C)$ is exact if there exists $r\geq 0$ such that $F(x,r,C)\geq val(P)$ for all $x\in Q$.

By extending the inner product and norm used in [26] to more general functions, we can obtain the following results.

Theorem 7. *For $\lambda\in Y^*$, suppose that there exist $\tau>0$ and $d(\lambda)\geq 0$ such that $|\phi(\lambda,y)|\leq d(\lambda)\sigma(y)$ for any $y\in\tau\mathbb{B}_Y$. Then λ is a generalized augmented Lagrange multiplier of (P) for $r>r(\lambda)$ if and only if the penalty function $F(x,r,\tau\mathbb{B})$ is exact and $L(\cdot,\lambda,r)$ is bounded from below in Q.*

Proof: Necessity. Suppose that $\lambda\in Y^*$ is a generalized augmented multiplier of (P). Then according to the nonnegative of $d(\lambda)$ and $\sigma(y)$, for $r>r(\lambda)$, we have

$$\begin{aligned}&f(x)+\delta_{\mathcal{K}}(G(x)+y)-\phi(\lambda,y)+r\sigma(y)\\ &\quad\leq f(x)+\delta_{\mathcal{K}}(G(x)+y)+(d(\lambda)+r)\sigma(y),\quad\forall y\in\tau\mathbb{B}.\end{aligned}$$

Taking the infimum over all $y\in\tau\mathbb{B}_Y$ yields

$$\begin{aligned}&L(x,\lambda,r)\\ &\quad\leq\inf_{y\in\tau\mathbb{B}_Y}\{f(x)+\delta_{\mathcal{K}}(G(x)+y)-\phi(\lambda,y)+r\lambda(y)\}\\ &\quad\leq\inf_{y\in\tau\mathbb{B}_Y}\{f(x)+\delta_{\mathcal{K}}(G(x)+y)+(d(\lambda)+r)\sigma(y)\}\\ &\quad=F(x,d(\lambda)+r,\tau\mathbb{B}_Y).\end{aligned} \tag{29}$$

By Proposition 1, we know $\inf_{x\in Q}L(x,\lambda,r)=val(P)$, which together with (29) implies

$$val(P)\leq F(x,d(\lambda)+r,\tau\mathbb{B}_Y).$$

Therefore, the penalty function $F(x,r,\tau\mathbb{B}_Y)$ is exact and $L(\cdot,\lambda,r)$ is bounded from below in Q.

Sufficiency. If $F(x,r,\tau\mathbb{B})$ is exact, then there exists r' satisfying $r'>d(\lambda)+r$ and

$$\begin{aligned}&F(x,r'-d(\lambda),\tau\mathbb{B})\\ &\quad=\inf_{y\in\tau\mathbb{B}}\{f(x)+\delta_{\mathcal{K}}(G(x)+y)+(r'-d(\lambda))\,\sigma(y)\}\\ &\quad\geq val(P),\quad\forall x\in Q.\end{aligned} \tag{30}$$

Because $\phi(\lambda,y)\leq d(\lambda)\sigma(y)$, we further have

$$\begin{aligned}&f(x)+\delta_{\mathcal{K}}(G(x))-\phi(\lambda,y)+r'\sigma(y)\\ &\quad\geq f(x)+\delta_{\mathcal{K}}(G(x)+y)+(r'-d(\lambda))\sigma(y).\end{aligned}$$

Taking the infimum overall $x\in Q,y\in\tau\mathbb{B}_Y$ and using (30) yields

$$\begin{aligned}&\inf_{x\in Q}\inf_{y\in\tau\mathbb{B}}\{f(x)+\delta_{\mathcal{K}}(G(x)+y)-\phi(\lambda,y)+r'\sigma(y)\}\\ &\quad\geq\inf_{x\in Q}\inf_{y\in\tau\mathbb{B}}\{f(x)+\delta_{\mathcal{K}}(G(x)+y)+(r'-d(\lambda))\sigma(y)\}\\ &\quad=\inf_{x\in Q}F(x,r'-d(\lambda),\tau\mathbb{B})\\ &\quad\geq val(P).\end{aligned}$$

This ensures

$$v(y)-\phi(\lambda,y)+r'\sigma(y)\geq v(0),\quad\forall y\in\tau\mathbb{B}.$$

Together with the boundedness of $L(\cdot, \lambda, r)$, we obtain that λ is a generalized augmented Lagrange multiplier by Theorem 3. ■

