Hamming Index of the Product of Two Graphs

Harshitha A, Swati Nayak, Sabitha D'Souza* and Pradeep G. Bhat

***Abstract*—Let $A(G)$ be the adjacency matrix of a graph G. Let $s(v_i)$ denote the row entries of $A(G)$ corresponding to the vertex v_i of G. The Hamming distance between the strings $s(u_i)$ and $s(v_i)$ is the number of positions in which their elements differ. The sum of Hamming distance between all the pairs of vertices is the Hamming index of a graph. In this paper, we study the Hamming distance between the strings generated by the adjacency matrix of various products of complete bipartite and complete graph. We also compute the Hamming index generated by the adjacency matrix of these graph products.**

***Index Terms*—Hamming index, Hamming distance, adjacency matrix, strings.**

I. Introduction

LET $(\mathbb{Z}_2, +)$ be an additive group of integers modulo 2, where $\mathbb{Z}_2 = \{0, 1\}$. For a positive integer n, $(\mathbb{Z}_2)^n = (\mathbb{Z}_2)\ (\mathbb{Z}_2)...(\mathbb{Z}_2)$ ($n-$factors). Each $s \in (\mathbb{Z}_2)^n$ is an $n-$tuple $s = (s_1, s_2, ..., s_n)$ with each s_i is either 0 or 1 which is referred as a string. The number of $1's$ in each string is known as weight of that string. The Hamming distance between two strings $s = (s_1, s_2, ..., s_n)$ and $t = (t_1, t_2, ..., t_n)$ of equal length is the number of positions at which $s_i \neq t_i, 1 \leq i \leq n$, denoted by $H_d(s, t)$. For the vertices u_i, v_j of a graph, $H_d(u_i, v_j) = H_d(s(u_i), s(v_j))$. A graph G is called a Hamming graph [1], [2], if each vertex $u_i \in V(G)$ can be labeled by a string $s(u_i)$ of fixed length such that $H_d(s(u_i), s(v_j)) = d_G(u_i, v_j), \forall u_i, v_j \in V(G)$, where $d_G(u_i, v_j)$ is the distance between the vertices u_i and v_j which is the number of edges in the shortest $u_i - v_j$ path.

Hamming weight analysis of bits has a significant role in information theory, coding theory and cryptography. The minimum Hamming distance is used as error detecting and error correcting codes. The Hamming distance is also used in biological systematic as a measure of genetic distance. For more information on Hamming graphs, one can refer [3], [4], [5]. Motivated by the work on Hamming graphs, in this paper, we study the Hamming index of the product of two graphs with respect to its adjacency matrix.

The paper is organized as follows. The authors present the preliminaries in section 2. The Hamming indices of different product graphs are evaluated in section 3.

Manuscript received October 27, 2021; revised April 07, 2022.

Harshitha A is a research scholar in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India, 576104 (e-mail: harshuarao@gmail.com).

Swati Nayak is an assistant professor in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India, 576104 (e-mail: swati.nayak@manipal.edu).

*Corresponding author: Sabitha D'Souza is an assistant professor - selection grade in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India, 576104 (Phone: 9449727376; e-mail: sabitha.dsouza@manipal.edu).

Pradeep G. Bhat is a professor in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India, 576104 (e-mail: pg.bhat@manipal.edu).

The research method used is document analysis to study the previous research. The work is developed from a qualitative descriptive approach with deductive reasoning.

II. Preliminaries

Let G be a simple graph. The degree of a vertex $u_i \in V(G)$ denoted as $deg_G u_i$ is the number of edges incident with u_i. We denote two vertices u_i adjacent to v_j by $u_i \sim v_j$ and u_i not adjacent to v_j by $u_i \nsim v_j$. Neighbors of a vertex u_i are the vertices v_j such that $u_i \sim v_j$. A common neighbor of two vertices u_i, v_j is a vertex w_k such that $w_k \sim u_i$ and $w_k \sim v_j$, denoted by $N(u_i, v_j : G)$ and a non-common neighbor of the vertices u_i, v_j is a vertex w_k such that $w_k \nsim u_i$ and $w_k \nsim v_j$. The adjacency matrix $A(G)$ of a graph G is a square matrix $A(G)$ of order n, whose elements are a_{ij}, where a_{ij} is 1 if $v_i \sim v_j$ and is 0 if $v_i \nsim v_j$ in G.

Definition 1: [6] The Hamming index $H_A(G)$ of a graph G of order n is defined as

$$H_A(G) = \sum_{1 \leq i < j \leq n} H_d(u_i, u_j).$$

Example 1:

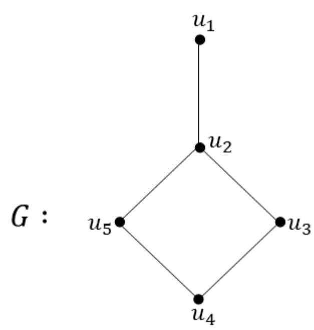

Figure 1. The Graph G.

Strings generated by $A(G)$ of graph G in Figure II are $s(u_1) = 01000$, $s(u_2) = 10101$, $s(u_3) = 01010$, $s(u_4) = 00101$, $s(u_5) = 01010$. Hamming index of G is $H_A(G) = 28$.

The graph products have been widely studied from different perspectives. These have applications in many branches like coding theory, network designs and chemical graph theory. Now we give definitions of various graph products. For more information on graph products, refer [7], [8], [9], [10].

The *corona product* $G \circ H$ of two graphs G and H is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining i^{th} vertex of G to each vertex in i^{th} copy of $H, i = 1, 2, \ldots, n$ [7].

The Cartesian, tensor, strong and symmetric product of the graphs G and H consist of $V(G) \times V(H)$ vertices.

In the *Cartesian product* $G\Box H$, two vertices (u_r, v_k), (u_s, v_l) are adjacent if and only if either
(i) $u_r = u_s$ and $u_k v_l \in E(H)$ or
(ii) $u_r u_s \in E(G)$ and $u_k = v_l$.

