Decoupled CNLF Modular Grad-Div Stabilized Scheme for the Fluid-Fluid Interaction Problems

Haiqiang Xiao, Chunya Wu, Feng Xue, Lingzhi Qian* and Huiping Cai *

***Abstract*—In this paper, we present a Crank-Nicolson leapfrog (CNLF) time stepping decoupled scheme for the fluid-fluid interaction problems. The scheme is based on the modular grad-div stabilized (MGDS) method which is very efficient for the fluid-fluid interaction problems. The algorithm is very simple to implement and retain the benefits of grad-div stabilization. The unconditional stability of the proposed scheme is proven rigorously. Numerical tests are given to verify the efficiency and robustness of the proposed scheme.**

***Index Terms*—Fluid-fluid interaction problems, Nonlinear interface condition, Crank-Nicolson leap-frog, Modular Grad-Div stabilization, Stability analysis.**

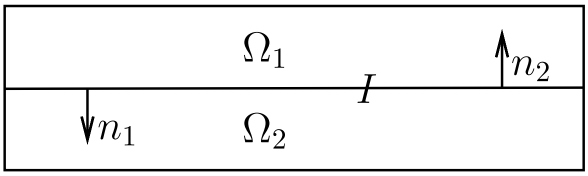

Fig. 1: Example of adjoining subdomains

I. Introduction

There are many problems in which different physical models, different parameter regimes, or different solution behaviors are coupled across interfaces. A model of two incompressible Newtonian fluids coupled across a common interface is studied in [13], [14]. Fig. 1 illustrates the subdomains, and the domain consists of two subdomains Ω_1 and Ω_2 coupled across an interface $I = \partial\Omega_1 \cap \partial\Omega_2$, where $\Omega_i \subset R^d, (d = 2, 3)$ is a bounded domain with piecewise smooth boundary $\partial\Omega_i$. We set $\Gamma_i = \partial\Omega_i \backslash I$, for $i = 1, 2$. The problem studied in this paper is: given $f_i : [0, T] \to H^1(\Omega_i)$ (i=1,2), find $\boldsymbol{u}_i : \overline{\Omega}_i \times [0, T] \to R^d$ and $p_i : \overline{\Omega}_i \times [0, T] \to R$ satisfying

$$\begin{cases} \boldsymbol{u}_{i,t} - \mu_i \Delta \boldsymbol{u}_i + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i + \nabla p_i = f_i \quad \text{in } \Omega_i, \\ -\mu_i \boldsymbol{n}_i \cdot \nabla \boldsymbol{u}_i \cdot \boldsymbol{\tau}_l = k |\boldsymbol{u}_i - \boldsymbol{u}_j| (\boldsymbol{u}_i - \boldsymbol{u}_j) \cdot \boldsymbol{\tau}_l \\ \qquad \text{on } I, \ i, j = 1, 2, \ i \neq j, \ l = 1, 2, \\ \boldsymbol{u}_i \cdot \boldsymbol{n}_i = 0 \quad \text{on } I, \ i = 1, 2, \\ \nabla \cdot \boldsymbol{u}_i = 0 \quad \text{in } \Omega_i, \\ \boldsymbol{u}_i(x, 0) = \boldsymbol{u}_i^0(x) \quad \text{in } \Omega_i, \\ \boldsymbol{u}_i = 0 \quad \text{on } \Gamma_i, \end{cases} \tag{1}$$

where $\boldsymbol{u}_{i,t} = \frac{\partial \boldsymbol{u}_i}{\partial t}$. Leap-frog scheme is widely used in the calculation of the atmosphere and oceans, since it preserves the wave energy conservation. The fully implicit Crank-Nicolson scheme is a popular second-order accurate format for the nonstationary Navier-Stokes equations. Recently, more and more scholars focus on this research topic, such as in the references [3], [4], [5], [7], [9], [10], [12], [14], [15]. Numerous works are also devoted to promoting the CNLF scheme to solve the geophysical flow, uncoupling groundwater-surface water flows [7], [8].

The grad-div stabilization method is an effective numerical method for improving the approximate quality of the fluid flow problems [2], [11]. Furthermore, the grad-div stabilization increases coupling, decreases sparsity, and makes preconditioning more difficult. Most aspects have been addressed, but the full resolution is still an open problem.

In this paper, the CNLF-MGDS scheme is presented and studied for the fluid-fluid interaction problems. The proposed scheme has the obvious advantage for computing the fluid-fluid interaction problems. The algorithm is very simple to implement, retain the benefits of grad-div stabilization. The unconditional stability is proven and ample numerical experiments are performed to illustrate the efficiency of the proposed scheme.

II. Preliminaries

To write the variational form of problem, we introduce the Sobolev spaces $W^{m,r}(\Omega)$ for all non-negative integers m and r equipped with the standard Sobolev norms $\|\cdot\|_{m,r}$. In particular, we write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ when $r = 2$.

