Solution of Fully Fuzzy Multi-objective Linear Fractional Programming Problems- A Gauss Elimination Approach

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Abstract—**A fuzzy version of Gauss Elimination Approach (GEA) for the solution of fully fuzzy multi objective linear fractional programming (FFMOLFP) problems involving triangular fuzzy numbers (TFNs) is presented in this article. Fully fuzzy linear fractional programming problem is first reduced to an equivalent fully fuzzy linear programming (FFLP) problem by suitable transformation and then the optimum value of each objective function is obtained individually with respect to the same set of constraints. Secondly by using all these objective values, the FFMOLFP problem is then converted to a single objective non fractional FFLP problem and its optimum solution is obtained which in turn provides the Pareto optimum solution the given FFMOLFP problem. To indicate the efficacy of the proposed procedure, a numerical illustration is given.**

Index Terms—**Multi-objective linear fractional programming, Pareto optimum, Gauss elimination method, TFNs, Parametric form, Fuzzy arithmetic, Ranking.**

I. Introduction

LINEAR fractional programming (LFP) problems are used to solve problems in organizations, industries, etc. When some or all the decision parameters of the problem are uncertain and are modeled as fuzzy numbers, then the underlying LFP problem becomes fuzzy linear fractional programming (FLFP) problem. When the problem posses more than one objective then it is a fuzzy multi-objective linear fractional programming (FMOLFP) problem. FMOLFP problems are applied to solve real world problems. Bellman and Zadeh [1] introduced a system for making decisions in a fuzzy environment. Guzel and Sivri [5] solved the MOLFP problem with multiple efficient solution. Nuran Guzel [13] introduced a new solution method for resolving multi-objective fractional programming problems by converting MOLFP problem into LPP. Farhana Akond Pramy [3] solved fuzzy MOLFP problem using graded mean integration method. Bhargava [2] solving linear programming problem with integer solution using Gauss method. Jain et al. [6],[7],[8] studied MOLFP problem by using the concept of bounds and used Gauss Elimination Method. Loganathan and Ganesan [10], [11] proposed a fuzzy approach to solve single and multi objective FFLFP problems. Rizk Allah [15] discussed two optimization approaches for bi-level MOLFP problems. Sapan Kumar Dasa [17] discussed FLFP problem using LU- decomposition method. Payan [14] proposed a linear modeling to solve MOLFP problem with fuzzy parameters. Surapati [19] discussed fuzzy MOLFPP with Taylor series method. Yonghong et al. [20] solved the generalized linear fractional programming using new linear relaxation technique.Yong-Hong Zhang and Chun-Feng Wang [21] proposed a new branch-and-reduce approach for solving generalized linear fractional programming problems. Rubi Arya and Pitam Singh [16] conducted a survey on fractional programming problems. Stephen Gbenga Fashoto and Sulaiman [18] presented an approach for solving a class of complexity in multi objective linear programming(MOLP) problems with fuzzy objective functions. Jiang and Qiu [9] proposed an optimization criterion of fuzzy linear problems.

In this paper, each linear fractional objective function is transformed to an equivalent linear objective function and is then converted to linear inequality and then the set of linear inequalities is solved using Gauss Elimination method. After getting the optimum value of each objective function, the FFMOLFP problem is then reduced into a single objective FFLP problem and then the FFLP problem is solved using Gauss Elimination method.

This article is organized as follows: Section 2 introduces fuzzy numbers and offers basic principles and outcomes, while Section 3 explains the mathematical formulation and methods for solving the FFMOLFP problem. The fourth section contains numerical examples, and the final section contains the conclusion.

II. PRELIMINARIES

Definition 2.1: A fuzzy number $\tilde{h}$ on R is a triangular fuzzy number if its membership function $\tilde{h} : R \to [0, 1]$ has the following characteristics:

$$\tilde{h}(x) = \begin{cases} \dfrac{x - h_1}{h_2 - h_1}, & \text{for } h_1 \leq x \leq h_2 \\ \dfrac{h_3 - x}{h_3 - h_2}, & \text{for } h_2 \leq x \leq h_3 \\ 0, & \text{otherwise.} \end{cases}$$

We denote this triangular fuzzy number as $\tilde{h} = (h_1, h_2, h_3)$. We use F(R) to denote the set of all triangular fuzzy numbers defined on R.

Definition 2.2: A triangular fuzzy number $\tilde{h} = (h_1, h_2, h_3)$ in $F(R)$ can also be represented as a pair $\tilde{h} = (\underline{h}, \overline{h})$ of functions $\underline{h}(a), \overline{h}(a)$ for $0 \leq a \leq 1$, which satisfies the following requirements:

- $\underline{h}(a)$ is a bounded monotonic increasing left continuous function.
- $\overline{h}(a)$ is a bounded monotonic decreasing left continuous function.

Manuscript received September 07, 2021; revised May 20, 2022.

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- $\underline{h}(a) \leq \overline{h}(a), 0 \leq a \leq 1$.

It is also represented by $\tilde{h} = (h_0, h_*, h^*)$ where $h_* = (h_0 - \underline{h})$, $h^* = (\overline{h} - h_0)$ are called the left fuzziness index function and the right fuzziness index function respectively. For an arbitrary triangular fuzzy number $\tilde{h} = (\underline{h}, \overline{h})$, the number $h_0 = \left(\frac{\underline{h}(1) + \overline{h}(1)}{2}\right)$ is said to be a location index number of $\tilde{h}$.