In the *tensor product* $G \times H$, two vertices $(u_r, v_k), (u_s, v_l)$ are adjacent if and only if $u_r \sim u_s$ and $v_k \sim v_l$ in G.

In the *strong product* $G \boxtimes H$, two vertices (u_r, v_k), (u_s, v_l) are adjacent if and only if
(i) $u_r = u_s$ and $v_k \sim v_l$ or
(ii) $v_k = v_l$ and $u_r \sim u_s$ or
(iii) $u_r \sim u_s$ and $v_k \sim v_l$ [8].

In the *composition* which is also called as *lexicographic product* $G[H]$, two vertices (u_r, v_k), (u_s, v_l) are adjacent if and only if
(i) $u_r \sim u_s$ or
(ii) $u_r = u_s$ and $v_k \sim v_l$ [11].

In the *symmetric product* $G \oplus H$, two vertices (u_r, v_k), (u_s, v_l) are adjacent if and only if either
(i) $u_r \sim u_s$ and $v_k \not\sim v_l$ or
(ii) $v_k \sim v_l$ and $u_r \not\sim u_s$ [12].

The *hierarchical product* $G \sqcap H$ is the graph with vertices (x_1, x_2), $x_1 \in V(G)$, $x_2 \in V(H)$ and edges $\{x_1x_2, y_1y_2\}$ where either
(i) $x_1 = y_1$ and $x_2 \sim y_2$ in G_2 or
(ii) $x_2 = 0$ and $x_1 \sim y_1$ in G_1 [13].

Theorem 1: [14] If two vertices v_i, v_j of a graph G have k common neighbors, then

$$H_d(v_i, v_j : G) = deg_G v_i + deg_G v_j - 2|N(v_i, v_j : G)|.$$

III. Main results

We now find the Hamming index of corona product of two graphs G_1 and G_2.

Theorem 2: For two graphs $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$,

$$\begin{aligned} H_A(G_1 \circ G_2) = {} & H_A(G_1) + n_1 H_A(G_2) + n_1 n_2 (2n_1 n_2 \\ & + 2n_1 - n_2 - 1) + 2n_1 m_2 (n_1 n_2 + n_1 - n_2 - 2) \\ & + 2m_1 n_2 (n_1 - 2). \end{aligned}$$

Proof:

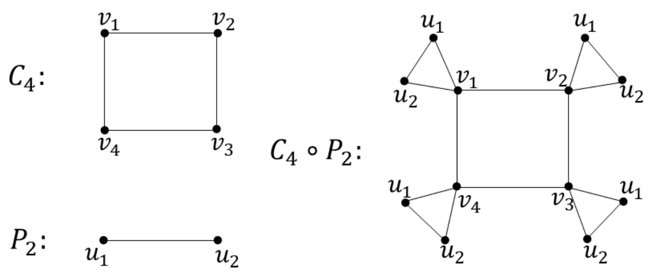

Figure 2. Corona product of the graphs C_4 and P_2.

Let $v_1, v_2, \ldots, v_{n_1} \in V(G_1)$ and $u_1, u_2, \ldots, u_{n_2} \in V(G_2)$. The Hamming index of $G_1 \circ G_2$ is given by,

$$\begin{aligned} H_A(G_1 \circ G_2) = {} & \sum_{v_i, v_j \in V(G_1)} H_d(v_i, v_j) \\ & + \sum_{u_i, u_j \in V(G_2)} H_d(u_i, u_j) \\ & + \sum_{\substack{v_i \in V(G_1), \\ u_j \in V(G_2)}} H_d(v_i, u_j). \end{aligned} \quad (1)$$

1) To determine $\sum_{v_i, v_j \in V(G_1)} H_d(v_i, v_j)$.
From Theorem 1, we have

$$\begin{aligned} H_d(v_i, v_j : G_1 \circ G_2) &= deg_{G_1} v_i + n_2 + deg_{G_1} v_j + n_2 \\ &\quad - 2|N(v_i, v_j : G_1)| \\ &= H_d(v_i, v_j : G_1) + 2n_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{v_i, v_j \in V(G_1)} H_d(v_i, v_j) &= \sum_{\binom{n_1}{2}} (H_d(v_i, v_j : G_1) + 2n_2) \\ &= H(G_1) + n_1 n_2 (n_1 - 1). \end{aligned} \quad (2)$$

2) To determine $\sum_{u_i, u_j \in V(G_2)} H_d(u_i, u_j)$.
There are n_1 copies of G_2 in $G_1 \circ G_2$ which is of the form $G_{2,i}, 1 \leq i \leq n_1$.
Case(i): Suppose both u_i and u_j belong to one of $V(G_{2,i})$, $1 \leq i \leq n_1$.
Then,

$$\begin{aligned} H_d(u_i, u_j : G_1 \circ G_2) &= deg_{G_2} u_i + 1 + deg_{G_2} u_j + 1 \\ &\quad - 2(|N(u_i, u_j : G_2)| + 1) \\ &= H_d(u_i, u_j : G_2). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{u_i, u_j \in V(G_{2,i})} H_d(u_i, u_j) &= \sum_{\binom{n_2}{2}} H_d(u_i, u_j : G_2) \\ &= H(G_2). \end{aligned}$$

Since there are n_1 copies of G_2,
$\sum_{u_i, u_j \in V(G_{2,i})} H_d(u_i, u_j) = n_1 H(G_2)$.
Case(ii): Suppose $u_i \in V(G_{2,i})$ and $u_j \in V(G_{2,j})$, $1 \leq i < j \leq n_2$.
Then,

$$\begin{aligned} H_d(u_i, u_j : G_1 \circ G_2) &= deg_{G_2} u_i + 1 + deg_{G_2} u_j \\ &\quad + 1 - 2(0) \\ &= deg_{G_2} u_i + deg_{G_2} u_j + 2. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{\substack{v_i \in V(G_{2,i}), \\ v_j \in V(G_{2,j})}} H_d(v_i, v_j) &= n_2 \binom{n_1}{2} 4m_2 + \sum_{\binom{n_1}{2} - n_1 \binom{n_2}{2}} 2 \\ &= 2n_1 n_2 (n_1 - 1) m_2 + n_1^2 n_2^2 \\ &\quad - n_1 n_2^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{u_i, u_j \in V(G_2)} H_d(u_i, u_j) &= n_1 H(G_2) \\ &\quad + 2n_1 n_2 m_2 (n_1 - 1) \\ &\quad + n_1^2 n_2^2 - n_1 n_2^2. \end{aligned} \quad (3)$$

3) To determine $\sum_{\substack{v_i \in V(G_1), \\ u_j \in V(G_2)}} H_d(v_i, u_j)$.