Let

$$X_i := \left\{ \boldsymbol{v}_i \in H^1(\Omega_i)^d : \boldsymbol{v}_i = 0 \text{ on } \Gamma_i = \partial\Omega_i \backslash I, \ \boldsymbol{v}_i \cdot \boldsymbol{n}_i = 0 \text{ on } I \right\},$$

$$Q_i := \left\{ q_i \in L^2(\Omega_i) : \int_{\Omega_i} q_i d\Omega_i = 0 \right\}.$$

Define $X = X_1 \times X_2$, $Q = Q_1 \times Q_2$, and $L^2(\Omega) = L^2(\Omega_1) \times L^2(\Omega_2)$. For $\boldsymbol{u}, \boldsymbol{v} \in X$ with $\boldsymbol{u} = [\boldsymbol{u}_1, \boldsymbol{u}_2]^T$ and $\boldsymbol{v} = [\boldsymbol{v}_1, \boldsymbol{v}_2]^T$, define the L^2 inner product

$$(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1,2} \int_{\Omega_i} \boldsymbol{u}_i \boldsymbol{v}_i dx,$$

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and the H^1 inner product

$$(\boldsymbol{u},\boldsymbol{v})_X=\sum_{i=1,2}\left(\int_{\Omega_i}\boldsymbol{u}_i\boldsymbol{v}_i dx+\int_{\Omega_i}\nabla\boldsymbol{u}_i\cdot\nabla\boldsymbol{v}_i dx\right),$$

and the induced norms $\|\boldsymbol{v}\|=(\boldsymbol{v},\boldsymbol{v})^{\frac{1}{2}}$, and $\|\boldsymbol{v}\|_X=(\boldsymbol{v},\boldsymbol{v})_X^{\frac{1}{2}}$, respectively.

Let $\mathcal{T}_i$ be a triangulation of Ω_i and $\mathcal{T}_h=\mathcal{T}_1\cup\mathcal{T}_2$, h be the mesh parameter of $\mathcal{T}_h$, i.e. h is the largest diameter of a simplex in either $\mathcal{T}_1$ or $\mathcal{T}_2$. We construct globally continuous conforming finite element spaces $X_{i,h}$ for velocity and $Q_{i,h}$ for pressure on the meshes $\mathcal{T}_i, i=1,\ 2$. Denote $X_h=X_{1,h}\times X_{2,h}$. As a consequence, $X_{i,h}\subset X_i$, $Q_{i,h}\subset Q_i$ and $X_h\subset X$, respectively.

A natural subdomain variational formulation for problem (1) is to find (for $i,j=1,2, i\neq j$) $\boldsymbol{u}_i:[0,T]\rightarrow X_i$ and $p_i:[0,T]\rightarrow Q_i$ satisfying

$$\begin{cases}(\boldsymbol{u}_{i,t},\boldsymbol{v}_i)_{\Omega_i}+\mu_i(\nabla\boldsymbol{u}_i,\nabla\boldsymbol{v}_i)_{\Omega_i}+(\boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i,\boldsymbol{v}_i)_{\Omega_i}-(p_i,\nabla\cdot\boldsymbol{v}_i)_{\Omega_i}\\ +\int_I k|\boldsymbol{u}_i-\boldsymbol{u}_j|(\boldsymbol{u}_i-\boldsymbol{u}_j)\boldsymbol{v}_i ds=(f_i,\boldsymbol{v}_i)_{\Omega_i},\quad\forall\boldsymbol{v}_i\in X_i,\\ (\nabla\cdot\boldsymbol{u}_i,q_i)_{\Omega_i}=0,\quad\forall q_i\in Q_i.\end{cases}\tag{2}$$

The natural monolithic variational formulation for problem (1) is to find $\boldsymbol{u}:[0,T]\rightarrow X$ and $\boldsymbol{p}:[0,T]\rightarrow Q$ satisfying

$$\begin{cases}(\boldsymbol{u}_t,\boldsymbol{v})+\mu(\nabla\boldsymbol{u},\nabla\boldsymbol{v})+(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})-(\boldsymbol{p},\nabla\cdot\boldsymbol{u})\\ +\int_I k|[\boldsymbol{u}]|[\boldsymbol{u}]\boldsymbol{v}ds=(\boldsymbol{f},\boldsymbol{v}),\quad\forall\boldsymbol{v}\in X,\\ (\nabla\cdot\boldsymbol{u},\boldsymbol{q})=0,\quad\forall\boldsymbol{q}\in Q.\end{cases}\tag{3}$$

where $[\cdot]$ denotes the jump across the interface I, $(\cdot\,,\cdot)$ is the $L^2(\Omega_1\cup\Omega_2)$ inner product and $\mu=\mu_i,\ \boldsymbol{f}=f_i,\boldsymbol{v}=\boldsymbol{v}_i$ in Ω_i.