Corollary 2. *For $\lambda \in Y^*$, suppose that there exist $\tau > 0$ and $d(\lambda) \geq 0$ such that $|\phi(\lambda, y)| \leq d(\lambda)\sigma(y)$ for any $y \in Y$. Then λ is a generalized augmented Lagrange multiplier of (P) for some $r > r(\lambda)$ if and only if the penalty function $F(x, r)$ is exact.*

Proof: The proof is similar to that of Theorem 7 by just replacing $\tau\mathbb{B}$ by Y. ■

Theorem 8. *Suppose that $\lambda \in Y^*$ is a generalized augmented Lagrange multiplier of (P). Choose a sequence $\{x_n\} \subset Q$ satisfying*

$$L(x_n, \lambda, r_n) \leq \inf_{x \in Q} L(x, \lambda, r_n) + \varepsilon_n, \ \forall n = 1, 2, \ldots, \quad (31)$$

where $r_n \to \infty$ and $\varepsilon_n \to 0$ as $n \to \infty$. Then any accumulate point of $\{x_n\}$ is a global optional solution of (P).

Proof: According to the definition of $L(x_n, \lambda, r_n)$, there exists $\{y_n\}$ such that

$$\begin{aligned} & f(x_n) + \delta_{\mathcal{K}}(G(x_n) + y_n) - \phi(\lambda, y_n) + r_n\sigma(y_n) \\ & \quad \leq L(x_n, \lambda, r_n) + \varepsilon_n, \ \forall n = 1, 2, \ldots. \end{aligned}$$

Since λ is a generalized augmented Lagrangian multiplier, then $\inf_{x \in Q} L(x, \lambda, r_n) = val(P)$ as $r_n \geq r(\lambda)$. So $L(x_n, \lambda, r_n) \leq val(P) + \varepsilon_n$ by (31). Hence

$$\begin{aligned} val(P) = & \inf_{x \in Q} L(x, \lambda, r(\lambda)) \\ \leq & f(x_n) + \delta_{\mathcal{K}}(G(x_n) + y_n) - \phi(\lambda, y_n) \\ & + r(\lambda)\sigma(y_n) \\ = & f(x_n) + \delta_{\mathcal{K}}(G(x_n) + y_n) - \phi(\lambda, y_n) \\ & + r_n\sigma(y_n) + (r(\lambda) - r_n)\sigma(y_n) \\ \leq & L(x_n, \lambda, r_n) + \varepsilon_n + (r(\lambda) - r_n)\sigma(y_n) \\ \leq & val(P) + 2\varepsilon_n + (r(\lambda) - r_n)\sigma(y_n), \end{aligned} \quad (32)$$

implying

$$0 \leq (r_n - r(\lambda))\sigma(y_n) \leq 2\varepsilon_n.$$

Thus $(r_n - r(\lambda))\sigma(y_n) \to 0$ and $y_n \to 0$ due to the valley property of σ. Using this fact to the second step in (32) leads to

$$G(x^*) \in \mathcal{K} \ \text{ and } \ f(x^*) = val(P).$$

So x^* is a global optimal solution. ■

For a given $\varepsilon \geq 0$, let us define

$$E_1(\varepsilon) := \{x \in Q | \text{dist}(G(x), \mathcal{K}) \leq \varepsilon\}$$

and

$$E_2(\varepsilon) := \{x \in Q | f(x) \leq v(0) + \varepsilon\}.$$

Theorem 9. *Assume that $\tilde{\lambda}$ satisfies $L(\cdot, \tilde{\lambda}, r)$ is bounded from below and the set $W(\tilde{\lambda}, r) := \{x \in Q | L(x, \tilde{\lambda}, r) \leq v(0)\}$ is bounded for some $r > 0$. If for each $\tilde{x} \in S$, $(\tilde{x}, \tilde{\lambda})$ is a local saddle point of $L(x, \lambda, r)$, then $\tilde{\lambda}$ is a generalized augmented Lagrange multiplier of (P).*

Proof: Pick $\tilde{x} \in S$. By assumption, $(\tilde{x}, \tilde{\lambda})$ is a local saddle point, i.e., there exist $\delta > 0$ and $\tilde{r} > 0$ such that

$$L(\tilde{x}, \lambda, \tilde{r}) \leq L(\tilde{x}, \tilde{\lambda}, \tilde{r}) \leq L(x, \tilde{\lambda}, \tilde{r}), \quad (33)$$

for all $x \in \mathbb{B}_X(\tilde{x}, \delta) \cap Q$ and $\lambda \in Y$. Taking into account of the first inequality above and (8), we know

$$L(\tilde{x}, \tilde{\lambda}, \tilde{r}) = f(\tilde{x}). \quad (34)$$

Because L is non-decreasing in r, for any $r \geq \tilde{r}$

$$L(\tilde{x}, \tilde{\lambda}, \tilde{r}) \leq L(\tilde{x}, \tilde{\lambda}, r) \leq f(\tilde{x}), \quad (35)$$

where the second inequality follows from (7) since $G(\tilde{x}) \in \mathcal{K}$. Combining (34) and (35) yields

$$L(\tilde{x}, \lambda, r) \leq f(\tilde{x}) \leq L(\tilde{x}, \tilde{\lambda}, r), \quad \forall r \geq \tilde{r}, \ \lambda \in Y^*. \quad (36)$$