Case(i): Consider $v_i \in V(G_1)$ and $u_j \in V(G_{2,i})$.

$$H_d(v_i, u_j : G_1 \circ G_2) = deg_{G_1} v_i + n_2 + deg_{G_2} u_j + 1 - 2(deg_{G_2} u_i).$$

Therefore,

$$\sum_{\substack{v_i \in V(G_1),\\ u_j \in V(G_{2,i})}} H_d(v_i, u_j) = \sum_{n_1 n_2} (deg_{G_1} v_i + deg_{G_2} u_j + n_2 + 1) - 2 \sum_{n_1 n_2} deg_{G_2} u_j.$$

Case(ii): Consider $v_i \in V(G_1)$ and $u_j \in V(G_{2,j})$ such that $v_i \sim v_j$ in G_1, where v_j is the vertex of G_1 which is joined to $G_{2,j}$ in $G_1 \circ G_2$. Now, the number of adjacent pairs of vertices in G_1 is equal to m_1. Then,

$$\begin{aligned} H_d(v_i, u_j : G_1 \circ G_2) &= deg_{G_1} v_i + n_2 + deg_{G_2} u_j + 1 - 2(1) \\ &= deg_{G_1} v_i + deg_{G_2} u_j + n_2 - 1. \end{aligned}$$

Hence,

$$\sum_{d_{G_1}(v_i, v_j)=1} H_d(v_i, u_j) = \sum_{2n_1 n_2} (deg_{G_1} v_i + deg_{G_2} u_j + n_2 - 1).$$

Case(iii): Consider $v_i \in V(G_1)$ and $u_j \in V(G_{2,j})$ such that the distance between v_i and v_j in G_1 is greater than or equal to 2, where v_j is the vertex of G_1 which is joined to $G_{2,j}$ in $G_1 \circ G_2$. We observe that, the number of non-adjacent pairs of vertices of G_1 equals to $\binom{n_1}{2} - m_1$.
Then,

$$\begin{aligned} H_d(v_i, u_j : G_1 \circ G_2) &= deg_{G_1} v_i + n_2 + deg_{G_2} u_j + 1 - 2(0) \\ &= deg_{G_1} v_i + deg_{G_2} u_j + n_2 + 1. \end{aligned}$$

Therefore,

$$\sum_{d_{G_1}(v_i, v_j) \geq 2} H_d(v_i, u_j) = \sum_{2n_2\left(\binom{n_1}{2} - m_1\right)} (deg_{G_1} v_i + deg_{G_2} u_j + n_2 + 1).$$

Thus,

$$\begin{aligned} \sum_{\substack{v_i \in V(G_1),\\ u_j \in V(G_2)}} H_d(v_i, u_j) &= \sum_{n_1 n_2} (deg_{G_1} v_i + deg_{G_2} u_j + n_2 + 1) - 2 \sum_{n_1 n_2} deg_{G_2} u_j \\ &+ \sum_{2m_1 n_2} (deg_{G_1} v_i + deg_{G_2} u_j + n_2 - 1) \\ &+ \sum_{2n_2\left(\binom{n_1}{2} - m_1\right)} (deg_{G_1} v_i + deg_{G_2} u_j + n_2 + 1). \end{aligned}$$

$$\begin{aligned} \sum_{\substack{v_i \in V(G_1),\\ u_j \in V(G_2)}} H_d(v_i, u_j) &= \sum_{n_1 n_2} (deg_{G_1} v_i + deg_{G_2} u_j) \\ &- 2 \sum_{n_1 n_2} deg_{G_2} u_j + n_1^2 n_2^2 \\ &+ n_1 n_2 - 4m_1 n_2 \\ &+ n_1 n_2 (n_1 - 1). \end{aligned}$$

But

$$\sum_{n_1 n_2} (deg_{G_1} v_i + deg_{G_2} u_j) = 2n_1 n_2 m_1 + 2n_1^2 m_2$$

and $\sum_{n_1 n_2} deg_{G_2} u_j = 2n_1 m_2$.
Hence,

$$\begin{aligned} \sum_{\substack{v_i \in V(G_1),\\ u_j \in V(G_2)}} H_d(v_i, u_j) &= 2n_1 n_2 m_1 + 2n_1^2 m_2 + n_1^2 n_2^2 \\ &+ n_1 n_2 - 4m_1 n_2 + n_1 n_2 (n_1 - 1) \\ &- 4n_1 m_2. \end{aligned} \quad (4)$$

On substituting equations (2), (3) and (4) in equation (1) we get,

$$\begin{aligned} H_A(G_1 \circ G_2) &= H_A(G_1) + n_1 n_2 (n_1 - 1) + n_1 H_A(G_2) \\ &+ 2n_1 n_2 m_2 (n_1 - 1) + n_1^2 n_2^2 - n_1 n_2^2 + 2n_1 n_2 m_1 \\ &+ 2n_1^2 m_2 + n_1^2 n_2^2 + n_1 n_2 - 4m_1 n_2 \\ &+ n_1 n_2 (n_1 - 1) - 4n_1 m_2 \\ &= H_A(G_1) + n_1 H_A(G_2) \\ &+ n_1 n_2 (2n_1 n_2 + 2n_1 - n_2 - 1) + 2n_1 m_2 (n_1 n_2 \\ &+ n_1 - n_2 - 2) + 2m_1 n_2 (n_1 - 2). \end{aligned}$$

■

We now compute the Hamming index of Cartesian, tensor, strong, symmetric, and hierarchical product of complete bipartite graph $K_{m,n}$ and complete graph K_n.