III. CNLF-MGDS scheme

We consider the fully discrete CNLF-MGDS scheme for the problem (1).

Algorithm 3.1 (CNLF-MGDS scheme) Let $\Delta t>0,\boldsymbol{f}\in H^{-1}(\Omega)$. Given $\boldsymbol{u}_{1,h}^{n-1},\boldsymbol{u}_{1,h}^{n}\in X_h$, $n\in\{1,2,\cdots,N-1\}$, find $\boldsymbol{u}_{1,h}^{n+1}\in X_{1,h}$ and $p_{1,h}^{n+1}\in Q_{1,h}$ satisfying

$$\begin{aligned}&\left(\frac{\boldsymbol{u}_{1,h}^{n+1}-\boldsymbol{u}_{1,h}^{n-1}}{2\Delta t},\boldsymbol{v}_1\right)_{\Omega_1}+\mu_1\left(\nabla\frac{\boldsymbol{u}_{1,h}^{n+1}+\boldsymbol{u}_{1,h}^{n-1}}{2},\nabla\boldsymbol{v}_1\right)_{\Omega_1}\\ &+\left(\boldsymbol{u}_{1,h}^{n}\cdot\nabla\frac{\boldsymbol{u}_{1,h}^{n+1}+\boldsymbol{u}_{1,h}^{n-1}}{2},\boldsymbol{v}_1\right)_{\Omega_1}-\left(p_{1,h}^{n+1},\nabla\cdot\boldsymbol{v}_1\right)_{\Omega_1}\\ &+\left(\nabla\cdot\frac{\boldsymbol{u}_{1,h}^{n+1}+\boldsymbol{u}_{1,h}^{n-1}}{2},q_1\right)_{\Omega_1}+\beta\left(\nabla\cdot\frac{u_{1,h}^{n+1}-u_{1,h}^{n-1}}{2\Delta t},\nabla\cdot\boldsymbol{v}_1\right)\\ &+\gamma\left(\nabla\cdot\frac{u_{1,h}^{n+1}+u_{1,h}^{n-1}}{2},\nabla\cdot\boldsymbol{v}_1\right)\\ &+k\int_I\left|[\boldsymbol{u}_h^n]\right|\frac{\boldsymbol{u}_{1,h}^{n+1}+\boldsymbol{u}_{1,h}^{n-1}}{2}\boldsymbol{v}_1 ds\\ &-k\int_I\left|[\boldsymbol{u}_h^n]\right|^{1/2}\left|[\boldsymbol{u}_h^{n-1}]\right|^{1/2}\frac{\boldsymbol{u}_{2,h}^{n}+\boldsymbol{u}_{2,h}^{n-2}}{2}\boldsymbol{v}_1 ds\\ &=\left(f_1(t^{n+1}),\boldsymbol{v}_1\right)_{\Omega_1},\quad\forall\boldsymbol{v}_1\in X_{1,h},\end{aligned}\tag{4}$$

$\boldsymbol{u}_{2,h}^{n+1}\in X_{2,h}$ and $p_{2,h}^{n+1}\in Q_{2,h}$ satisfying

$$\begin{aligned}&\left(\frac{\boldsymbol{u}_{2,h}^{n+1}-\boldsymbol{u}_{2,h}^{n-1}}{2\Delta t},\boldsymbol{v}_2\right)_{\Omega_2}+\mu_2\left(\nabla\frac{\boldsymbol{u}_{2,h}^{n+1}+\boldsymbol{u}_{2,h}^{n-1}}{2},\nabla\boldsymbol{v}_2\right)_{\Omega_2}\\ &+\left(\boldsymbol{u}_{2,h}^{n}\cdot\nabla\frac{\boldsymbol{u}_{2,h}^{n+1}+\boldsymbol{u}_{2,h}^{n-1}}{2},\boldsymbol{v}_2\right)_{\Omega_2}-\left(p_{2,h}^{n+1},\nabla\cdot\boldsymbol{v}_2\right)_{\Omega_2}\\ &+\left(\nabla\cdot\frac{\boldsymbol{u}_{2,h}^{n+1}+\boldsymbol{u}_{2,h}^{n-1}}{2},q_2\right)_{\Omega_2}+\beta\left(\nabla\cdot\frac{u_{2,h}^{n+1}-u_{2,h}^{n-1}}{2\Delta t},\nabla\cdot\boldsymbol{v}_2\right)\\ &+\gamma\left(\nabla\cdot\frac{u_{2,h}^{n+1}+u_{2,h}^{n-1}}{2},\nabla\cdot\boldsymbol{v}_2\right)\\ &+k\int_I\left|[\boldsymbol{u}_h^n]\right|\frac{\boldsymbol{u}_{2,h}^{n+1}+\boldsymbol{u}_{2,h}^{n-1}}{2}\boldsymbol{v}_2 ds\\ &-k\int_I\left|[\boldsymbol{u}_h^n]\right|^{1/2}\left|[\boldsymbol{u}_h^{n-1}]\right|^{1/2}\frac{\boldsymbol{u}_{1,h}^{n}+\boldsymbol{u}_{1,h}^{n-2}}{2}\boldsymbol{v}_2 ds\\ &=\left(f_2(t^{n+1}),\boldsymbol{v}_2\right)_{\Omega_1},\quad\forall\boldsymbol{v}_2\in X_{2,h}.\end{aligned}\tag{5}$$

where β and γ are the positive grad-div stabilization parameters.