A. Ranking of TFNs

Several approaches for the ranking of fuzzy numbers have been proposed in the literature. An efficient approach for comparing the fuzzy numbers is by the use of a ranking function based on their graded means. We define the magnitude of the triangular fuzzy number $\tilde{h} = (h_0, h_*, h^*)$ by

$$R(\tilde{h}) = \left(\frac{h^* + 4h_0 - h_*}{4}\right) = \left(\frac{\underline{h} + \overline{h} + h_0}{4}\right).$$

For any two triangular fuzzy numbers $\tilde{h} = (h_0, h_*, h^*)$ and $\tilde{k} = (k_0, k_*, k^*)$ in F(R) we have

- $\tilde{h} \succ \tilde{k} \Leftrightarrow R(\tilde{h}) > R(\tilde{k})$
- $\tilde{h} \prec \tilde{k} \Leftrightarrow R(\tilde{h}) < R(\tilde{k})$
- $\tilde{h} \approx \tilde{k} \Leftrightarrow R(\tilde{h}) = R(\tilde{k})$

If $R(\tilde{h}) \geq R(\tilde{0})$, then the triangular fuzzy number $\tilde{h} = (h_0, h_*, h^*)$ is said to be non-negative and is denoted by $\tilde{h} \succeq \tilde{0}$.

B. Arithmetic Operations of TFN's

Ming Ma et al.[12] have proposed a new fuzzy arithmetic based upon both location index and fuzziness index functions with ordinary arithmetic and lattice rule. i.e. $\nabla = \{+.-, \times, \div\}$ the arithmetic operations on the fuzzy numbers are defined by

$$\begin{aligned}\tilde{h}\nabla\tilde{k} &= (h_0, h_*, h^*)\nabla(k_0, k_*, k^*)\\ &= (h_0\nabla k_0, \max(h_*, k_*), \max(h^*, k^*))\end{aligned}$$

Note: Division is possible only when the location index number of the denominator fuzzy number is non-zero.

III. FULLY FUZZY MOLFP PROBLEM

The general form of FFMOLFP problem is given by

$$\begin{aligned}\mathbf{P}_I : \max \tilde{\mathbf{z}}(\tilde{h}) =&[\tilde{z}_i(\tilde{h})], i = 1, 2..., n\\ \text{subject to } &\tilde{A}\tilde{\mathbf{h}} \preceq \tilde{\mathbf{b}}\\ \text{and } &\tilde{\mathbf{h}} \succeq \tilde{\mathbf{0}}.\end{aligned} \tag{1}$$

where $\tilde{z}_i(\tilde{h})$= $\frac{f_i(\tilde{h})}{g_i(\tilde{h})}$ = $\frac{\tilde{\mathbf{c}}_i\tilde{\mathbf{h}} + \tilde{\alpha}_i}{\tilde{\mathbf{d}}_i\tilde{\mathbf{h}} + \tilde{\beta}_i}$, $\tilde{A} = (\tilde{a}_{ij})_{(m\times n)}$, $\tilde{\mathbf{h}} = (\tilde{h}_1, \tilde{h}_2, ..., \tilde{h}_n)$, $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_m)$, $\tilde{\mathbf{c}} = (\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_n)$, $\tilde{\mathbf{d}} = (\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_n)$, $\tilde{\alpha}_i, \tilde{\beta}_i \in F(R)$.

Definition 3.1: Let $\tilde{S}$ be the set of all feasible solutions of the FFMOLFP problem (1). A feasible solution $\tilde{\mathbf{h}}^*$ is said to be Pareto optimum solution of the FFMOLFP problem (1), if there does not exist another feasible solution $\tilde{\mathbf{h}} \in \tilde{S}$ such that $\tilde{z}_i(\tilde{\mathbf{h}}) \succeq \tilde{z}_i(\tilde{\mathbf{h}}^*)$ for all i and $\tilde{z}_j(\tilde{\mathbf{h}}) \succ \tilde{z}_j(\tilde{\mathbf{h}}^*)$ atleast one j.

A. Transformation of FFMOLFPP into FFMOLPP

The FFMOLFP prroblem (1) is transformed in to a non-fractional FFMOLPP as

$$\begin{aligned}\mathbf{P}_{II} : &\max_{\tilde{h}\in S}\{f_i(\tilde{h}) - \tilde{z}_i{}^*(g_i(\tilde{h}))\}\\ &\text{subject to } \tilde{A}\tilde{\mathbf{h}} \preceq \tilde{\mathbf{b}}\\ &\text{and } \tilde{\mathbf{h}} \succeq \tilde{\mathbf{0}}.\end{aligned} \tag{2}$$

where $\tilde{z}_i{}^*$ is the optimum value of $\tilde{z}_i(\tilde{h})$, $i = 1, 2, 3, \cdots, n$.