Theorem 3:

$$\begin{aligned} H_A(K_{m,n} \square K_p) &= mnp(2p^2 + 2mp + 2np - m - n \\ &- 6p + 4) + mp(mp^2 - p^2 + 2p - mp - 1) \\ &+ np(np^2 - p^2 + 2p - np - 1). \end{aligned}$$

Proof:

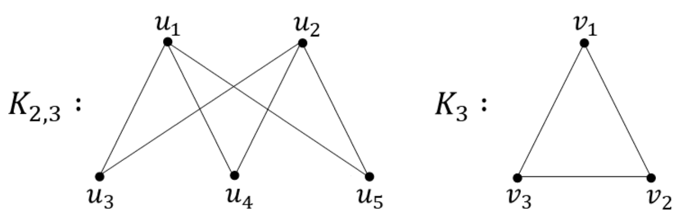

Figure 3. Graphs $K_{2,3}$ and K_3.

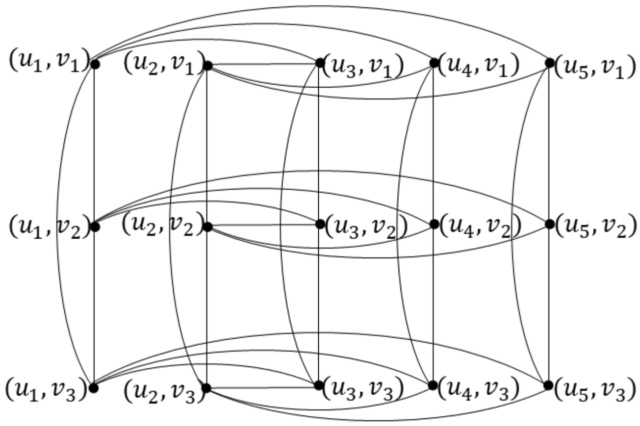

Figure 4. Cartesian product of the graphs $K_{2,3}$ and K_3.

Let $u_1, u_2, \ldots, u_m, u_{m+1}, u_{m+2}, \ldots, u_n$ be vertices of $K_{m,n}$ and $v_1, v_2, \ldots, v_p$ be vertices of K_p. Then the vertices of Cartesian product $K_{m,n} \Box K_p$ are (u_i, v_j), $i = 1, 2, \ldots, m, m+1, m+2, \ldots, n$ and $1 \le j \le p$.

By adding all the Hamming distances listed in Table I,

$$\begin{aligned} H_A(K_{m,n} \Box K_p) &= \binom{m}{2} p(2p-2) + \binom{n}{2} p(2p-2) \\ &+ \binom{p}{2} m(2n+2) + \binom{p}{2} n(2m+2) \\ &+ \binom{m}{2} p(p-1)(2n+2p-2) \\ &+ \binom{n}{2} p(p-1)(2m+2p-2) \\ &+ mnp(m+n+2p-2) \\ &+ mnp(p-1)(m+n+2p-6) \\ &= mnp(2p^2+2pm+2pn-6p-m \\ &- n+4) + mp(mp^2-p^2+2p-pm-1) \\ &+ np(np^2-p^2+2p-pn-1). \end{aligned}$$

■

Corollary 1: Hamming index of a Book graph $B_m(G)$ of order $2m+2$ is $10m^2+4m+2$.

Proof:

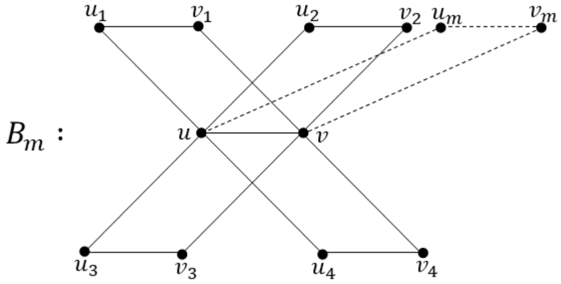

Figure 5. Book graph.

The Book graph $B_m(G)$ is a Cartesian product of star graph $K_{1,m}$ and complete graph K_2. Substituting the values of m, n and p in Theorem 3, we get

$H_A(B_m(G)) = 10m^2+4m+2$. ■

Theorem 4:

$$H_A(K_{m,n} \times K_p) = mnp(mp^2+np^2-m-n).$$

Proof:

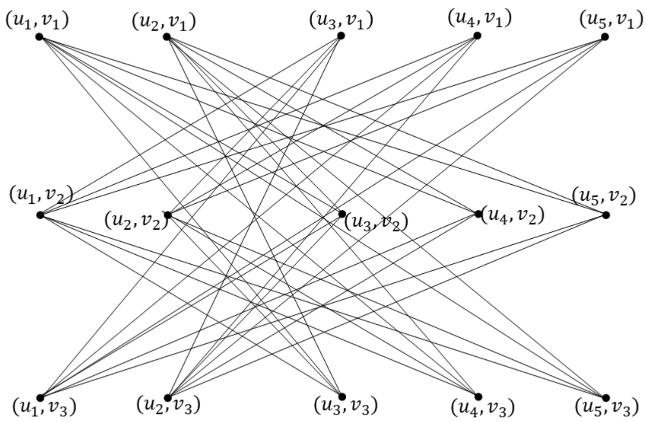

Figure 6. Tensor product of the graphs $K_{2,3}$ and K_3.

