Wiener Index of Corona of Wheel Related Graphs with Wheel Graph

K Arathi Bhat*

***Abstract*—The Wiener index W is the sum of distances between all pairs of vertices of a (connected) graph. The corona of two graphs G and H is denoted by $G \circ H$. The graphs which could be obtained by modifying the given wheel graph are also known as wheel related graphs. In this paper, we shall obtain the formula for $W(G \circ W_{1,n-1})$, when G is a wheel related graph and $W_{1,n-1}$ is a wheel graph on n vertices.**

***Index Terms*—Wiener index, distance, corona of two graphs, wheel related graphs**

I. Introduction

We consider graphs which are finite, undirected, connected, without loops and multiple edges. The vertex and edge sets of a graph G are $V(G)$ and $E(G)$, respectively. The distance $d_G(u, v)$ or $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between u and v in G. For any two vertices v_i and $v_j, i \neq j$ in a graph G, $v_i \sim v_j$ denotes that the vertices are adjacent in the graph G, and $v_i \nsim v_j$ denotes that the vertices are not adjacent in the graph G.

A graph invariant is a real number related to a graph G which is invariant under graph isomorphism, that is it does not depend on the labeling or the pictorial representation of a graph. In chemistry, graph invariants are known as topological indices. Topological indices have many applications as tools for modeling chemical and other properties of molecules. The Wiener index is the first and most studied of the distance-based topological indices, both from a theoretical point of view and applications. This index was the first topological index to be used in chemistry which is defined as follows.

Definition I.1. *The Wiener index is defined as the sum of distances between all unordered pair of vertices of a graph G, i.e.,*

$$W(G) = \sum_{u,v \in V(G)} d(u, v)$$

where d(u,v) is the distance between u and v in G.

In current times the researchers are more fascinated towards the graph operations such as product. Among many existent graph operations researchers are showing more interest in corona product because of its structure. Since corona product one graph to itself creates replicas of the graph which generates a star like structure. Also, different important class of graphs can be obtained by corona product of some general and particular graphs. The formal mathematical definition of corona product of two graphs is as follows.

Definition I.2. *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices. The corona $G_1 \circ G_2$ is obtained by taking one copy of G_1 and n_1 copies of G_2, and by joining each vertex of the i^{th} copy of G_2 to the i^{th} vertex of G_1, $i = 1, 2, \ldots, n_1$.*

For other graph theoretic terminologies used here, we refer [1].

Several papers contributed to determine the Wiener index of special graphs ([2], [3], [4]). Computing topological indices of graph operations has been the object of some papers [5], [6]. Yeh and Gutman [7] computed the Wiener index in the case of graphs that are obtained by means of certain binary operations (such as product, join, and composition) on pairs of graphs. Stevanović [8] generalized their results and computed the Wiener polynomial (and hence Wiener and hyper-Wiener indices) of product, join, and composition of graphs. Authors of the article [9], determined the Wiener index of graphs which are constructed by some operations such as Mycielski's construction, generalized hierarchical product and t-th subdivision of graphs. The explicit formula for the Wiener index of the corona of two graphs are available in the article [10]. The formula for $W(C_m \circ C_n)$ is obtained by the authors of the article [11]. In this article, we obtain the Wiener index of the corona of wheel related graph and wheel graph.

II. Wheel Related Graphs

The helm graph, gear graph, friendship graph, flower graph, sunflower graph and fan graph which could be obtained by modifying the given wheel graph are also known as wheel related graphs. We obtain Wiener index of these wheel related graphs.

We consider the wheel graph $W_{1,n}$ with peripheral vertices $v_1, v_2, \ldots, v_n$ and central vertex v_{n+1}. The peripheral vertices are such that $v_i \sim v_{i+1},\ 1 \leq i \leq n-1$ and $v_n \sim v_1$. The central vertex $v_{n+1} \sim v_i,\ 1 \leq i \leq n$. The following note gives the Wiener index of a wheel graph.

Note II.1. *Wiener index of wheel graph $W_{1,n}$ is given by $n^2 - n$.*

A. Helm Graph

The helm graph is the graph H_n obtained from the wheel $W_{1,n}$ by adding the pendant vertices $\{u_1, u_2, \ldots, u_n\}$ where, $u_i \sim v_i$, $1 \leq i \leq n$.

The helm graph H_n has $2n + 1$ vertices and $3n$ edges as shown in Figure 1.