B. Transformation of FFMOLPP into FFLPP

Guzel et al.[5] have transformed the given MOLFPP in to a single objective LPP and then obtained the Pareto optimum solution of MOLFPP from the optimum solution of the transformed LPP. We transform the given FFMOLPP (2) in to a single objective FFLPP as

$$\begin{aligned}\mathbf{P}_{III} : \max \tilde{z} = &\left\{\sum_{i=1}^{n}[f_i(\tilde{h}) - \tilde{z}_i^*(g_i(\tilde{h}))]/\tilde{h} \in \tilde{S}\right\}\\ &\text{subject to } \tilde{A}\tilde{\mathbf{h}} \preceq \tilde{\mathbf{b}}\\ &\text{and } \tilde{\mathbf{h}} \succeq \tilde{\mathbf{0}}.\end{aligned} \tag{3}$$

where $\tilde{z}_i^*$ is the optimal value of objective function $\tilde{z}_i$, that is

$\tilde{z}_i^* = z_i(\tilde{h}_i^*)$= $\max\left\{\frac{f_i(\tilde{h})}{g_i(\tilde{h})}/\tilde{h} \in \tilde{S}\right\}$ and $i = 1, 2, ...n$.

Theorem 3.1: [11] $\tilde{z}^* \approx \frac{f(\tilde{h^*})}{g(\tilde{h^*})} \approx \max\left\{\frac{f(\tilde{h})}{g(\tilde{h})}/\tilde{h} \in \tilde{S}\right\}$ if and only if $F(\tilde{z}^*, \tilde{h}^*) \approx \max\{f(\tilde{h}^*) - \tilde{z}^* g(\tilde{h}^*)/\tilde{h}^* \in \tilde{S}\} \approx \tilde{0}$.

Theorem 3.2: If $\tilde{\mathbf{h}}^*$ is an optimum solution of the problem (3), then $\tilde{\mathbf{h}}^*$ is a pareto optimum solution of the problem (1).

Proof: Assume that $\tilde{\mathbf{h}}^*$ is an optimum solution of the problem (3). Suppose that $\tilde{\mathbf{h}}^*$ is not a Pareto optimum solution of the problem (1). Then there exists a $\tilde{\mathbf{h}} \in \tilde{S}$ such that $\tilde{z}_i(\tilde{\mathbf{h}}) \succeq \tilde{z}_i(\tilde{\mathbf{h}}^*)$ for all $i = 1, 2, ..., n$ and $\tilde{z}_j(\tilde{\mathbf{h}}) \succ \tilde{z}_j(\tilde{\mathbf{h}}^*)$ for at least one $j = 1, 2, ..., n$.

That is $\frac{f_i(\tilde{\mathbf{h}})}{g_i(\tilde{\mathbf{h}})} \succeq \frac{f_j(\tilde{\mathbf{h}}^*)}{g_j(\tilde{\mathbf{h}}^*)}$ for all i and $\frac{f_j(\tilde{\mathbf{h}})}{g_j(\tilde{\mathbf{h}})} \succ \frac{f_j(\tilde{\mathbf{h}}^*)}{g_j(\tilde{\mathbf{h}}^*)}$ for at least one j.

$\Rightarrow \frac{f_i(\tilde{\mathbf{h}})}{g_i(\tilde{\mathbf{h}})} \succeq \tilde{z}_i^*$ for all i and $\frac{f_j(\tilde{h})}{g_j(\tilde{h})} \succ \tilde{z}_j^*$ for at least one j.

$\Rightarrow f_i(\tilde{\mathbf{h}}) - \tilde{z}_i^* g_i(\tilde{\mathbf{h}}) \succeq \tilde{0}$ for all i and $f_j(\tilde{\mathbf{h}}) - \tilde{z}_j^* g_j(\tilde{\mathbf{h}}) \succ \tilde{0}$ for at least one j.

Summing over n, we get

$$\sum_{i=1}^{n}[f_i(\tilde{\mathbf{h}}) - \tilde{z}_i^* g_i(\tilde{\mathbf{h}})] \succeq \tilde{0}, \text{ for all } i \text{ and}$$

$$\sum_{j=1}^{n}[f_j(\tilde{\mathbf{h}}) - \tilde{z}_j^* g_j(\tilde{\mathbf{h}})] \succ \tilde{0}, \text{ for at least one } j.$$

$$\Rightarrow \sum_{i=1}^{n}[f_i(\tilde{\mathbf{h}}) - \tilde{z}_i^* g_i(\tilde{\mathbf{h}})] \succeq \sum_{i=1}^{n}[f_i(\tilde{\mathbf{h}}^*) - \tilde{z}_i^* g_i(\tilde{\mathbf{h}}^*)]$$

for all i and

$$\sum_{j=1}^{n}[f_j(\tilde{\mathbf{h}}) - \tilde{z}_j^* g_j(\tilde{\mathbf{h}})] \succ \sum_{i=1}^{n}[f_j(\tilde{\mathbf{h}}^*) - \tilde{z}_j^* g_j(\tilde{h}^*)]$$

for at least one j, which is a contradiction to our assumption that $\tilde{\mathbf{h}}^*$ is an optimum solution of the problem (3). Hence $\tilde{\mathbf{h}}^*$ be a Pareto optimum solution of the problem (1).