Let $u_1, u_2, \ldots, u_m, u_{m+1}, u_{m+2}, \ldots, u_n$ be vertices of $K_{m,n}$ and $v_1, v_2, \ldots, v_p$ be vertices of K_p. Then the vertices of tensor product $K_{m,n} \times K_p$ are (u_i, v_j), $i = 1, 2, \ldots, m, m+1, m+2, \ldots, n$ and $1 \le j \le p$.

Thus, from Table II,

$$\begin{aligned} H_A(K_{m,n} \times K_p) &= \binom{m}{2} p(2p-2) + 2m \binom{p}{2} \\ &+ p(p-1) \binom{m}{2} (2p-2) + 2mn \binom{p}{2} \\ &+ 2np(p-1) \binom{m}{2} + 2mn \binom{p}{2} \\ &+ 2mp(p-1) \binom{n}{2} \\ &+ mnp^2(mp+np-m-n) \\ &= mnp(mp^2+np^2-m-n). \end{aligned}$$

■

Theorem 5:

$$\begin{aligned} H_A(K_{m,n} \boxtimes K_p) &= mnp(2p^2+2pm+2pn-6p-m-n \\ &+ 4) + mp(mp^2-p^2+2p-pm-1) \\ &+ np(np^2-p^2+2p-pn-1). \end{aligned}$$

Proof:

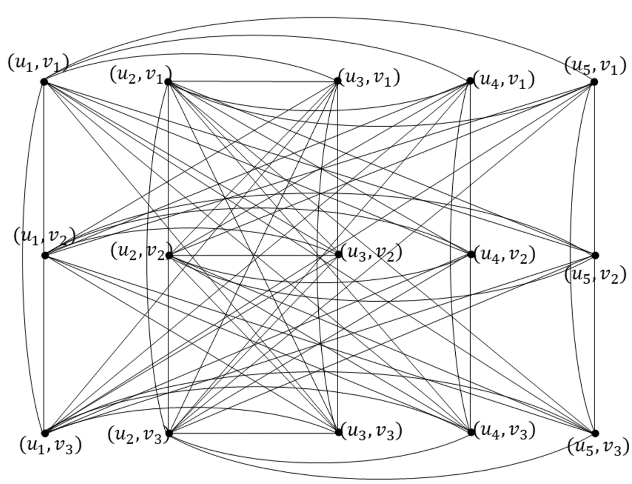

Figure 7. Strong product of the graphs $K_{2,3}$ and K_3.

Let $u_1, u_2, \ldots, u_m, u_{m+1}, u_{m+2}, \ldots, u_n$ be vertices of $K_{m,n}$ and $v_1, v_2, \ldots, v_p$ be vertices of K_p. Then the vertices of strong product $K_{m,n} \boxtimes K_p$ are (u_i, v_j), $i = 1, 2, \ldots, m, m+1, m+2, \ldots, n$ and $1 \le j \le p$.

From Table III, we get

$$\begin{aligned} H_A(K_{m,n} \boxtimes K_p) &= p\binom{n}{2}(2p-2) + 2n\binom{p}{2} \\ &+ p(p-1)(2p-2)\binom{n}{2} + p\binom{n}{2}(2p-2) \\ &+ 2n\binom{p}{2} + p(p-1)(2p-2)\binom{n}{2} \\ &+ mnp^2(mp+np-2p+2) \\ &= mnp(np^2+mp^2-2p^2+2p) \\ &+ mp(mp^2-p^2+2p-mp-1) \\ &+ np(np^2-p^2+2p-np-1). \end{aligned}$$

■

Remark 1: The composition and strong product of $K_{m,n}$ and K_p are isomorphic since every pair of vertices of K_p are adjacent. Therefore, $H_A(K_{m,n}[K_p]) = H_A(K_{m,n} \boxtimes K_p)$. That is,

$$\begin{aligned} H_A(K_{m,n}[K_p]) &= mnp(np^2+mp^2-2p^2+2p) \\ &+ mp(mp^2-p^2+2p-mp-1) \\ &+ np(np^2-p^2+2p-np-1). \end{aligned}$$

Theorem 6:

$$\begin{aligned} H_A(K_{m,n} \oplus K_p) &= mnp(mp^2+np^2+mp+np-m-n \\ &- 4p+4) + mp(2m^2p-2m^2-2mp \\ &+ 2m+p-1) + np(2n^2p-2n^2-2np \\ &+ 2n+p-1). \end{aligned}$$

Proof:

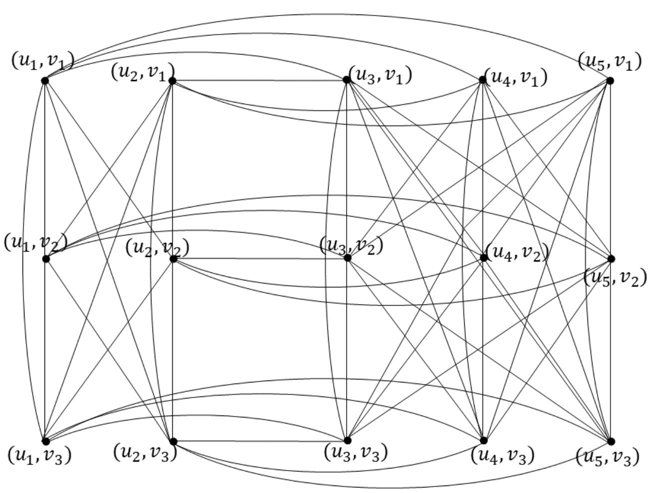

Figure 8. Symmetric product of the graphs $K_{2,3}$ and K_3.

Let $u_1, u_2, \ldots, u_m, u_{m+1}, u_{m+2}, \ldots, u_n$ be vertices of $K_{m,n}$ and $v_1, v_2, \ldots, v_p$ be vertices of K_p. Then the vertices of symmetric product $K_{m,n} \oplus K_p$ are (u_i, v_j), $i = 1, 2, \ldots, m, m+1, m+2, \ldots, n$ and $1 \le j \le p$.