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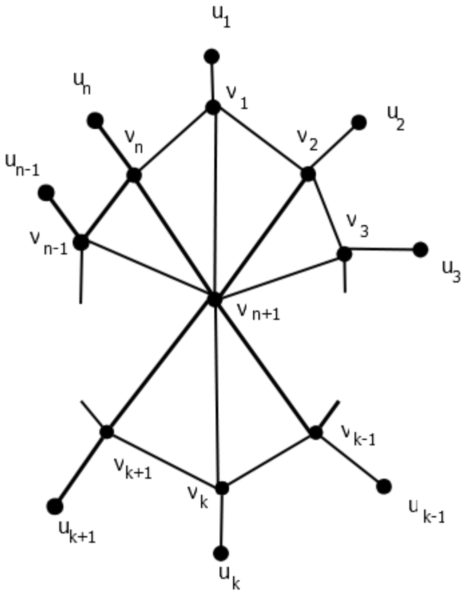

Fig. 1: Helm graph H_n

Theorem II.2. $W(H_n) = 6n(n-1)$.

Proof: For each u_i, $1 \leq i \leq n$, we note the following; $d(u_i, v_i) = 1$, $d(u_i, v_j) = 2, j = i-1, i+1, n+1$, $d(u_i, v_j) = 3, j \neq i, i-1, i+1$, $d(u_i, u_k) = 3, k = i-1, i+1$ and $d(u_i, u_k) = 4, k \neq i-1, i, i+1$.
Similarly, for each vertex v_i, $1 \leq i \leq n$, we note the following;
$d(v_i, v_k) = 1, k = i-1, i+1, n+1$, $d(v_i, u_i) = 2, j = i-1, i+1, n+1$, $d(u_i, v_j) = 3, j \neq i, i-1, i+1$, $d(u_i, u_k) = 3, k = i-1, i+1$ and $d(u_i, u_k) = 4, k \neq i-1, i, i+1$.
Also, $d(v_{n+1}, v_i) = 1, 1 \leq i \leq n$ and $d(v_{n+1}, u_i) = 2, 1 \leq i \leq n$. Hence the result follows. ■

B. Gear Graph

A gear graph, denoted by G_n is a graph obtained from the wheel graph $W_{1,n}$, by removing the edge $v_i v_{i+1}$ and adding vertex u_i, where each u_i is made adjacent to v_i and v_{i+1} for all $i, 1 \leq i \leq n-1$. Also, remove the edge $v_n v_1$ and add the vertex u_n and the vertex u_n is made adjacent to v_n and v_1. Thus, G_n has $2n+1$ vertices and $3n$ edges as shown in Figure 2.

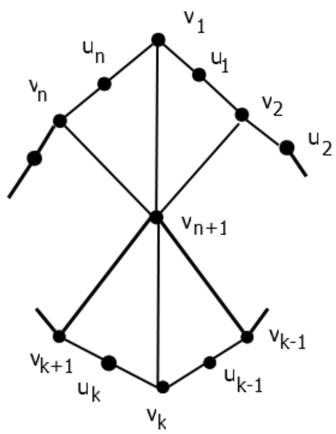

Fig. 2: Gear graph G_n

Theorem II.3. $W(G_n) = 6n(n-1)$.

Proof: For any vertex $v_i, 1 \leq i \leq n$, there are three vertices at distance one, $n-1$ vertices at distance two and $n-2$ vertices at distance three. Similarly, for any vertex $u_i, 1 \leq i \leq n$, there are two vertices at distance one, three vertices at distance two, $n-2$ vertices at distance three and $n-3$ vertices at distance four. From the vertex v_{n+1}, there are n vertices at distance one and n vertices at distance two. Hence the result follows. ■

Note II.4. *We note that* $W(H_n) = W(G_n)$.

C. Friendship Graph

The friendship graph F_n is a graph with $2n+1$ vertices and $3n$ edges, obtained from the wheel graph $W_{1,2n}$, by removing every alternating edges on the circumference of the wheel as shown in Figure 3. Let the vertices of wheel graph $W_{1,2n}$ be $v_1, v_2, \ldots, v_{2n}, v_{2n+1}$. The edges $v_i v_{i+1}, i = 2, 4, \ldots, 2n-2$ and the edge $v_{2n} v_1$ is removed from the wheel graph $W_{1,2n}$.

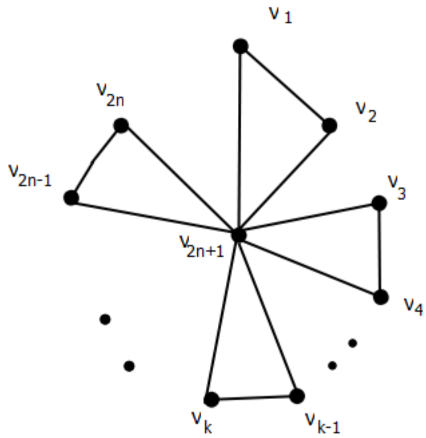

Fig. 3: Friendship graph F_n

Theorem II.5. $W(F_n) = n(4n-1)$.