C. Algorithm

Step 1: Given a FFMOLFPP. Express all the fuzzy numbers in their parametric form.
Step 2: Convert each linear fractional objective function to an equivalent non-fractional linear objective function in terms of y, t using $\dfrac{1}{g(\tilde{h})} \preceq \tilde{t}$ and then $\tilde{y} \approx \tilde{h}\tilde{t}$.
Step 3: Convert each linear objective function into a linear inequality in terms of z, y, t using $\tilde{z} \preceq \max \tilde{z}$. Hence each linear fractional objective function along with the same set of constraints provides a system of linear inequalities.
Step 4: Compute the optimum value of each fractional objective function using the Gauss Elimination approach by solving the systems obtained in step 3.
Step 5: Using the optimum value of each objective function obtained in step 4, convert the given FFMOLFPP into a single objective FFLPP.
Step 6: Transform the FFLPP obtained in step 5 to a system of linear inequalities using $\tilde{z} \preceq \max \tilde{z}$.
Step 7: Solve the systems obtained in step 6 using the Gauss Elimination approach for the optimum solution of the single objective FFLPP obtained in step 5, which in turn provides the pareto optimum solution of the given FFMOLFPP.

IV. NUMERICAL EXAMPLE

Example 1: Consider a FFMOLFPP discussed by Surapati Pramanik and Indrani Maiti [19]

$$\begin{aligned} \max \tilde{z}_1 &\approx \frac{\tilde{2}\tilde{h}_1 + 4\tilde{h}_2 + \tilde{5}}{\tilde{2}\tilde{h}_1 + 6} \\ \max \tilde{z}_2 &\approx \frac{\tilde{h}_1 + \tilde{6}\tilde{h}_2 + 50}{\tilde{h}_1 + \tilde{1}\tilde{h}_2 + \tilde{8}} \\ \text{subject to } & \tilde{2}\tilde{h}_1 + 2\tilde{h}_2 \preceq \tilde{140} \\ & \tilde{h}_2 \succeq \tilde{8} \\ & \tilde{h}_1 \succeq \tilde{16} \\ \text{and } & \tilde{h}_1, \tilde{h}_2 \succeq \tilde{0} \end{aligned} \tag{4}$$

Solution: We assume that all the fuzzy numbers are triangular fuzzy numbers. Here $4 = \tilde{4} = (4, 0, 0), \tilde{5} = (5, 5 - 5a, 5 - 5a), 2 = \tilde{2} = (2, 1 - a, 1 - a), 6 = \tilde{6} = (6, 0, 0), \tilde{8} = (8, 3 - 3a, 3 - 3a), \tilde{140} = (140, 1 - a, 1 - a), 1 = \tilde{1} = (1, 0, 0), 50 = \tilde{50} = (50, 0, 0)$ and $\tilde{16} = (16, 3 - 3a, 3 - 3a)$.

Sub-problem I

$$\begin{aligned} \max \tilde{z}_1 &\approx \frac{\tilde{2}\tilde{h}_1 + \tilde{4}\tilde{h}_2 + \tilde{5}}{\tilde{2}\tilde{h}_1 + \tilde{6}} \\ \text{subject to } & \tilde{2}\tilde{h}_1 + \tilde{2}\tilde{h}_2 \preceq \tilde{140} \\ & \tilde{h}_2 \succeq \tilde{8} \\ & \tilde{h}_1 \succeq \tilde{16} \\ \text{and } & \tilde{h}_1,\ \tilde{h}_2 \succeq \tilde{0} \end{aligned} \tag{5}$$

The corresponding system of linear inequalities is given by

$$\begin{aligned} &\tilde{z}_1 - (2, 1-a, 1-a)\tilde{y}_1 - (4, 0, 0)\tilde{y}_2 \\ &\qquad - (5, 5-5a, 5-5a)\tilde{t} \preceq (0, 0, 0) \\ &- (2, 1-a, 1-a)\tilde{y}_1 - (6, 0, 0)\tilde{t} \preceq -(1, 0, 0) \\ &(2, 1-a, 1-a)\tilde{y}_1 + (2, 1-a, 1-a)\tilde{y}_2 \\ &\qquad - (140, 1-a, 1-a)\tilde{t} \preceq (0, 0, 0) \\ &- (1, 0, 0)\tilde{y}_2 + (8, 3-3a, 3-3a)\tilde{t} \preceq (0, 0, 0) \\ &- (1, 0, 0)\tilde{y}_1 + (16, 3-3a, 3-3a)\tilde{t} \preceq (0, 0, 0) \\ &- (1, 0, 0)\tilde{y}_1 \preceq (0, 0, 0) \end{aligned} \tag{6}$$

Applying fuzzy version of GEA, eliminating $\tilde{y}_1$ in (6), we have the reduced system

$$\begin{aligned} &(4, 1-a, 1-a)\tilde{y}_2 - (1, 5-5a, 5-5a)\tilde{t} \\ &\qquad - (1, 1-a, 1-a)\tilde{z}_1 \preceq -(1, 1-a, 1-a) \\ &- (2, 1-a, 1-a)\tilde{y}_2 - (145, 5-5a, 5-5a)\tilde{t} \\ &\qquad + (1, 1-a, 1-a)\tilde{z}_1 \preceq (0, 1-a, 1-a) \\ &- (1, 1-a, 1-a)\tilde{y}_2 + (8, 5-5a, 5-5a)\tilde{t} \\ &\qquad + (0, 1-a, 1-a)\tilde{z}_1 \preceq (0, 1-a, 1-a) \\ &(2, 1-a, 1-a)\tilde{y}_2 + (1.85, 5-5a, 5-5a)\tilde{t} \\ &\qquad - (0.5, 1-a, 1-a)\tilde{z}_1 \preceq (0, 1-a, 1-a) \\ &(2, 1-a, 1-a)\tilde{y}_2 + (2.5, 5-5a, 5-5a)\tilde{t} \\ &\qquad - (0.5, 1-a, 1-a)\tilde{z}_1 \preceq (0, 1-a, 1-a) \\ &- (1, 0, 0)\tilde{y}_2 \preceq (0, 0, 0) \end{aligned} \tag{7}$$