By adding all the Hamming distances from Table IV,

$$\begin{aligned} H_A(K_{m,n} \oplus K_p) &= p\binom{m}{2}(2mp-2m-2p+2) \\ &+ m\binom{p}{2}(2m+2n) \\ &+ p(p-1)\binom{m}{2}(2m+2n) \\ &+ p(2np-2n-2p+2)\binom{n}{2} \\ &+ n(2m+2n)\binom{p}{2} \\ &+ p(p-1)\binom{n}{2}(2m+2n) \\ &+ mnp(mp+np) \\ &+ mnp(p-1)(mp+np-4) \\ &= mnp(2p^2+2pm+2pn-6p-m-n \\ &+ 4) + mp(mp^2-p^2+2p-pm-1) \\ &+ np(np^2-p^2+2p-pn-1). \end{aligned}$$

■

Theorem 7:

$$\begin{aligned} H_A(K_{m,n} \sqcap K_p) &= p^2(p-1)(m^2+n^2) \\ &+ mn(m+n)(2p-1) \\ &- p(p-1)^2(m+n) \\ &+ 2mn(p-1)(p^2-2). \end{aligned}$$

Proof:

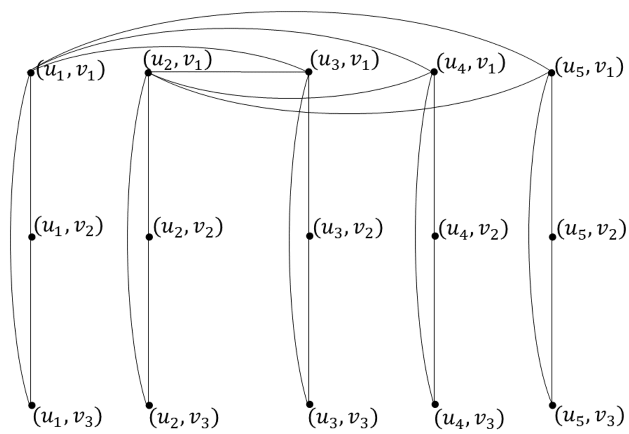

Figure 9. Hierarchical product of the graphs $K_{2,3}$ and K_3 with u_1 and v_1 as the root vertices.

Let $u_1, u_2, \ldots, u_m, u_{m+1}, u_{m+2}, \ldots, u_n$ be vertices of $K_{m,n}$ with any one vertex say, u_1 as the root vertex and $v_1, v_2, \ldots, v_p$ be vertices of K_p with any vertex say, v_1 as the root vertex. Then the vertices of hierarchical product $K_{m,n} \sqcap K_p$ are (u_i, v_j), $i = 1, 2, \ldots, m, m+1, m+2, \ldots, n$ and $1 \le j \le p$.

Thus from Table V,

$$\begin{aligned} H_A(K_{m,n} \sqcap K_p) &= 2p^2(p-1)(m^2+n^2) + mn(m+n) \\ &(2p-1) - p(p-1)^2(m+n) \\ &+ 2mn(p-1)(p^2-2). \end{aligned}$$

■

Table I
HAMMING DISTANCE BETWEEN ALL PAIRS OF VERTICES OF $K_{m,n}\Box K_p$.

Pairs of vertices	Degree of vertices	Number of pairs of vertices	Number of common neighbors	Hamming distance between vertices
(u_r, v_k), (u_s, v_l), $1 \le r, s \le m$, $1 \le k, l \le p$, $k = l$	$n+p-1$	$p\binom{m}{2}$	n	$2p-2$
(u_r, v_k), (u_s, v_l), $1 \le r,\ s \le m$, $1 \le k, l \le p$, $r = s$		$m\binom{p}{2}$	$p-2$	$2n+2$
(u_r, v_k), (u_s, v_l), $1 \le r, s \le m$, $1 \le k, l \le p$, $r, s \ne k, l$		$p(p-1)\binom{m}{2}$	0	$2n+2p-2$
(u_r, v_k), (u_s, v_l), $m+1 \le r, s \le n$, $1 \le k, l \le p$, $k = l$	$m+p-1$	$p\binom{n}{2}$	m	$2p-2$
(u_r, v_k), (u_s, v_l), $m+1 \le r, s \le n$, $1 \le k, l \le p$, $r = s$		$n\binom{p}{2}$	$p-2$	$2m+2$
(u_r, v_k), (u_s, v_l), $m+1 \le r, s \le n$, $1 \le k, l \le p$, $r, s \ne k, l$		$p(p-1)\binom{n}{2}$	0	$2m+2p-2$
(u_r, v_k), $1 \le r \le m$, $1 \le k \le p$	$n+p-1$	mnp	0	Hamming distance between one vertex of degree $n+p-1$ and another vertex of degree $m+p-1$ with $v_k = v_l$ is $m+n+2p-2$
(u_s, v_l), $m+1 \le s \le n$, $1 \le l \le p$	$m+p-1$			
(u_r, v_k), $1 \le r \le m$, $1 \le k \le p$	$n+p-1$	$mnp(p-1)$	0	Hamming distance between one vertex of degree $n+p-1$ and another vertex of degree $m+p-1$ with $v_k \ne v_l$ is $m+n+2p-2$
(u_s, v_l), $m+1 \le s \le n$, $1 \le l \le p$	$m+p-1$			

Table II
HAMMING DISTANCE BETWEEN ALL PAIRS OF VERTICES OF $K_{m,n} \times K_p$.