Proof: Let the vertices of friendship graph be $v_1, v_2, \ldots, v_{2n}, v_{2n+1}$. From each v_i, $1 \leq i \leq 2n$ there are 2 vertices at distance one and $2(n-1)$ vertices at distance two. From v_{2n+1}, all the $2n$ vertices are at distance one. Hence the result follows. ■

D. Flower Graph

The flower graph $Fl_n^{(3)}$ is obtained from the wheel graph $W_{1,n}$, by adding the vertices u_i, and the vertex u_i is made adjacent with v_i and v_{n+1}, for all i, $1 \leq i \leq n$ as shown in Figure 4.

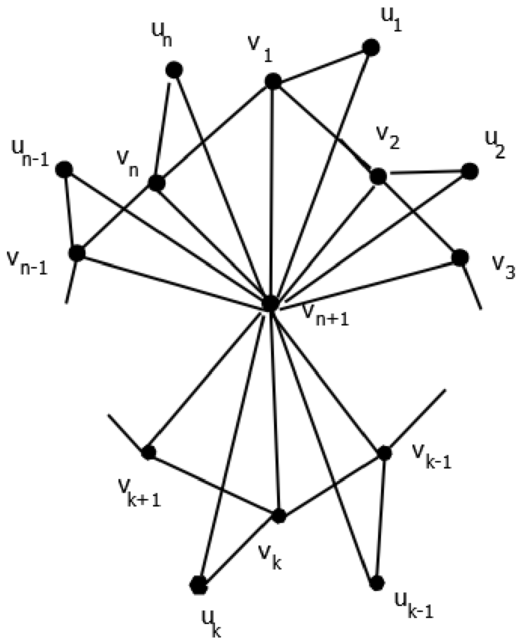

Fig. 4: Flower graph $Fl_n^{(3)}$

Theorem II.6. $W(Fl_n^{(3)}) = n(4n-2)$.

Proof: Let the vertices of flower graph be $v_1, v_2, \ldots, v_n, u_1, \ldots, u_n, v_{n+1}$. From each $v_i,\ 1 \le i \le n$ there are four vertices at distance one, $2(n-3)+2$ vertices at distance two. From each $u_i,\ 1 \le i \le n$ there are two vertices at distance one, $2(n-1)$ vertices at distance two. From v_{n+1}, all the $2n$ vertices are at distance one. Hence, $W(Fl_n^{(3)}) = n(4n-2)$. ■

E. Sunflower Graph

The sunflower graph SF_n is obtained from the wheel graph $W_{1,n}$, by adding the vertices u_i, where each u_i is adjacent to v_i and v_{i+1}, for all $i,\ 1 \le i \le n-1$ and adding the vertex u_n which is made adjacent to v_n and v_1 as shown in Figure 5.

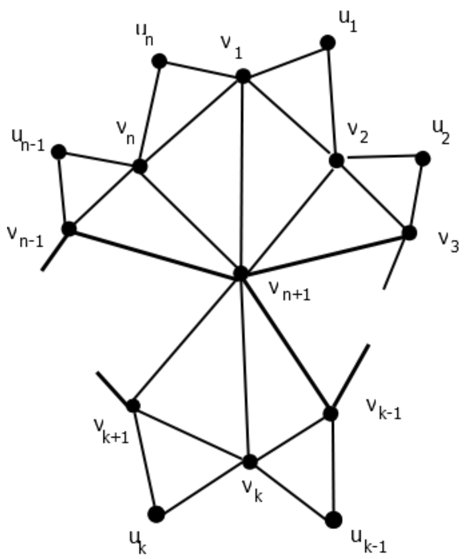

Fig. 5: Sunflower graph SF_n

Theorem II.7. $W(SF_n) = n(6n-10)$.

Proof: Let the vertices of sunflower graph be $v_1, v_2, \ldots, v_n, u_1, \ldots, u_n, v_{n+1}$. From each $v_i, 1 \le i \le n$ there are five vertices at distance one, $(n-3)+2$ vertices at distance two and $n-4$ vertices at distance three. From each $u_i, 1 \le i \le n$ there are two vertices at distance one, five vertices at distance two, $n-2$ vertices at distance three and $n-5$ vertices at distance four. From v_{n+1}, the vertices $v_i, 1 \le i \le n$ are at distance one and a the vertices $u_i,\ 1 \le i \le n$ are at distance two. Hence, we get the desired result. ■

F. Fan Graph

The fan graph $F_{1,n}$ is obtained from the Wheel graph $W_{1,n}$ by removing the edge between the vertices v_1 and v_n. Fan graph $F_{1,n}$ has $n+1$ vertices and $2n-1$ edges as shown in Figure 6.