It should be noted that this algorithm requires three initial values $\boldsymbol{u}_h^0$, $\boldsymbol{u}_h^1$ and $\boldsymbol{u}_h^2$. The implicit solution at the first and second time step and the initial data are used to start Algorithm 3.1. The implicit solution can be obtained from the semi-implicit scheme.

Algorithm 3.2 (Semi-implicit scheme [1]) Let $\Delta t>0,\boldsymbol{f}\in H^{-1}(\Omega)$. Given $\boldsymbol{u}_h^n\in X_h$, $n\in\{0,1,\cdots,N-1\}$, find $\boldsymbol{u}_h^{n+1}\in X_h$ and $\boldsymbol{p}_h^{n+1}\in Q_h$ satisfying

$$\begin{aligned}&\left(\frac{\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^n}{\Delta t},\boldsymbol{v}\right)+\mu\left(\nabla\frac{\boldsymbol{u}_h^{n+1}+\boldsymbol{u}_h^n}{2},\nabla\boldsymbol{v}\right)\\ &+\left(\frac{\boldsymbol{u}_h^{n+1}+\boldsymbol{u}_h^n}{2}\cdot\nabla\frac{\boldsymbol{u}_h^{n+1}+\boldsymbol{u}_h^n}{2},\boldsymbol{v}\right)-\left(\boldsymbol{p}_h^{n+1},\nabla\cdot\boldsymbol{v}\right)\\ &+\left(\nabla\cdot\frac{\boldsymbol{u}_h^{n+1}+\boldsymbol{u}_h^n}{2},\boldsymbol{q}\right)+k\int_I\left|[\boldsymbol{u}_h^n]\right|[\boldsymbol{u}_h^{n+1}]\boldsymbol{v}ds\\ &=\left(\frac{\boldsymbol{f}(t^{n+1})+\boldsymbol{f}(t^n)}{2},\boldsymbol{v}\right),\quad\forall\boldsymbol{v}\in X_h,\forall\boldsymbol{q}\in Q_h.\end{aligned}\tag{6}$$

IV. Stability analysis

Theorem 4.1: Let $\boldsymbol{u}_{1,h}^{n+1}\in X_{1,h}$ and $\boldsymbol{u}_{2,h}^{n+1}\in X_{2,h}$ satisfy (4) and (5), respectively. Then

$$\begin{aligned}&\left\|\boldsymbol{u}_h^{n+1}\right\|^2+\left\|\boldsymbol{u}_h^n\right\|^2+\beta\left(\left\|\nabla\cdot\boldsymbol{u}_h^{n+1}\right\|^2+\left\|\nabla\cdot\boldsymbol{u}_h^n\right\|^2\right)\\ &+\gamma\Delta t\sum_{j=2}^{n}\left\|\nabla\cdot\frac{\boldsymbol{u}_{1,h}^{j+1}+\boldsymbol{u}_{1,h}^{j-1}}{2}\right\|^2\\ &+\frac{k\Delta t}{2}\int_I\left|[\boldsymbol{u}_h^n]\right|\left(|\boldsymbol{u}_{1,h}^{n+1}+\boldsymbol{u}_{1,h}^{n-1}|^2+|\boldsymbol{u}_{2,h}^{n+1}+\boldsymbol{u}_{2,h}^{n-1}|^2\right)ds\\ &+\frac{k\Delta t}{2}\sum_{j=2}^{n}\int_I\left|\left(\boldsymbol{u}_{1,h}^{j+1}+\boldsymbol{u}_{1,h}^{j-1}\right)\left|[\boldsymbol{u}_h^j]\right|^{1/2}-\left(\boldsymbol{u}_{2,h}^{j}+\boldsymbol{u}_{2,h}^{j-2}\right)\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2}\right|^2 ds\\ &+\frac{k\Delta t}{2}\sum_{j=2}^{n}\int_I\left|\left(\boldsymbol{u}_{2,h}^{j+1}+\boldsymbol{u}_{2,h}^{j-1}\right)\left|[\boldsymbol{u}_h^j]\right|^{1/2}-\left(\boldsymbol{u}_{1,h}^{j}+\boldsymbol{u}_{1,h}^{j-2}\right)\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2}\right|^2 ds\\ &\leq\left\|\boldsymbol{u}_h^2\right\|^2+\left\|\boldsymbol{u}_h^1\right\|^2+\frac{k\Delta t}{2}\int_I\left|[\boldsymbol{u}_h^1]\right|\left(\left|\boldsymbol{u}_{1,h}^2+\boldsymbol{u}_{1,h}^0\right|^2+\left|\boldsymbol{u}_{2,h}^2+\boldsymbol{u}_{2,h}^0\right|^2\right)ds\\ &+\beta\left(\left\|\nabla\cdot\boldsymbol{u}_h^2\right\|^2+\left\|\nabla\cdot\boldsymbol{u}_h^1\right\|^2\right)\\ &+\sum_{j=2}^{n}\left(\frac{\Delta t}{\mu_1}\left\|\boldsymbol{f}_1(t^j)\right\|_{H^{-1}(\Omega_1)}^2+\frac{\Delta t}{\mu_2}\left\|\boldsymbol{f}_2(t^j)\right\|_{H^{-1}(\Omega_2)}^2\right).\end{aligned}\tag{7}$$