Pairs of vertices	Degree of vertices	Number of pairs of vertices	Number of common neighbors	Hamming distance between vertices
(u_r, v_k), (u_s, v_l), $1 \le r, s \le m$, $1 \le k, l \le p$, $k = l$	$np+p-1$	$p\binom{m}{2}$	np	$2p-2$
(u_r, v_k), (u_s, v_l), $1 \le r, s \le m$, $1 \le k, l \le p$, $r = s$		$m\binom{p}{2}$	$p-2+np$	2
(u_r, v_k), (u_s, v_l), $1 \le r, s \le m$, $1 \le k, l \le p$, $r, s \ne k, l$		$p(p-1)\binom{m}{2}$	np	$2p-2$
(u_r, v_k), (u_s, v_l), $1 \le r, s \le m$, $1 \le k, l \le p$, $k = l$	$n(p-1)$	$p\binom{m}{2}$	$n(p-1)$	0
(u_r, v_k), (u_s, v_l), $1 \le r, s \le m$, $1 \le k, l \le p$, $r = s$		$m\binom{p}{2}$	$n(p-2)$	$2n$
(u_r, v_k), (u_s, v_l), $1 \le r \le m$, $1 \le k \le p$, $r, s \ne k, l$		$p(p-1)\binom{m}{2}$	$n(p-2)$	$2n$
(u_r, v_k), (u_s, v_l), $m+1 \le r, s \le n$, $1 \le k, l \le p$, $k = l$	$m(p-1)$	$p\binom{n}{2}$	$m(p-1)$	0
(u_r, v_k), (u_s, v_l), $m+1 \le r \le n$, $1 \le k \le p$, $r = s$		$n\binom{p}{2}$	$m(p-2)$	$2m$
(u_r, v_k), (u_s, v_l), $m+1 \le r \le n$, $1 \le k \le p$, $r, s \ne k, l$		$p(p-1)\binom{n}{2}$	$m(p-2)$	$2m$
(u_r, v_k), $1 \le r \le m$, $1 \le k \le p$	$n(p-1)$	mnp^2	0	Hamming distance between one vertex of degree $n(p-1)$ and another vertex of degree $m(p-1)$ is $mp+np-m-n$
(u_s, v_l), $m+1 \le s \le n$, $1 \le l \le p$	$m(p-1)$			

Table III
HAMMING DISTANCE BETWEEN ALL PAIRS OF VERTICES OF $K_{m,n} \boxtimes K_p$.

Pairs of vertices	Degree of vertices	Number of pairs of vertices	Number of common neighbors	Hamming distance between vertices
(u_r, v_k), (u_s, v_l), $m+1 \leq r, s \leq n$, $1 \leq k, l \leq p$, $k = l$	$mp + p - 1$	$p\binom{n}{2}$	mp	$2p - 2$
(u_r, v_k), (u_s, v_l), $m+1 \leq r, s \leq n$, $1 \leq k, l \leq p$, $r = s$		$n\binom{p}{2}$	$p - 2 + mp$	2
(u_r, v_k), (u_s, v_l), $m+1 \leq r, s \leq n$, $1 \leq k, l \leq p$, $r, s \neq k, l$		$p(p-1)\binom{n}{2}$	mp	$2p - 2$
(u_r, v_k), (u_s, v_l), $m+1 \leq r, s \leq n$, $1 \leq k, l \leq p$, $k = l$	$mp + p - 1$	$p\binom{n}{2}$	mp	$2p - 2$
(u_r, v_k), (u_s, v_l), $m+1 \leq r, s \leq n$, $1 \leq k, l \leq p$, $r = s$		$n\binom{p}{2}$	$p - 2 + mp$	2
(u_r, v_k), (u_s, v_l), $m+1 \leq r, s \leq n$, $1 \leq k, l \leq p$, $r, s \neq k, l$		$p(p-1)\binom{n}{2}$	mp	$2p - 2$
(u_r, v_k), $1 \leq r \leq m$, $1 \leq k \leq p$	$np + p - 1$	mnp^2	$2p - 2$	Hamming distance between one vertex of degree $np + p - 1$ and another vertex of degree $mp + p - 1$ is $mp + np - 2p + 2$
(u_s, v_l), $m+1 \leq s \leq n$, $1 \leq l \leq p$	$mp + p - 1$			

Table IV
HAMMING DISTANCE BETWEEN ALL PAIRS OF VERTICES OF $K_{m,n} \oplus K_p$.

Pairs of vertices	Degree of vertices	Number of pairs of vertices	Number of common neighbors	Hamming distance between vertices
$(u_r, v_k), (u_s, v_l)$, $1 \leq r, s \leq m$, $1 \leq k, l \leq p$, $k = l$	$n + m(p-1)$	$p\binom{m}{2}$	$n + p - 1$	$2mp - 2m - 2p + 2$
(u_r, v_k), (u_s, v_l), $1 \leq r, s \leq m$, $1 \leq k, l \leq p$, $r = s$		$m\binom{p}{2}$	$m(p-2)$	$2m + 2n$
(u_r, v_k), (u_s, v_l), $1 \leq r, s \leq m$, $1 \leq k, l \leq p$, $r, s \neq k, l$		$p(p-1)\binom{m}{2}$	$m(p-2)$	$2m + 2n$
$(u_r, v_k), (u_s, v_l)$, $m+1 \leq r, s \leq n$, $1 \leq k, l \leq p$, $k = l$	$m + n(p-1)$	$p\binom{n}{2}$		$2np - 2n - 2p + 2$
(u_r, v_k), (u_s, v_l), $m+1 \leq r, s \leq n$, $1 \leq k, l \leq p$, $r = s$		$n\binom{p}{2}$	$n(p-2)$	$2m + 2n$
(u_r, v_k), (u_s, v_l), $m+1 \leq r, s \leq n$, $1 \leq k, l \leq p$, $r, s \neq k, l$		$p(p-1)\binom{n}{2}$	$n(p-2)$	$2m + 2n$
(u_r, v_k), $1 \leq r \leq m$, $1 \leq k \leq p$	$n + m(p-1)$	mnp	0	Hamming distance between one vertex of degree $n + m(p-1)$ and another vertex of degree $m + n(p-1)$ with $v_k = v_l$ is $mp + np$
(u_s, v_l), $m+1 \leq s \leq n$, $1 \leq l \leq p$	$m + n(p-1)$			
(u_r, v_k), $1 \leq r \leq m$, $1 \leq k \leq p$	$n + m(p-1)$	$mnp(p-1)$	2	Hamming distance between one vertex of degree $n + m(p-1)$ and another vertex of degree $m + n(p-1)$ with $v_k \neq v_l$ is $mp + np - 4$
(u_s, v_l), $m+1 \leq s \leq n$, $1 \leq l \leq p$	$m + n(p-1)$			

Table V
HAMMING DISTANCE BETWEEN ALL PAIRS OF VERTICES OF $K_{m,n} \sqcap K_p$.