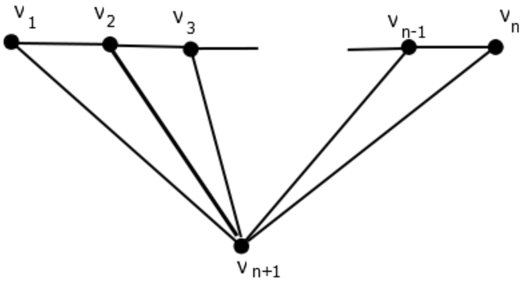

Fig. 6: Fan graph $F_{1,n}$

Theorem II.8. $W(F_{1,n}) = n^2 - n + 1$.

Proof: The number of edges in $F_{1,n}$ is $2n-1$. There are $\dfrac{n(n-1)}{2} - (n-1)$ pairs of vertices at distance two. Hence the result follows. ■

III. Wheel related graphs corona with wheel graph

In this section we obtain the expression for Wiener index of various wheel related graphs corona with the wheel graph. First we note the following observation.

Note III.1. *Let G and H be two graphs on m and n vertices respectively. Then, $W(G \circ H) = W(G) + mW(H) + K$, where K is equal to sum of distances of all pairs of vertices such that one vertex is from G and the other vertex is from any copies of the graph H or two vertices belongs to two different copies of H.*

A. Helm graph H_{n_1} corona with wheel graph W_{1,n_2-1}

In the following theorem we determine the expression for Wiener index of helm graph H_{n_1} corona with wheel graph W_{1,n_2-1}.

Theorem III.2. *Wiener index of helm graph H_{n_1} corona with wheel graph W_{1,n_2-1} is equal to $W(H_{n_1}) + (2n_1+1)W(W_{1,n_2-1}) + n_2 + 10n_1^2n_2^2 + 16n_1^2n_2 - 4n_1n_2^2 - 8n_1n_2$.*

Proof: From Note III.1, we have $W(H_{n_1} \circ W_{1,n_2-1}) = W(H_{n_1}) + (2n_1+1)W(W_{1,n_2-1}) + K$.

To find the value of K, we will consider sum of distances from all pairs of vertices such that one vertex is from the helm graph and the other vertex is from any copies of the wheel graph or two vertices belongs to two different copies of wheel graph.
Let $v_1, v_2, \ldots, v_{n_1}$ be pendant vertices and $v_{n_1+1}, v_{n_1+2}, \ldots, v_{2n_1}$ be the peripheral vertices and v_{2n_1+1} be the central vertex of helm graph H_{n_1}. The vertices in the i^{th} copy of wheel graph W_{1,n_2-1} are labeled as $v_{ij}, 1 \leq i \leq 2n_1+1, 1 \leq j \leq n_2$.
For any vertex $v_{ij},\ 1 \leq i \leq n_1,\ 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, one vertex at distance two, n_2+3 vertices at distance three, $3n_2+(n_1-3)+2$ vertices at distance four, $(n_1-3)+(2+(n_1-3))n_2$ vertices at distance five and $(n_1-3)n_2$ vertices at distance six, which lies in the other copies of the wheel graph or the helm graph.
For any vertex $v_{ij},\ n_1+1 \leq i \leq 2n_1,\ 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, four vertices at distance two, $(n_1-3)+2+4n_2$ vertices at distance three, $(2+(n_1-3))n_2+(n_1-3)$ vertices at distance four and $(n_1-3)n_2$ vertices at distance five, which lies in the other copies of the wheel graph or the helm graph.
For any vertex $v_j, 1 \leq j \leq n_1$, there are n_2 vertices at distance one, n_2 vertices at distance two, $3n_2$ vertices at distance three, $(2+(n_1-3))n_2$ vertices at distance four and $(n_1-3)n_2$ vertices at distance five.
Similarly, for any vertex $v_j,\ n_1+1 \leq j \leq 2n_1$, there are n_2 vertices at distance one, $4n_2$ vertices at distance two, $(2+n_1-3)n_2$ vertices at distance three and $n_2(n_1-3)$ vertices at distance four.
For any vertex $v_{2n_1+1j},\ 1 \leq j \leq n_2$, there is one vertex at distance one, n_1 vertices at distance two, $n_1n_2+n_1$ vertices at distance three and n_1n_2 vertices at distance four.
For the vertex v_{2n+1}, there are n_2 vertices at distance one, n_1n_2 vertices at distance two and n_1n_2 vertices at distance three.
Hence, $K = n_2 + 10n_1^2n_2^2 + 16n_1^2n_2 - 4n_1n_2^2 - 8n_1n_2$, from which the desired result follows. ■

Corollary III.3. $W(H_{n_1} \circ W_{1,n_2-1}) = 10n_1^2n_2^2 + 16n_1^2n_2 - 2n_1 - 14n_1n_2 - 2n_1n_2^2 - 2n_2 + 6n_1^2 + n_2^2 + 2$.

Proof: Proof follows by Theorem III.2, Theorem II.2 and Note II.1. ■

B. Gear graph G_{n_1} corona with wheel graph W_{1,n_2-1}

We obtain the expression for Wiener index of gear graph G_{n_1} corona with wheel graph W_{1,n_2-1} in the following theorem.