Proof. Let $j \in \{2, \cdots, N-1\}$ and consider (4) with $n = j$. Taking $(\boldsymbol{v}_1, q_1) = \left(\frac{\boldsymbol{u}_{1,h}^{j+1}+\boldsymbol{u}_{1,h}^{j-1}}{2}, p_{1,h}^{j+1}\right)$ in (4) to get

$$\left(\frac{\boldsymbol{u}_{1,h}^{j+1} - \boldsymbol{u}_{1,h}^{j-1}}{2\Delta t}, \frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\right)_{\Omega_1} + \frac{\mu_1}{4}\left\|\nabla\left(\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}\right)\right\|_{\Omega_1}^2$$
$$+ k\int_I \left(\frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\right)^2 \left|[\boldsymbol{u}_h^j]\right| ds$$
$$- k\int_I \frac{\boldsymbol{u}_{2,h}^{j} + \boldsymbol{u}_{2,h}^{j-2}}{2}\frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\left|[\boldsymbol{u}_h^j]\right|^{1/2}\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2} ds$$
$$= \left(f_1(t^j), \frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\right)_{\Omega_1}$$
$$+ \beta\left(\nabla\cdot\frac{\boldsymbol{u}_{1,h}^{j+1} - \boldsymbol{u}_{1,h}^{j-1}}{2\Delta t}, \frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\right)$$
$$+ \gamma\left(\nabla\cdot\frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}, \frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\right). \quad (8)$$

Similarly, taking $(\boldsymbol{v}_2, q_2) = \left(\frac{\boldsymbol{u}_{2,h}^{j+1}+\boldsymbol{u}_{2,h}^{j-1}}{2}, p_{2,h}^{j+1}\right)$ in (5). Applying formula for the difference of squares to the first terms on the left-hand sides of the above relations and adding together, it follows that

$$\frac{1}{4\Delta t}\left(\left\|\boldsymbol{u}_{1,h}^{j+1}\right\|_{\Omega_1}^2 - \left\|\boldsymbol{u}_{1,h}^{j-1}\right\|_{\Omega_1}^2 + \left\|\boldsymbol{u}_{2,h}^{j+1}\right\|_{\Omega_2}^2 - \left\|\boldsymbol{u}_{2,h}^{j-1}\right\|_{\Omega_2}^2\right)$$
$$+ \frac{\mu_1}{4}\left\|\nabla\left(\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}\right)\right\|_{\Omega_1}^2 + \frac{\mu_2}{4}\left\|\nabla\left(\boldsymbol{u}_{2,h}^{j+1} + \boldsymbol{u}_{2,h}^{j-1}\right)\right\|_{\Omega_2}^2$$
$$+ k\int_I\left(\frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\right)^2\left|[\boldsymbol{u}_h^j]\right| ds$$
$$+ k\int_I\left(\frac{\boldsymbol{u}_{2,h}^{j+1} + \boldsymbol{u}_{2,h}^{j-1}}{2}\right)^2\left|[\boldsymbol{u}_h^j]\right| ds$$
$$- k\int_I \frac{\boldsymbol{u}_{2,h}^{j} + \boldsymbol{u}_{2,h}^{j-2}}{2}\frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\left|[\boldsymbol{u}_h^j]\right|^{1/2}\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2} ds$$
$$- k\int_I \frac{\boldsymbol{u}_{1,h}^{j} + \boldsymbol{u}_{1,h}^{j-2}}{2}\frac{\boldsymbol{u}_{2,h}^{j+1} + \boldsymbol{u}_{2,h}^{j-1}}{2}\left|[\boldsymbol{u}_h^j]\right|^{1/2}\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2} ds$$
$$+ \frac{\beta}{4\Delta t}\left(\left\|\nabla\cdot\boldsymbol{u}_{1,h}^{j+1}\right\|_{\Omega_1}^2 - \left\|\nabla\cdot\boldsymbol{u}_{1,h}^{j-1}\right\|_{\Omega_1}^2 + \left\|\nabla\cdot\boldsymbol{u}_{2,h}^{j+1}\right\|_{\Omega_2}^2\right.$$
$$\left.- \left\|\nabla\cdot\boldsymbol{u}_{2,h}^{j-1}\right\|_{\Omega_2}^2\right) + \frac{\gamma}{4}\left\|\nabla\cdot\frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\right\|^2$$
$$= \left(f_1(t^j), \frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\right)_{\Omega_1} + \left(f_2(t^j), \frac{\boldsymbol{u}_{2,h}^{j+1} + \boldsymbol{u}_{2,h}^{j-1}}{2}\right)_{\Omega_2}. \quad (9)$$