Eliminating $\tilde{y}_2$ in (7), we have the reduced system

$$\begin{aligned} &- (145.5, 5-5a, 5-5a)\tilde{t} + (0.5, 1-a, 1-a)\tilde{z}_1 \\ &\qquad \preceq -(0.5, 1-a, 1-a) \\ &(19, 5-5a, 5-5a)\tilde{t} + (0, 1-a, 1-a)\tilde{z}_1 \\ &\qquad \preceq (0.5, 1-a, 1-a) \\ &(3, 5-5a, 5-5a)\tilde{t} + (0, 1-a, 1-a)\tilde{z}_1 \\ &\qquad \preceq (0.5, 1-a, 1-a) \\ &(7.75, 5-5a, 5-5a)\tilde{t} - (0.25, 1-a, 1-a)\tilde{z}_1 \\ &\qquad \preceq -(0.25, 1-a, 1-a) \\ &- (0.25, 5-5a, 5-5a)\tilde{t} - (0.25, 1-a, 1-a)\tilde{z}_1 \\ &\qquad \preceq -(0.25, 1-a, 1-a) \\ &- (1, 0, 0)\tilde{t} \preceq (0, 0, 0) \end{aligned} \tag{8}$$

Eliminating $\tilde{t}$ in (8), we have the reduced system

$$\begin{aligned} &\tilde{z}_1 \preceq (6.658, 5-5a, 5-5a) \\ &\tilde{z}_1 \preceq (47.497, 5-5a, 5-5a) \\ &\tilde{z}_1 \succeq (1.238, 5-5a, 5-5a) \\ &\tilde{z}_1 \succeq (0.993, 5-5a, 5-5a) \\ &\tilde{z}_1 \succeq -(1, 5-5a, 5-5a) \end{aligned} \tag{9}$$

which implies $\tilde{z}_1 \preceq (6.658, 5 - 5a, 5 - 5a) \Rightarrow \max \tilde{z}_1 \approx (6.658, 5 - 5a, 5 - 5a)$.
That is, on applying the fuzzy version of GEA, the solution of the system (6) is given by $\tilde{y}_1 = (0.4, 5 - 5a, 5 - 5a), \tilde{y}_2 = (0.4, 5 - 5a, 5 - 5a), \tilde{t} = (0.026, 5 - 5a, 5 - 5a)$ and $\max \tilde{z}_1 \approx (6.658, 5 - 5a, 5 - 5a)$ which provides the optimum solution of (5) as $\tilde{h}_1 = (15.38, 5 - 5a, 5 - 5a), \tilde{h}_2 = (53.8, 5 - 5a, 5 - 5a)$

and $\max \tilde{z}_1 \approx (6.658, 5-5a, 5-5a)$.

Sub-problem II

$$\begin{aligned} \max \tilde{z}_2 \approx & \frac{\tilde{1}\tilde{h}_1 + \tilde{6}\tilde{h}_2 + \tilde{50}}{\tilde{1}\tilde{h}_1 + \tilde{1}\tilde{h}_2 + \tilde{8}} \\ \text{subject to } & \tilde{2}\tilde{h}_1 + \tilde{2}\tilde{h}_2 \preceq \tilde{140} \\ & \tilde{h}_2 \succeq \tilde{8} \\ & \tilde{h}_1 \succeq \tilde{16} \\ \text{and } & \tilde{h}_1,\ \tilde{h}_2 \succeq \tilde{0} \end{aligned} \tag{10}$$

The corresponding system of linear inequalities is given by

$$\begin{aligned} \tilde{z}_2 - (1,0,0)\tilde{y}_1 - (6,3-3a,3-3a)\tilde{y}_2 & \\ - (50,0,0)\tilde{t} \preceq (0,0,0) & \\ -(1,0,0)\tilde{y}_1 - (1,2-2a,2-2a)\tilde{y}_2 & \\ - (8,5-5a,5-5a)\tilde{t} \preceq -(1,0,0) & \\ (2,1-a,1-a)\tilde{y}_1 + (2,0,0)\tilde{y}_2 & \\ - (140,1-a,1-a)\tilde{t} \preceq (0,0,0) & \\ -(1,0,0)\tilde{y}_2 + (8,3-3a,3-3a)\tilde{t} \preceq (0,0,0) & \\ -(1,0,0)\tilde{y}_1 + (16,3-3a,3-3a)\tilde{t} \preceq (0,0,0) & \\ -(1,0,0)\tilde{y}_1 \preceq (0,0,0) & \end{aligned} \tag{11}$$

On applying the fuzzy version of GEA, the solution of the system (11) is given by $\tilde{y}_1 = (0.2, 5-5a, 5-5a), \tilde{y}_2 = (0.69, 5-5a, 5-5a), \tilde{t} = (0.0128, 5-5a, 5-5a)$ and $\max \tilde{z}_2 \approx (5, 5-5a, 5-5a)$. which provides the optimum solution of (10) as $\tilde{h}_1 = (15.6, 5-5a, 5-5a), \tilde{h}_2 = (54, 5-5a, 5-5a)$ and $\max \tilde{z}_2 \approx (5, 5-5a, 5-5a)$.