Theorem III.4. *Wiener index of gear graph G_{n_1} corona with wheel graph W_{1,n_2-1} is equal to $W(G_{n_1}) + (2n_1+1)W(W_{1,n_2-1}) + n_2 + 10n_1^2n_2^2 + 16n_1^2n_2 - 4n_1n_2^2 - 8n_1n_2$.*

Proof: From Note III.1, we have $W(G_{n_1} \circ W_{1,n_2-1}) = W(G_{n_1}) + (2n_1+1)W(W_{1,n_2-1}) + K$.
To find the value of K we will consider sum of all pairs of distances such that one vertex is from the gear graph and the other vertex is from any copies of the wheel graph or two vertices belongs to two different copies of wheel graph.
Let $v_1, v_2, \ldots, v_{n_1}$ be peripheral vertices of gear graph G_{n_1}. By removing the edges v_iv_{i+1} and adding new vertices be labeled as v_{n_1+i} where each v_{n_1+i} is adjacent to v_i and v_{i+1} for all $i, 1 \leq i \leq n_1-1$. Also, by removing the edge $v_{n_1}v_1$ and adding the new vertex be labeled as v_{2n_1} and v_{2n_1} is made adjacent to v_{n_1} and v_1. The central vertex be labeled as v_{2n_1+1} which is adjacent to $v_i,\ 1 \leq i \leq n_1$. The vertices in the i^{th} copy of wheel graph W_{1,n_2-1} are labeled as $v_{ij},\ 1 \leq i \leq 2n_1+1, 1 \leq j \leq n_2$.
For any vertex $v_{ij}, 1 \leq i \leq n_1,\ 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, three vertices at distance two, $(3n_2+n_1-1)$ vertices at distance three, $(n_1-2)+(n_1-1)n_2$ vertices at distance four and $(n_1-2)n_2$ vertices at distance five, which lies in the other copies of the wheel graph or the gear graph.
For any vertex $v_{ij}, n_1+1 \leq i \leq 2n_1,\ 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, two vertices at distance two, $2n_2+3$ vertices at distance three, $3n_2+(n_1-2)$ vertices at distance four, $(n_1-2)n_2+n_1-3$ vertices at distance five and $(n_1-3)n_2$ vertices at distance six.
For any vertex $v_j,\ 1 \leq j \leq n_1$, we note the following.
There are n_2 vertices at distance one, $3n_2$ vertices at distance two, $(n_1-1)n_2$ vertices at distance three and $(n_1-2)n_2$ vertices at distance four.
For any vertex $v_j,\ n_1+1 \leq j \leq 2n_1$, we note the following.
There are n_2 vertices at distance one, $2n_2$ vertices at distance two, $3n_2$ vertices at distance three, $n_2(n_1-2)$ vertices at distance four and $n_2(n_1-3)$ vertices at distance five.
For any vertex $v_{2n_1+1j},\ 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, n_1 vertices at distance two, $n_1n_2+n_1$ vertices at distance three and n_1n_2 vertices at distance four.
For the vertex v_{2n+1}, we note the following;
There are n_2 vertices at distance one, n_1n_2 vertices at distance two and n_1n_2 vertices at distance three.
Hence, $K = n_2 + 10n_1^2n_2^2 + 16n_1^2n_2 - 4n_1n_2^2 - 8n_1n_2$. ■

Corollary III.5. $W(G_{n_1} \circ W_{1,n_2-1}) = 16n_1^2n_2 + 10n_1^2n_2^2 - 2n_1 - 14n_1n_2 - 2n_1n_2^2 - 2n_2 + 6n_1^2 + n_2^2 + 2$.

Proof: Proof follows by Theorem III.4, Theorem II.3 and Note II.1. ■

Note III.6. *One can note that $W(H_{n_1} \circ W_{1,n_2-1}) = W(G_{n_1} \circ W_{1,n_2-1})$.*

C. Friendship graph F_{n_1} corona with wheel graph $W_{,n_2-1}$

We obtain the expression for Wiener index of friendship graph F_{n_1} corona with wheel graph W_{1,n_2-1} in the following theorem.