The following two interface integrals may be dealt with the following way

$$\int_I\left(\frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\right)^2\left|[\boldsymbol{u}_h^j]\right| ds$$
$$- k\int_I \frac{\boldsymbol{u}_{2,h}^{j} + \boldsymbol{u}_{2,h}^{j-2}}{2}\frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\left|[\boldsymbol{u}_h^j]\right|^{1/2}\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2} ds$$
$$= \frac{k}{4}\int_I\left(\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}\right)\left|[\boldsymbol{u}_h^j]\right|^{1/2}.$$
$$\left(\left(\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}\right)\left|[\boldsymbol{u}_h^j]\right|^{1/2} - \left(\boldsymbol{u}_{2,h}^{j} + \boldsymbol{u}_{2,h}^{j-2}\right)\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2}\right) ds$$
$$= \frac{k}{8}\int_I\left|\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}\right|^2\left|[\boldsymbol{u}_h^j]\right| ds$$
$$- \frac{k}{8}\int_I\left|\boldsymbol{u}_{2,h}^{j} + \boldsymbol{u}_{2,h}^{j-2}\right|^2\left|[\boldsymbol{u}_h^{j-1}]\right| ds$$
$$+ \frac{k}{8}\int_I\left|\left(\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}\right)\left|[\boldsymbol{u}_h^j]\right|^{1/2} - \left(\boldsymbol{u}_{2,h}^{j} + \boldsymbol{u}_{2,h}^{j-2}\right)\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2}\right|^2 ds.$$

The remaining two interface integrals are treated analogously and the expressions are inserted into (9), we get

$$\frac{1}{4\Delta t}\left(\left\|\boldsymbol{u}_{1,h}^{j+1}\right\|_{\Omega_1}^2 - \left\|\boldsymbol{u}_{1,h}^{j-1}\right\|_{\Omega_1}^2 + \left\|\boldsymbol{u}_{2,h}^{j+1}\right\|_{\Omega_2}^2 - \left\|\boldsymbol{u}_{2,h}^{j-1}\right\|_{\Omega_2}^2\right)$$
$$+ \frac{\mu_1}{4}\left\|\nabla\left(\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}\right)\right\|_{\Omega_1}^2 + \frac{\mu_2}{4}\left\|\nabla\left(\boldsymbol{u}_{2,h}^{j+1} + \boldsymbol{u}_{2,h}^{j-1}\right)\right\|_{\Omega_2}^2$$
$$+ \frac{\beta}{4\Delta t}\left(\left\|\nabla\cdot\boldsymbol{u}_{1,h}^{j+1}\right\|_{\Omega_1}^2 - \left\|\nabla\cdot\boldsymbol{u}_{1,h}^{j-1}\right\|_{\Omega_1}^2 + \left\|\nabla\cdot\boldsymbol{u}_{2,h}^{j+1}\right\|_{\Omega_2}^2\right.$$
$$\left.- \left\|\nabla\cdot\boldsymbol{u}_{2,h}^{j-1}\right\|_{\Omega_2}^2\right) + \frac{\gamma}{4}\left\|\nabla\cdot\frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\right\|^2$$
$$+ \frac{k}{8}\int_I\left|\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}\right|^2\left|[\boldsymbol{u}_h^j]\right| ds - \frac{k}{8}\int_I\left|\boldsymbol{u}_{2,h}^{j} + \boldsymbol{u}_{2,h}^{j-2}\right|^2\left|[\boldsymbol{u}_h^{j-1}]\right| ds$$
$$+ \frac{k}{8}\int_I\left|\left(\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}\right)\left|[\boldsymbol{u}_h^j]\right|^{1/2} - \left(\boldsymbol{u}_{2,h}^{j} + \boldsymbol{u}_{2,h}^{j-2}\right)\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2}\right|^2 ds$$
$$+ \frac{k}{8}\int_I\left|\boldsymbol{u}_{2,h}^{j+1} + \boldsymbol{u}_{2,h}^{j-1}\right|^2\left|[\boldsymbol{u}_h^j]\right| ds - \frac{k}{8}\int_I\left|\boldsymbol{u}_{1,h}^{j} + \boldsymbol{u}_{1,h}^{j-2}\right|^2\left|[\boldsymbol{u}_h^{j-1}]\right| ds$$
$$+ \frac{k}{8}\int_I\left|\left(\boldsymbol{u}_{2,h}^{j+1} + \boldsymbol{u}_{2,h}^{j-1}\right)\left|[\boldsymbol{u}_h^j]\right|^{1/2} - \left(\boldsymbol{u}_{1,h}^{j} + \boldsymbol{u}_{1,h}^{j-2}\right)\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2}\right|^2 ds$$
$$\leq \frac{1}{\mu_1}\left\|\boldsymbol{f}_1(t^j)\right\|_{H^{-1}(\Omega_1)}^2 + \frac{1}{\mu_2}\left\|\boldsymbol{f}_2(t^j)\right\|_{H^{-1}(\Omega_2)}^2$$
$$+ \frac{\mu_1}{4}\left\|\nabla\left(\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}\right)\right\|_{\Omega_1}^2 + \frac{\mu_2}{4}\left\|\nabla\left(\boldsymbol{u}_{2,h}^{j+1} + \boldsymbol{u}_{2,h}^{j-1}\right)\right\|_{\Omega_2}^2 \quad (10)$$