Using the optimum values of the above subproblems (5) and (10), the given FFMOLFPP (4) is converted to an equivalent single objective FFLPP as

$$\begin{aligned} & \max \tilde{z} \approx -(15.316, 5-5a, 5-5a)\tilde{h}_1 \\ & + (5, 5-5a, 5-5a)\tilde{h}_2 - (24.918, 5-5a, 5-5a) \\ & \text{subject to} \\ & (2,1-a,1-a)\tilde{h}_1 + (2,1-a,1-a)\tilde{h}_2 \\ & \quad \preceq (140, 1-a, 1-a) \\ & -(1,2-2a,2-2a)\tilde{h}_2 \preceq -(8,3-3a,3-3a) \\ & -(1,2-2a,2-2a)\tilde{h}_1 \preceq -(16,3-3a,3-3a) \\ & -(1,0,0)\tilde{h}_1 \preceq (0,0,0) \end{aligned} \tag{12}$$

It can be reduced to a system of linear inequalities as

$$\begin{aligned} & \tilde{Z} + (15.316, 5-5a, 5-5a)\tilde{h}_1 \\ & -(5,5-5a,5-5a)\tilde{h}_2 \preceq -(24.918, 5-5a, 5-5a) \\ & (2,1-a,1-a)\tilde{h}_1 + (2,0,0)\tilde{h}_2 \\ & \quad \preceq (140, 1-a, 1-a) \\ & -(1,0,0)\tilde{h}_2 \preceq -(8,3-3a,3-3a) \\ & -(1,0,0)\tilde{h}_1 \preceq -(16,3-3a,3-3a) \\ & -(1,0,0)\tilde{h}_1 \preceq (0,0,0) \end{aligned} \tag{13}$$

On applying the fuzzy version of GEA, the system (13) is solved which in turn provides the optimum solution of the FFLPP (12) as $\tilde{h}_1 = (16, 5-5a, 5-5a), \tilde{h}_2 = (54, 5-5a, 5-5a)$. By theorem (3.2) the optimum solution of the FFLPP (12) is the Pareto optimum solution of the given FFMOLFPP (4).

Hence the Pareto optimum solution of the given FFMOLFPP (4) is $\tilde{h}_1 = (16, 5-5a, 5-5a), \tilde{h}_2 = (54, 5-5a, 5-5a)$ with $\max \tilde{z}_1 \approx (6.694, 5-5a, 5-5a)$ and $\max \tilde{z}_2 \approx (5.008, 5-5a, 5-5a)$.

For the same problem, the authors Surapati Pramanik et al.[19] have obtained only the crisp solution.

Also the following tables provides different solutions based on the preference of the decision makers by choosing suitable value for a.

TABLE I
$\tilde{h}_1, \tilde{h}_2$ FOR DIFFERENT VALUES OF "a"

Value of a	$\tilde{h}_1$	$\tilde{h}_2$
a=1	(16,16,16)	(54,54,54)
a=0.5	(13.5,16,18.5)	(51.5,54,56.5)
a=0.25	(12.25,16,19.75)	(50.25,54,57.75)
a=0	(11,16,21)	(49,54,59)

TABLE II
$\tilde{z}_1, \tilde{z}_2$ FOR DIFFERENT VALUES OF "a"

Value of a	$\tilde{z}_1$	$\tilde{z}_2$
a=1	(6.694,6.694,6.694)	(5.008,5.008,5.008)
a=0.5	(4.194,6.694,9.194)	(2.508,5.008,7.508)
a=0.25	(2.944,6.694,10.444)	(1.258,5.008,8.758)
a=0	(1.694,6.694,11.694)	(0.008,5.008,10.008)

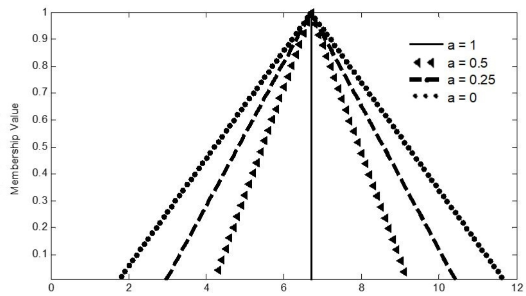

Fig. 1. Graphical representation of the optimal values of $\tilde{z}_1$ for different 'a'

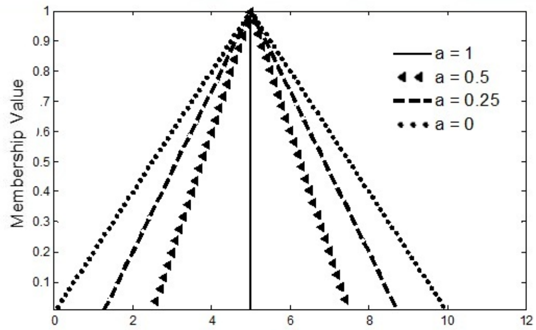

Fig. 2. Graphical representation of the optimal values of $\tilde{z}_2$ for different 'a'