Next let us show that

$$L(\tilde{x}, \tilde{\lambda}, r) \leq L(x, \tilde{\lambda}, r), \quad \forall x \in Q \backslash \mathbb{B}_X(\tilde{x}, \delta), \quad (37)$$

whenever r is sufficiently large. We prove it by contradiction. If (37) is invalid, then there exist $r_k \to +\infty$ and $x_k \in Q \backslash \mathbb{B}_X(\tilde{x}, \delta)$ such that

$$L(x_k, \tilde{\lambda}, r_k) < L(\tilde{x}, \tilde{\lambda}, r_k) = f(\tilde{x}) = v(0), \quad (38)$$

where the first equation follows from the fact that $L(\tilde{x}, \tilde{\lambda}, r) = f(\tilde{x})$ for all $r \geq \tilde{r}$ by (35) and (36).

According to (38), we know $x_k \in \{x \in Q | L(x, \tilde{\lambda}, r_k) < v(0)\} \subset W(\tilde{\lambda}, r)$. Since $W(\tilde{\lambda}, r)$ is bounded by assumption, there exists a cluster point of sequence $\{x_k\}$, say $\overline{x}$.

For any $x \in Q \setminus E_1(\varepsilon)$ we have $dist(G(x), \mathcal{K}) > \varepsilon$. Hence for any $y \in \mathcal{K} - G(x)$, we have

$$\|y\| \geq dist(G(x), \mathcal{K}) > \varepsilon.$$

Since σ has valley property, then there exists $\eta > 0$ such that $\sigma(u) \geq \eta$ for all $\|u\| \geq \varepsilon$. Hence

$$\begin{aligned} & L(x, \tilde{\lambda}, r') \\ & = \inf_{y \in \mathcal{K} - G(x)} \{f(x) - \phi(\tilde{\lambda}, y) + r'\sigma(y)\} \\ & = \inf_{y \in \mathcal{K} - G(x)} \{f(x) - \phi(\tilde{\lambda}, y) + r\sigma(y) + (r' - r)\sigma(y)\} \\ & \geq \inf_{y \in \mathcal{K} - G(x)} \{f(x) - \phi(\tilde{\lambda}, y) + r\sigma(y)\} \\ & \quad + (r' - r) \inf_{y \in \mathcal{K} - G(x)} \sigma(y) \\ & \geq L(x, \tilde{\lambda}, r) + (r' - r)\eta. \end{aligned}$$

Taking the limit as $r' \to \infty$ yields

$$\begin{aligned} & \lim_{r' \to +\infty} \inf_{x \in Q \setminus E_1(\varepsilon)} L(x, \tilde{\lambda}, r) \\ & \quad \geq \inf_{x \in Q} L(x, \tilde{\lambda}, r) + \lim_{r' \to +\infty} (r' - r)\eta \\ & \quad = +\infty. \end{aligned}$$

Now we show that $W(\tilde{\lambda}, r) \subset E_1(\varepsilon)$ as r sufficiently large. If not there must exist $r'_k \to +\infty$ and $x_k \in Q$ satisfy $x_k \in W(\tilde{\lambda}, r'_k)$ while $x_k \notin E_1(\varepsilon)$. So by replacing r' by r_k in (39) we have

$$\begin{aligned} v(0) & \geq \liminf_{k \to +\infty} L(x_k, \tilde{\lambda}, r_k) \\ & \geq \liminf_{k \to +\infty} \inf_{x \in Q \setminus E_1(\varepsilon_0)} L(x, \tilde{\lambda}, r_k) \\ & = +\infty, \end{aligned}$$

which contradicts the finiteness of $v(0)$. Thus $W(\tilde{\lambda}, r) \subseteq E_1(\varepsilon)$ as $r > 0$ sufficiently large.