Theorem III.7. *Wiener index of friendship graph F_{n_1} corona with wheel graph W_{1,n_2-1} is equal to $W(F_{n_1}) + (2n_1+1)W(W_{1,n_2-1}) + n_2 + 8n_1^2n_2^2 + 12n_1^2n_2 + n_1n_2^2 + 2n_1n_2$.*

Proof: From Note III.1, we have $W(F_{n_1} \circ W_{1,n_2-1}) = W(F_{n_1}) + (2n_1+1)W(W_{1,n_2-1}) + K$.
To find the value of K we will consider sum of distances from all pairs of vertices, such that the other vertex belongs

to the other copies of the wheel graph or the friendship graph.
Let $v_1, v_2, \ldots, v_{2n_1}$ be the vertices of friendship graph F_{n_1} and v_{2n_1+1} be the central vertex. Let the vertices in the i^{th} copy of wheel graph W_{1,n_2-1} be labeled as $v_{ij},\ 1 \leq i \leq 2n_1+1, 1 \leq j \leq n_2$.
For any vertex $v_{ij},\ 1 \leq i \leq 2n_1,\ 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, two vertices at distance two, $(2n_2+2(n_1-1))$ vertices at distance three and $2(n_1-1)n_2$ vertices at distance four which lies in the other copies of the wheel graph or the friendship graph.
For any vertex $v_{2n_1+1j},\ 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, $2n_1$ vertices at distance two and $2n_1n_2$ vertices at distance three which are in the other copies of the wheel graph or the friendship graph.
For any $v_i, 1 \leq i \leq 2n_1$, there are n_2 vertices at distance one, $2n_2$ vertices at distance two and $2(n_1-1)n_2$ vertices at distance three.
For the vertex v_{2n_1+1}, there are n_2 vertices at distance one and $2n_1n_2$ vertices at distance two.
Hence, $K = n_2 + 8n_1^2n_2^2 + 12n_1^2n_2 + n_1n_2^2 + 2n_1n_2$.

■

Corollary III.8. $W(F_{n_1} \circ W_{1,n_2-1}) = 12n_1^2n_2 + 8n_1^2n_2^2 + 3n_1 - 4n_1n_2 + 3n_1n_2^2 - 2n_2 + 4n_1^2 + n_2^2 + 2$.

Proof: Proof follows by Theorem III.7, Theorem II.5 and Note II.1. ■

D. Flower graph $Fl_{n_1}^{(3)}$ corona with wheel W_{1,n_2-1}

We obtain the expression for Wiener index of flower graph $Fl_{n_1}^{(3)}$ corona with wheel graph W_{1,n_2-1} in the following theorem.

Theorem III.9. *Wiener index of flower graph $Fl_{n_1}^{(3)}$ corona with wheel W_{1,n_2-1} is equal to $W(Fl_{n_1}^{(3)}) + (2n_1+1)W(W_{1,n_2-1}) + 12n_1^2n_2 + 8n_1^2n_2^2 + n_2$.*

Proof: From Note III.1, we have $W(Fl_{n_1}^{(3)} \circ W_{1,n_2-1}) = W(Fl_{n_1}^{(3)}) + (2n_1+1)W(W_{1,n_2-1}) + K$.
To find the value of K we will consider sum of distances from all pairs of vertices such that, such that the other vertex belongs to the other copies of the wheel graph or the flower graph. Let $v_1, v_2, \ldots, v_{2n_1}, v_{2n_1+1}$ be the vertices of the flower graph $Fl_{n_1}^{(3)}$. The vertices $v_i,\ 1 \leq i \leq n_1$ is adjacent to v_{i+n_1} and the central vertex $v_{2n_1+1} \sim v_i,\ 1 \leq i \leq 2n_1$. The vertices $v_{n_1+i},\ 1 \leq i \leq n_1-1$ is adjacent to v_{n_1+i+1} and $v_{2n_1-1} \sim v_{2n_1}$. Let the vertices in the i^{th} copy of wheel graph W_{1,n_2-1} be labeled as $v_{ij},\ 1 \leq i \leq 2n_1+1, 1 \leq j \leq n_2$.
For any vertex $v_{ij},\ 1 \leq i \leq n_1, 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, two vertices at distance two, $(2n_2+2(n_1-1))$ vertices at distance three and $2(n_1-1)n_2$ vertices at distance four, which lies in the other copies of the wheel graph or the flower graph.
For any vertex $v_{ij},\ n_1+1 \leq i \leq 2n_1,\ 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, four vertices at distance two, $(4n_2+2(n_1-3)+2)$ vertices at distance three and $(2(n_1-3)+2)n_2$ vertices at distance four, which lies in the other copies of the wheel graph or the flower graph.
For any vertex $v_{2n_1+1j}, 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, $2n_1$ vertices at distance two and $2n_1n_2$ vertices at distance three, which lies in the other copies of the wheel graph or the flower graph.
For any $v_i,\ 1 \leq i \leq n_1$, there are n_2 vertices at distance one, $2n_2$ vertices at distance two and $2(n_1-1)n_2$ vertices at distance three.
For any $v_i,\ n_1+1 \leq i \leq 2n_1$, there are n_2 vertices at distance one, $4n_2$ vertices at distance two and $(2(n_1-3)+2)n_2$ vertices at distance three.
For the vertex v_{2n_1+1}, there are n_2 vertices at distance one and $2n_1n_2$ vertices at distance two.
Hence, $K = 12n_1^2n_2 + 8n_1^2n_2^2 + n_2$.