Example 2:
Consider a FFMOLFPP discussed by Durga Prasad Dash et al.[4]

$$\begin{aligned} \max \tilde{z}_1 \approx & \frac{\tilde{6}\tilde{h}_1 + \tilde{5}\tilde{h}_2}{\tilde{2}\tilde{h}_1 + \tilde{7}} \\ \max \tilde{z}_2 \approx & \frac{\tilde{2}\tilde{h}_1 + \tilde{3}\tilde{h}_2}{\tilde{1}\tilde{h}_1 + \tilde{1}\tilde{h}_2 + \tilde{7}} \\ \text{subject to } & \tilde{1}\tilde{h}_1 + \tilde{2}\tilde{h}_2 \preceq \tilde{3} \\ & \tilde{3}\tilde{h}_1 + \tilde{2}\tilde{h}_2 \preceq \tilde{6} \\ \text{and } & \tilde{h}_1, \tilde{h}_2 \succeq \tilde{0} \end{aligned} \tag{14}$$

Solution: We assume that all the fuzzy numbers are triangular fuzzy numbers. Here $\tilde{5} = (5, 0.1-0.1a, 0.1-0.1a), 2 = \tilde{2} = (2, 0.1-0.1a, 0.1-0.1a), \tilde{6} = (6, 0.5-0.5a, 0.5-0.5a), \tilde{7} = (7, 0.3-0.3a, 0.3-0.3a), \tilde{3} = (3, 0.1-0.1a, 0.1-0.1a)$ and $1 = \tilde{1} = (1, 0, 0)$.

Sub-problem I

$$\begin{aligned} \max \tilde{z}_1 \approx & \frac{\tilde{6}\tilde{h}_1 + \tilde{5}\tilde{h}_2}{\tilde{2}\tilde{h}_1 + \tilde{7}} \\ \text{subject to } & \tilde{1}\tilde{h}_1 + \tilde{2}\tilde{h}_2 \preceq \tilde{3} \\ & \tilde{3}\tilde{h}_1 + \tilde{2}\tilde{h}_2 \preceq \tilde{6} \\ \text{and } & \tilde{h}_1, \tilde{h}_2 \succeq \tilde{0} \end{aligned} \tag{15}$$

The corresponding system of linear inequalities is given by

$$\begin{aligned} & \tilde{z}_1 - (6, 0.5-0.5a, 0.5-0.5a)\tilde{y}_1 \\ & \quad - (5, 0.1-0.1a, 0.1-0.1a)\tilde{y}_2 \preceq (0,0,0) \\ & - (2, 0.1-0.1a, 0.1-0.1a)\tilde{y}_1 \\ & \quad - (7, 0.3-0.3a, 0.3-0.3a)\tilde{t} \preceq -(1,0,0) \\ & (1, 0.1-0.1a, 0.1-0.1a)\tilde{y}_1 + (2,0,0)\tilde{y}_2 \\ & \quad - (3, 0.1-0.1a, 0.1-0.1a)\tilde{t} \preceq (0,0,0) \\ & (3, 0.1-0.1a, 0.1-0.1a)\tilde{y}_1 + (2,0,0)\tilde{y}_2 \\ & \quad - (6, 0.1-0.1a, 0.1-0.1a)\tilde{t} \preceq (0,0,0) \\ & - (1,0,0)\tilde{y}_1 \preceq (0,0,0) \end{aligned} \tag{16}$$

On applying the fuzzy version of GEA, the solution of the system (16) is $\tilde{y}_1 = (0.15, 0.5-0.5a, 0.5-0.5a), \tilde{y}_2 = (0.07, 0.5-0.5a, 0.5-0.5a), \tilde{t} = (0.1, 0.5-0.5a, 0.5-0.5a)$ and $\tilde{z}_1 = (1.27, 0.5-0.5a, 0.5-0.5a)$ which provides the optimum solution of (15) as $\tilde{h}_1 = (1.5, 0.5-0.5a, 0.5-0.5a), \tilde{h}_2 = (0.7, 0.5-0.5a, 0.5-0.5a)$ and $\max \tilde{z}_1 \approx (1.27, 0.5-0.5a, 0.5-0.5a)$.

Sub-problem II

$$\begin{aligned} \max \tilde{z}_2 \approx & \frac{\tilde{2}\tilde{h}_1 + \tilde{3}\tilde{h}_2}{\tilde{1}\tilde{h}_1 + \tilde{1}\tilde{h}_2 + \tilde{7}} \\ \text{subject to } & \tilde{1}\tilde{h}_1 + \tilde{2}\tilde{h}_2 \preceq \tilde{3} \\ & \tilde{3}\tilde{h}_1 + \tilde{2}\tilde{h}_2 \preceq \tilde{6} \\ \text{and } & \tilde{h}_1, \tilde{h}_2 \succeq \tilde{0} \end{aligned} \tag{17}$$