Similarly we can show that $W(\tilde{\lambda}, r) \subseteq E_2(\varepsilon)$ as r sufficiently large. In fact, if there exist $r_k \to \infty$ and $x_k \in Q$

such that $L(x_k, \tilde{\lambda}, r_k) \leq v(0)$, but $x \notin E_2(\varepsilon)$, then according to (7) we can find $\varepsilon_0 > 0$ and $y_k \in \mathcal{K} - G(x_k)$ such that

$$\begin{aligned} v(0) + \frac{\varepsilon}{2} &\geq L(x_k, \tilde{\lambda}, r_k) + \frac{\varepsilon}{2} \\ &\geq f(x_k) - \phi(\tilde{\lambda}, y_k) + r_k \sigma(y_k) \\ &\geq L(x_k, \tilde{\lambda}, r_0) + (r_k - r_0)\sigma(y_k). \end{aligned} \tag{39}$$

Hence $\sigma(y_k) \to 0$ as $k \to 0$ since $r_k \to +\infty$. So $y_k \to 0$ by the valley-at-zero property of σ, which in turn implies $\phi(\tilde{\lambda}, y_k) \to 0$. It follows from (39) that

$$v(0) + \frac{\varepsilon}{2} \geq f(x_k) - \phi(\tilde{\lambda}, y_k).$$

Since $y_k \to 0$ as shown above, then $f(x_k) \leq v(0) + \varepsilon$. So $x_k \in E_2(\varepsilon)$. This proves $W(\tilde{\lambda}, r) \subseteq E_2(\varepsilon)$.

From the above discussion, we know that

$$W(\lambda, r) \subseteq E_1(\varepsilon) \cap E_2(\varepsilon).$$

So $\overline{x} \in E_1(\varepsilon) \cap E_2(\varepsilon)$. By the arbitrariness of $\varepsilon > 0$, we have $\overline{x} \in E_1(0) \cap E_2(0)$, implying $\overline{x} \in S$. Hence $(\bar{x}, \tilde{\lambda})$ is also a local saddle point of $L(x, \lambda, r)$ for some $\overline{r} > 0$, i.e., there exists $\overline{\delta} > 0$ such that

$$L(\overline{x}, \lambda, \overline{r}) \leq L(\overline{x}, \tilde{\lambda}, \overline{r}) \leq L(x, \tilde{\lambda}, \overline{r}), \tag{40}$$

for all $x \in B_X(\overline{x}, \overline{\delta}) \cap Q$ and $\lambda \in Y^*$. By Lemma 1

$$L(\overline{x}, \tilde{\lambda}, \overline{r}) = f(\overline{x}) = val(P). \tag{41}$$

Since $x_k \in \mathbb{B}_X(\tilde{x}, \overline{\delta})$ and $r_k \geq \overline{r}$ for k large enough, by (40) and (41)

$$f(\tilde{x}) = f(\overline{x}) = L(\overline{x}, \tilde{\lambda}, \overline{r}) \leq L(x_k, \tilde{\lambda}, r_k),$$

which contradicts (38). So (37) holds. Putting (33), (36) and (37) together shows that $(\tilde{x}, \tilde{\lambda})$ is a global saddle point of $L(x, \lambda, r)$ for some sufficiently large r. Recalling to Theorem 2 obtains that $\tilde{\lambda}$ is a generalized augmented Lagrange multiplier of (P). ■

REFERENCES

[1] J. Rckmann and Alexander Shapiro, "Augmented Lagrangians in semi-infinite programming," Mathematical Programming, vol. 116, no. 1, pp499-512, 2009.

[2] R. S. Burachik, X. Q. Yang and Y. Y. Zhou, "Existence of augmented Lagrange multipliers for semi-infinite programming problems," Journal of Optimization Theory and Applications, vol. 173, no. 2, pp471-503, 2017.

[3] J. C. Zhou and J. S. Chen, "On the existence of saddle points for nonlinear second-order cone programming problems," Journal of Global Optimization, vol. 62, no. 3, pp459-480, 2015.

[4] C. Kan and W. Song, "Second-order conditions for existence of augmented Lagrange multipliers for eigenvalue composite optimization problems," Journal of Global Optimization, vol. 63, no. 1, pp77-97, 2015.

[5] M. V. Dolgopolik, "Augmented Lagrangian functions for cone constrained optimization: the existence of global saddle points and exact penalty property," Journal of Global Optimization, vol. 71, no. 2, pp237-296, 2018.

[6] A. Shapiro and J. Sun, "Some properties of the augmented Lagrangian in cone constrained optimization," Mathematics of Operations Research, vol. 29, no. 3, pp479-491, 2004.

[7] Y. Y. Zhou, J. C. Zhou and X. Q. Yang, "Existence of augmented Lagrange multipliers for cone constrained optimization problems," Journal of Global Optimization, vol. 58, no. 2, pp243-260, 2014.