■

Corollary III.10. $W(Fl_{n_1}^{(3)} \circ W_{1,n_2-1}) = 12n_1^2n_2 + 8n_1^2n_2^2 + 2n_1 - 6n_1n_2 + 2n_1n_2^2 - 2n_2 + 4n_1^2 + n_2^2 + 2$.

Proof: Proof follows by Theorem III.9, Theorem II.6 and Note II.1. ■

E. Sunflower graph SF_{n_1} corona with wheel W_{1,n_2-1}

The expression for Wiener index of sunflower graph SF_{n_1} corona with wheel graph W_{1,n_2-1} is discussed in the following theorem.

Theorem III.11. *Wiener index of sunflower graph SF_{n_1} corona with wheel graph W_{1,n_2-1} is equal to $W(SF_{n_1}) + (2n_1+1)W(W_{1,n_2-1}) + n_2 + 10n_1^2n_2^2 + 16n_1^2n_2 - 8n_1n_2^2 - 16n_1n_2$.*

Proof: From Note III.1, we have $W(SF_{n_1} \circ W_{1,n_2-1}) = W(SFn_1) + (2n_1+1)W(W_{1,n_2-1}) + K$.
To find the value of K we will consider sum of distances from all pairs of vertices such that one vertex is from the sunflower graph and the other vertex is from any copies of the wheel graph or two vertices belongs to two different copies of wheel graph.
Let $v_1, v_2, \ldots, v_{2n_1}, v_{2n_1+1}$ be the vertices of the sunflower graph SF_{n_1}. The vertices $v_i, 1 \leq i \leq n_1$ is adjacent to v_{i+n_1} and $v_{i+n_1+1}, 1 \leq i \leq n_1-1$ and v_{n_1} is adjacent to v_{2n_1} and v_{n_1+1}. The vertices $v_{n_1+i}, 1 \leq i \leq n_1-1$ is adjacent to v_{n_1+i+1} and $v_{2n_1} \sim v_{n_1+1}$. The central vertex $v_{2n_1+1} \sim v_{n_1+i}, 1 \leq i \leq n_1$. The vertices in the i^{th} copy of wheel graph W_{1,n_2-1} are labeled as $v_{ij}, 1 \leq i \leq 2n_1+1, 1 \leq j \leq n_2$.
For any vertex $v_{ij}, 1 \leq i \leq n_1, 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, two vertices at distance two, $(2n_2+5)$ vertices at distance three, $(n_1-2)+5n_2$ vertices at distance four, $(n_1-2)n_2+n_1-5$ vertices at distance 5 and $(n_1-5)n_2$ vertices at distance six, which lies in the other copies of the wheel graph or the sunflower graph.
For any vertex $v_{ij}, n_1+1 \leq i \leq 2n_1, 1 \leq j \leq n_2$, we note the following.
There is one vertex at distance one, five vertices at distance two, $(5n_2+(n_1-3)+2)$ vertices at distance three, $(n_1-4)+(n_1-1)n_2$ vertices at distance four and $(n_1-4)n_2$ vertices at distance five, which lies in the other copies of the wheel graph or the sunflower graph.
For any vertex $v_{2n_1+1j}, 1 \leq j \leq n_2$, we note the following.

There is one vertex at distance one, n_1 vertices at distance two, $n_1n_2+n_1$ vertices at distance three and n_1n_2 vertices at distance four, such that the other vertex belongs to the other copies of the wheel graph or the sunflower graph.
For any $v_i, 1\le i\le n_1$, there are n_2 vertices at distance one, $2n_2$ vertices at distance two, $5n_2$ vertices at distance three, $(n_1-2)n_2$ vertices at distance four and $(n_1-5)n_2$ vertices at distance five, which lies in the other copies of the wheel graph.
From any $v_i,\ n_1+1\le i\le 2n_1$, there are n_2 vertices at distance one, $5n_2$ vertices at distance two $(n_1-1)n_2$ vertices at distance three and $(n_1-4)n_2$ vertices at distance four in any copies of wheel graph.
From the vertex v_{2n_1+1}, there are n_2 vertices at distance one, n_1n_2 vertices at distance two and n_1n_2 vertices at distance three in any copies of wheel graph.
Hence, $K=16n_1^2n_2+10n_1^2n_2^2+n_2-8n_1n_2^2-16n_1n_2$. ■

Corollary III.12. $W(SF_{n_1}\circ W_{1,n_2-1}) = 16n_1^2n_2+10n_1^2n_2^2-6n_1-22n_1n_2-6n_1n_2^2-2n_2+6n_1^2+n_2^2+2$.

Proof: Proof follows by Theorem III.11, Theorem II.7 and Note II.1. ■

F. Fan graph F_{1,n_1} corona with wheel W_{1,n_2-1}

We consider an example for the corona of fan graph $F_{1,5}$ and wheel graph $W_{1,3}$ as shown in Figure 7.