Pairs of vertices	Degree of vertices	Number of pairs of vertices	Number of common neighbors	Hamming distance between vertices
(u_r, v_1), (u_s, v_1), $1 \leq r, s \leq m$	$2n + 2p - 2$	$\binom{m}{2}$	n	$2(p-1)$
(u_r, v_1), (u_s, v_1), $m+1 \leq r, s \leq n$	$2m + 2p - 2$	$\binom{n}{2}$	m	$2(p-1)$
(u_r, v_1), (u_s, v_1), $1 \leq r \leq m$, $m+1 \leq s \leq n$	$m + n + 2p - 2$	mn	0	$m + n + 2p - 2$
(u_r, v_k), (u_r, v_l), $1 \leq r \leq n$, $2 \leq k, l \leq p$	$2p - 2$	$(m+n)\binom{p-1}{2}$	$p - 2$	2
(u_r, v_k), (u_s, v_l), $1 \leq r, s \leq n$, $r \neq s$, $2 \leq k, l \leq p$	$2p - 2$	$(p-1)^2\binom{m+n}{2}$	0	$2(p-1)$
(u_r, v_1), (u_r, v_l), $1 \leq r \leq m$, $2 \leq l \leq p$	$n + 2p - 2$	$m(p-1)$	$p - 2$	$n + 2$
(u_r, v_1), (u_r, v_l), $m+1 \leq r \leq n$, $2 \leq l \leq p$	$m + 2p - 2$	$n(p-1)$	$p - 2$	$m + 2$
(u_r, v_1), (u_s, v_l), $1 \leq r, s \leq m$, $r \neq s$, $2 \leq l \leq p$	$n + 2p - 2$	$m(m-1)(p-1)$	0	$n + 2p - 2$
(u_r, v_1), (u_s, v_l), $m+1 \leq r, s \leq n$, $r \neq s$, $2 \leq l \leq p$	$m + 2p - 2$	$n(n-1)(p-1)$	0	$m + 2p - 2$
(u_r, v_1), (u_s, v_l), $1 \leq r \leq m$, $m+1 \leq s \leq n$, $2 \leq l \leq p$	$n + 2p - 2$	$mn(p-1)$	1	$n + 2p - 4$
(u_r, v_1), (u_s, v_l), $m+1 \leq r \leq n$, $1 \leq s \leq m$, $2 \leq l \leq p$	$m + 2p - 2$	$mn(p-1)$	1	$m + 2p - 4$

IV. CONCLUSION

The results of this article give the sum of Hamming distances between all pairs of strings generated by the adjacency matrix of a graph. Hamming distance between the pair of vertices depends on the degree and number of common neighbors. In this paper, we have studied Hamming index of Cartesian, tensor, strong, symmetric and hierarchical products of complete bipartite and complete graphs.

REFERENCES

[1] W. Imrich and S. Klavzar, "A simple O(mn) algorithm for recognizing Hamming graphs," *Bulletin of the institute of combinatorics and its applications*, vol. 9, pp. 45-56, 1993.

[2] A. B. Ganagi and H. S. Ramane, "Hamming distance between the strings generated by adjacency matrix of a graph and their sum," *Algebra and discrete mathematics*, vol. 22, no. 1, pp. 82-93, 2016.

[3] S. Bang, R. R. Van Dam and J. H. Koolen, "Spectral characterization of the Hamming graphs," *Linear algebra and its applications*, vol. 429, no. 11-12, pp. 2678-2686, 2008.

[4] V. Chepoi, "d-convexity and isometric subgraphs of Hamming graphs," *Cybernetics*, vol. 1, pp. 6-9, 1988.

[5] W. Imrich and S. Klavzar, "Recognizing Hamming graphs in linear time and space," *Information processing letters*, vol. 63, no. 2, pp. 91-95, 1997.

[6] H. S. Ramane and A. B. Ganagi, "Hamming index of class of graphs," *International journal of current engineering and technology, Special Issue 1* , pp. 205-208, 2013.

[7] R. Frucht and F. Harary, "On the corona of two graphs," *Aequationes mathematicae*, vol. 4, pp. 322-325, 1970.

[8] G. Sabidussi, "Graph multiplication," *Mathematische zeitschrift*, vol. 72, no. 1, pp. 446-457, 1959.

[9] P. M. Weichsel, "The kronecker product of graphs," *Proceedings of the American mathematical society*, vol. 13, no. 1, pp. 47-52, 1962.

[10] M. Yuan, F. Wen, and R. Wang, "Merging The A- and Q-spectrum of Three Kinds of Graph Product," *IAENG International journal of applied mathematics*, vol. 51, no. 4, pp. 990-994, 2021.

[11] G. Sabidussi, "The lexicographic product of graphs," *Duke mathematical journal*, vol. 28, no. 4, pp. 573-578, 1961.

[12] R. Balakrishnan, K. Ranganathan, *A textbook of graph theory*. New York, NY, USA: Springer, 2000.

[13] L. Barriere, F. Comellas, C. Dalfó, M. Fiol, "The hierarchical product of graphs," *Discrete Applied Mathematics*, vol. 157, no. 1, pp. 36-48, 2009.

[14] R. L. Pasaribu, Mardiningsih and S. Suwilo, "Hamming index of thorn and double graphs," *Bulletin of mathematics*, vol. 10, no. 1, pp. 25-32, 2018.