Multiply through (10) by $4\Delta t$, and summing over $j = 2, \cdots, n$, then we have

$$\left\|\boldsymbol{u}_h^{n+1}\right\|^2 - \left\|\boldsymbol{u}_h^2\right\|^2 + \left\|\boldsymbol{u}_h^n\right\|^2 - \left\|\boldsymbol{u}_h^1\right\|^2$$
$$+ \beta\left(\left\|\nabla\cdot\boldsymbol{u}_h^{n+1}\right\|^2 - \left\|\nabla\cdot\boldsymbol{u}_h^2\right\|^2 + \left\|\nabla\cdot\boldsymbol{u}_h^n\right\|^2 - \left\|\nabla\cdot\boldsymbol{u}_h^1\right\|^2\right)$$
$$+ \frac{k\Delta t}{2}\int_I\left|[\boldsymbol{u}_h^n]\right|\left|\boldsymbol{u}_{1,h}^{n+1} + \boldsymbol{u}_{1,h}^{n-1}\right|^2 ds + \frac{k\Delta t}{2}\int_I\left|[\boldsymbol{u}_h^n]\right|\left|\boldsymbol{u}_{2,h}^{n+1} + \boldsymbol{u}_{2,h}^{n-1}\right|^2 ds$$
$$- \frac{k\Delta t}{2}\int_I\left|[\boldsymbol{u}_h^1]\right|\left|\boldsymbol{u}_{2,h}^{2} + \boldsymbol{u}_{2,h}^{0}\right|^2 ds - \frac{k\Delta t}{2}\int_I\left|[\boldsymbol{u}_h^n]\right|\left|\boldsymbol{u}_{1,h}^{2} + \boldsymbol{u}_{1,h}^{0}\right|^2 ds$$
$$+ \frac{k\Delta t}{2}\sum_{j=2}^{n}\int_I\left|\left(\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}\right)\left|[\boldsymbol{u}_h^j]\right|^{1/2} - \left(\boldsymbol{u}_{2,h}^{j} + \boldsymbol{u}_{2,h}^{j-2}\right)\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2}\right|^2 ds$$
$$+ \frac{k\Delta t}{2}\sum_{j=2}^{n}\int_I\left|\left(\boldsymbol{u}_{2,h}^{j+1} + \boldsymbol{u}_{2,h}^{j-1}\right)\left|[\boldsymbol{u}_h^j]\right|^{1/2} - \left(\boldsymbol{u}_{1,h}^{j} + \boldsymbol{u}_{1,h}^{j-2}\right)\left|[\boldsymbol{u}_h^{j-1}]\right|^{1/2}\right|^2 ds$$
$$+ \gamma\Delta t\sum_{j=2}^{n}\left\|\nabla\cdot\frac{\boldsymbol{u}_{1,h}^{j+1} + \boldsymbol{u}_{1,h}^{j-1}}{2}\right\|^2$$
$$\leq \sum_{j=2}^{n}\left(\frac{\Delta t}{\mu_1}\left\|\boldsymbol{f}_1(t^j)\right\|_{H^{-1}(\Omega_1)}^2 + \frac{\Delta t}{\mu_2}\left\|\boldsymbol{f}_2(t^j)\right\|_{H^{-1}(\Omega_2)}^2\right). \quad (11)$$

Then, we can complete the proof.