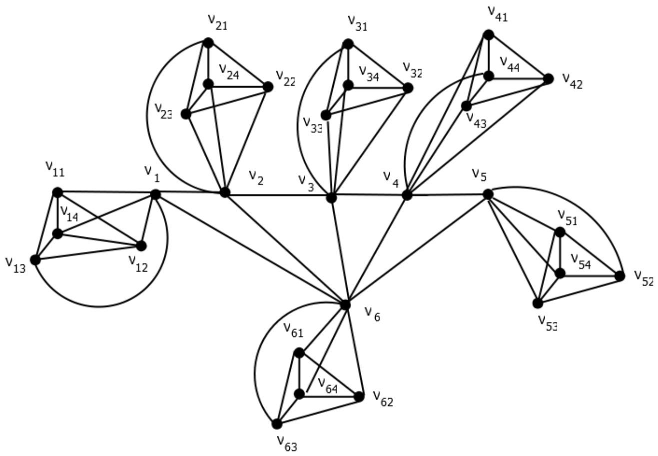

Fig. 7: $F_{1,5}\circ W_{1,3}$

The expression for Wiener index of the corona of fan graph F_{1,n_1} and wheel graph W_{1,n_2-1} is discussed in the following theorem.

Theorem III.13. *Wiener index of Fan graph F_{1,n_1} corona with wheel graph W_{1,n_2-1} is equal to $W(F_{1,n_1})+(n_1+1)W(W_{1,n_2-1})+2n_1^2n_2^2+n_2^2+3n_1^2n_2+3n_2$.*

Proof: From Note III.1, we have $W(F_{1,n_1}\circ W_{1,n_2-1}) = W(F_{1,n_1})+(n_1+1)W(W_{1,n_2-1})+K$.
To find the value of K we will consider sum of distances from all pairs of vertices such that one vertex is from the fan graph and the other vertex is from any copies of the wheel graph or two vertices belongs to two different copies of wheel graph.
Let $v_1, v_2, \dots, v_{n_1}, v_{n_1+1}$ be the vertices of the fan graph F_{1,n_1}. The vertex $v_i\sim v_{i+1}, 1\le i\le n_1-1$ and the central vertex $v_{n_1+1}\sim v_i, 1\le i\le n_1$. The vertices in the i^{th} copy of wheel graph W_{1,n_2-1} are labeled as $v_{ij}, 1\le i\le n_1+1, 1\le j\le n_2$.
For any vertex $v_{ij}, i=1,n_1,\ 1\le j\le n_2$, we note the following.
There is one vertex at distance one, two vertices at distance two, $2n_2+n_1-2$ vertices at distance three and $(n_1-2)n_2$ vertices at distance four which lies in the other copies of the wheel graph or the fan graph.
For any vertex $v_{ij}, 2\le i\le n_1-1,\ 1\le j\le n_2$, we note the following.
There is one vertex at distance one, three vertices at distance two, $3n_2+n_1-3$ vertices at distance three and $(n_1-3)n_2$ vertices at distance four which lies in the other copies of the wheel graph or the fan graph.
For any vertex $v_{n_1+1j}, 1\le j\le n_2$, we note the following.
There is one vertex at distance one, n_1 vertices at distance two and n_1n_2 vertices at distance three, such that the other vertex belongs to the other copies of the wheel graph or the fan graph.
For any $v_i, i=1,n_1$, there are n_2 vertices at distance one, $2n_2$ vertices at distance two, and $(n_1-2)n_2$ vertices at distance three, which lies in the other copies of the wheel graph.
From any $v_i,\ 2\le i\le n_1-1$, there are n_2 vertices at distance one, $3n_2$ vertices at distance two and $(n_1-3)n_2$ vertices at distance three in any copies of wheel graph.
From the vertex v_{n_1+1}, there are n_2 vertices at distance one and n_1n_2 vertices at distance two in any copies of wheel graph.
Hence, $K=2n_1^2n_2^2+n_2^2+3n_1^2n_2+3n_2$. ■

Corollary III.14. $W(F_{1,n_1}\circ W_{1,n_2-1}) = 3n_1^2n_2+2n_1^2n_2^2+n_1-3n_1n_2+n_1n_2^2+n_1^2+2n_2^2+3$.

Proof: Proof follows by Theorem III.13, Theorem II.8 and Note II.1. ■

IV. Conclusions

Weiner index is an important topological index in chemistry since Harold Wiener defined it in 1947. Scientists found that there is a very close relation between the physical and chemical characteristics of many compounds and the topological structure of that. So, in this article, the Wiener index of the corona of wheel related graph and wheel graph is discussed. For further study, other topological indices of these corona product of graphs can be computed.

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