[8] Y. Wang, J. C. Zhou and J. Y. Tang, "Existence of generalized augmented Lagrange multipliers for constrained optimization problems," Mathematical and Computational Applications, vol. 25, no. 2, pp1-20, 2020.

[9] E. G. Birgin and J. M. Martinez, "Practical augmented Lagrangian methods for constrained optimization," Society for Industrial and Applied Mathematics, 2014.

[10] M. V. Dolgopolik, "Minimax exactness and global saddle points of nonlinear augmented lagrangians," Journal of Applied and Numerical Optimization, vol. 3, no. 1, pp61-83, 2021.

[11] D. Fernandez and M. V. Solodov, "On the cost of solving augmented Lagrangian subproblems," Mathematical Programming, vol. 182, no. 1, pp37-55, 2020.

[12] X. Gu, A. Shabbir and S. S. Dey, "Exact augmented Lagrangian duality for mixed integer quadratic programming," SIAM Journal on Optimization, vol. 30, no. 1, pp781-797, 2020.

[13] C. Kanzow and D. Steck, "On error bounds and multiplier methods for variational problems in Banach spaces," SIAM Journal on Optimization, vol. 56, no. 3, pp1716-1738, 2018.

[14] Y. F. Lian, J. C. Zhou, J. T. Tang and X. Liu, "On the existence of saddle points for l_1-minimization problems," IAENG International Journal of Applied Mathematics, vol. 51, no. 1, pp48-56, 2021.

[15] C. Y. Wang, X. Q. Yang and X. M. Yang, "Nonlinear augmented Lagrangian and duality theory," Mathematics of Operations Research, vol. 38, no. 4, pp740-760, 2013.

[16] R. Andreani, G. Haeser, M. L. Schuverdt, L. D. Secchin and P. J. S. Silva, "On scaled stopping criteria for a safeguarded augmented Lagrangian method with theoretical guarantees," Mathematical Programming Computation, vol. 14, no.1, pp121-146, 2022.

[17] E. G. Birgin, G. Haeser and A. Ramos, "Augmented Lagrangians with constrained subproblems and convergence to second-order stationary points," Computational Optimization and Application, vol. 69, no. 1, pp51-75, 2018.

[18] B. Dandurand, N. Boland, J. Christiansen, A. Eberhard and F. Oliveira, "A parallelizable augmented Lagrangian method applied to large-scale non-convex-constrained optimization problems," Mathematical Programming, vol. 175, no. 1, pp503-536, 2019.

[19] M. V. Dolgopolik, "A unified approach to the global exactness of penalty and augmented Lagrangian functions I: parametric exactness," Journal of Optimization Theory and Applications, vol. 176, no. 3, pp728-744, 2018.

[20] M. V. Dolgopolik, "A unified approach to the global exactness of penalty and augmented Lagrangian functions II: extended exactness," Journal of Optimization Theory and Applications, vol. 176, no. 3, pp745-762, 2018.

[21] D. Fernandez and M. V. Solodov, "Local convergence of exact and inexact augmented Lagrangian methods under the second-order sufficient optimality condition,"

SIAM Journal on Optimization, vol. 22, no. 2, pp384-407, 2012.

[22] A. F. Izmailov, A. S. Kurennoy and M. V. Solodov, "Local convergence of the method of multipliers for variational and optimization problems under the non-criticality assumption," Computational Optimization and Application, vol. 60, no. 1, pp111-140, 2015.

[23] C. Kanzow and D. Steck, "Quasi-variational inequalities in Banach spaces: theory and augmented Lagrangian methods," SIAM Journal on Optimization, vol. 29, no. 4, pp3174-3200, 2019.

[24] C. Kanzow and D. Steck, "Improved local convergence results for augmented Lagrangian methods in C^2-cone reducible constrained optimization," Mathematical Programming, vol. 177, no. 1, pp425-438, 2019.

[25] G. H. Lan and R. D. C. Monteiro, "Iteration-complexity of first-order augmented Lagrangian methods for convex programming," Mathematical Programming, vol. 155, no. 1, pp511-547, 2016.

[26] M. V. Dolgopolik, "Existence of augmented Lagrange multipliers: reduction to exact penalty functions and localization principle," Mathematical Programming, vol. 166, no. 1, pp297-326, 2017.

[27] R. T. Rockafellar, "Convex Analysis," Princeton University Press, Princeton, 1970.

[28] R. T. Rockafellar and R. J. B. Wets, "Variational Analysis," Springer, Berlin, 1998.