V. Numerical experiments

In this section, we present some numerical experiments to illustrate the theoretical results obtained in the previous section and show the efficiency of the proposed scheme. Assume $\Omega_1 = [0,1]\times[0,1]$, and $\Omega_2 = [0,1]\times[-1,0]$, so I is the portion of the x-axis from 0 to 1. Then $\boldsymbol{n}_1 = [0,-1]^T$

TABLE I: $\max_n \|u_h^n\|_0^2$ of the CNLF-MGDS scheme

$\frac{1}{h}$ \ $\frac{1}{\Delta t}$	2^6	2^5	2^4	2^3	2^2
2^4	0.004251	0.004116	0.003864	0.003389	0.002613
2^5	0.004458	0.004318	0.004040	0.003567	0.002763
2^6	0.004711	0.004565	0.004282	0.003779	0.002936
2^7	0.005034	0.004879	0.004582	0.004043	0.003147

TABLE II: $\max_n \|\nabla u_h^n\|_0^2$ of the CNLF-MGDS scheme

$\frac{1}{h}$ \ $\frac{1}{\Delta t}$	2^6	2^5	2^4	2^3	2^2
2^4	0.340311	0.333486	0.320475	0.296828	0.257739
2^5	0.232813	0.225833	0.212516	0.188284	0.148125
2^6	0.229010	0.221924	0.208402	0.183782	0.144392
2^7	0.231230	0.224084	0.210445	0.185603	0.144359

and $\boldsymbol{n}_2 = [0, 1]^T$. For a, μ_1, μ_2, and k all arbitrary positive constants, the right-hand side function f_1 is chosen to ensure that

$$\begin{cases} u_{1,1}(t, x, y) = ax^2(1 - x)^2(1 - y)\exp(-t), \\ u_{1,2}(t, x, y) = axy(-2 + y + 6x - 3xy - 4x^2 + 2x^2y)\exp(-t), \\ p_1(t, x, y) = \cos(\pi x)\sin(\pi y)\exp(-t). \end{cases}$$

Similarly, the right-hand side function f_2 is chosen to ensure that

$$\begin{cases} u_{2,1}(t, x, y) = ax^2(1 - x)^2(1 + y)\exp(-t), \\ u_{2,2}(t, x, y) = axy(-2 - y + 6x + 3xy - 4x^2 - 2x^2y)\exp(-t), \\ p_2(t, x, y) = \cos(\pi x)\sin(\pi y)\exp(-t). \end{cases}$$

The spatial discretization is accomplished using the MINI-element. The parameter values are chosen $a = 1$, $\mu_1 = 1.0 \times 10^{-3}$, $\mu_2 = 1.0$, $k = 100$, $T = 1$. In order to validate Theorem 4.1 with higher density ratio, we compute the maximum values of $\|u_h^n\|_0^2$, and $\|\nabla u_h^n\|_0^2$, respectively with different time steps in Tables I-II, and compare the values with different space meshes under the same time step. When one fixes a time step, the norms increase monotonically but the increased values reduce quickly. This phenomenon indicates that the CNLF-MGDS scheme is unconditional stable on time step size $\triangle t$. Furthermore, the numerical test results, including errors and convergence orders of velocity and pressure, obtained by the proposed scheme at $t_n = 1$ when $\Delta t = h$, are shown in Table III and Table IV.

TABLE III: The convergence of velocity of CNLF-MGDS scheme

$h = \Delta t$	$Err(\boldsymbol{u}_1)$	Rate	$Err(\boldsymbol{u}_2)$	Rate
$h = \frac{1}{4}$	1.27e-2	-	1.25e-2	-
$h = \frac{1}{8}$	8.43e-3	0.60	8.45e-3	0.56
$h = \frac{1}{16}$	2.85e-3	1.57	2.85e-3	1.57
$h = \frac{1}{32}$	5.96e-4	2.26	5.96e-4	2.26
$h = \frac{1}{64}$	7.48e-5	3.00	7.48e-5	3.00

TABLE IV: The convergence of pressure of CNLF-MGDS scheme

$h = \Delta t$	$Err(p_1)$	Rate	$Err(p_2)$	Rate
$h = \frac{1}{4}$	2.31e-1	-	2.52e-1	-
$h = \frac{1}{8}$	1.18e-1	0.97	1.22e-1	1.04
$h = \frac{1}{16}$	5.14e-2	1.20	5.10e-2	1.26
$h = \frac{1}{32}$	2.35e-2	1.13	2.26e-2	1.18
$h = \frac{1}{64}$	1.12e-2	1.07	1.05e-2	1.11

VI. Conclusion

In this work, we present a CNLF time stepping decoupled scheme for the fluid-fluid interaction problems based on the modular grad-div stabilized method. The proposed scheme is very efficient for the fluid-fluid interaction problems. The proposed scheme has the obvious advantage for computing the fluid-fluid interaction problems. The unconditional stability of the proposed scheme is proven. Numerical tests are proposed to verify the efficiency and robustness of the proposed method. Moreover, the modular grad-div stabilization and higher order time marching methods for fluid-fluid interaction are need to further study.

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