The corresponding system of linear inequalities is given by

$$\begin{aligned} & \tilde{z}_2 - (2, 0.1-0.1a, 0.1-0.1a)\tilde{y}_1 \\ & \quad - (3, 0.3-0.3a, 0.3-0.3a)\tilde{y}_2 \preceq (0,0,0) \\ & - (1, 0.2-0.2a, 0.2-0.2a)\tilde{y}_1 \\ & \quad - (1, 0.1-0.1a, 0.1-0.1a)\tilde{y}_2 \\ & \quad - (7, 0.1-0.1a, 0.1-0.1a)\tilde{t} \preceq -(1,0,0) \\ & (1, 0.1-0.1a, 0.1-0.1a)\tilde{y}_1 + (2,0,0)\tilde{y}_2 \\ & \quad - (3, 0.1-0.1a, 0.1-0.1a)\tilde{t} \preceq (0,0,0) \\ & (3, 0.1-0.1a, 0.1-0.1a)\tilde{y}_1 + (2,0,0)\tilde{y}_2 \\ & \quad - (6, 0.1-0.1a, 0.1-0.1a)\tilde{t} \preceq (0,0,0) \\ & - (1,0,0)\tilde{y}_1 \preceq (0,0,0) \end{aligned} \tag{18}$$

On applying the fuzzy version of GEA, the solution of the system (18) is $\tilde{y}_1 = (0.164, 0.3-0.3a, 0.3-0.3a), \tilde{y}_2 = (0.08, 0.3-0.3a, 0.3-0.3a), \tilde{t} = (0.108, 0.3-0.3a, 0.3-0.3a)$ and $\tilde{z}_2 = (0.568, 0.3-0.3a, 0.3-0.3a)$ which provides the optimum solution of (17) as $\tilde{h}_1 = (1.5, 0.3-0.3a, 0.3-0.3a), \tilde{h}_2 = (0.7, 0.3-0.3a, 0.3-0.3a)$ and $\max \tilde{z}_2 \approx (0.568, 0.3-0.3a, 0.3-0.3a)$.

Using the optimum values of the above subproblems (15) and (17), the given FFMOLFPP (14) is converted to an equivalent single objective FFLPP as

$$\begin{aligned} & \max \tilde{z} \approx (4.892, 0.5-0.5a, 0.5-0.5a)\tilde{h}_1 \\ & \quad + (7.432, 0.3-0.3a, 0.3-0.3a)\tilde{h}_2 \\ & \quad - (12.866, 0.5-0.5a, 0.5-0.5a) \\ & \text{subject to} \\ & (1, 0.1-0.1a, 0.1-0.1a)\tilde{h}_1 + (2,0,0)\tilde{h}_2 \\ & \quad \preceq (3, 0.1-0.1a, 0.1-0.1a) \\ & (3, 0.1-0.1a, 0.1-0.1a)\tilde{h}_1 + (2,0,0)\tilde{h}_2 \\ & \quad \preceq (6, 0.1-0.1a, 0.1-0.1a) \\ & - (1,0,0)\tilde{h}_1 \preceq (0,0,0) \\ & - (1,0,0)\tilde{h}_2 \preceq (0,0,0) \end{aligned} \tag{19}$$

It can be reduced to a system of linear inequalities as

$$\begin{aligned} & \tilde{z} - (4.892, 0.5-0.5a, 0.5-0.5a)\tilde{h}_1 \\ & \quad - (7.432, 0.3-0.3a, 0.3-0.3a)\tilde{h}_2 \\ & \quad \preceq -(12.866, 0.5-0.5a, 0.5-0.5a) \\ & (1, 0.1-0.1a, 0.1-0.1a)\tilde{h}_1 + (2,0,0)\tilde{h}_2 \\ & \quad \preceq (3, 0.1-0.1a, 0.1-0.1a) \\ & (3, 0.1-0.1a, 0.1-0.1a)\tilde{h}_1 + (2,0,0)\tilde{h}_2 \\ & \quad \preceq (6, 0.1-0.1a, 0.1-0.1a) \\ & - (1,0,0)\tilde{h}_1 \preceq (0,0,0) \\ & - (1,0,0)\tilde{h}_2 \preceq (0,0,0) \end{aligned} \tag{20}$$

On applying the fuzzy version of GEA, the system (20) is solved which in turn provides the optimum solution of the FFLPP (19) as $\tilde{h}_1 = (1.5, 0.5-0.5a, 0.5-0.5a), \tilde{h}_2 = (0.75, 0.5-0.5a, 0.5-0.5a)$.

By theorem (3.2) the optimum solution of the FFLPP (19) is the Pareto optimum solution of the given FFMOLFPP (14).

Hence the Pareto optimum solution of the given FF-MOLFPP (14) is $\tilde{h}_1 = (1.5, 0.5-0.5a, 0.5-0.5a), \tilde{h}_2 = (0.75, 0.5-0.5a, 0.5-0.5a)$ with $\max \tilde{z}_1 \approx (1.27, 0.5-$

$0.5a, 0.5-0.5a)$ and $\max \tilde{z}_2 \approx (0.568, 0.5-0.5a, 0.5-0.5a)$. For the same problem, the authors Durga Prasad Dash et al.[4] have obtained only the crisp solution.
Also the following tables provides different solutions based on the preference of the decision makers by choosing suitable value for a.

TABLE III
$\tilde{h}_1, \tilde{h}_2$ FOR DIFFERENT VALUES OF "a"

Value of a	$\tilde{h}_1$	$\tilde{h}_2$
a=1	(1.5,1.5,1.5)	(0.75,0.75,0.75)
a=0.5	(1.25,1.5,1.75)	(0.5,0.75,1)
a=0.25	(1.125,1.5,1.875)	(0.375,0.75,0.125)
a=0	(1,1.5,2)	(0.25,0.75,1.25)

TABLE IV
$\tilde{z}_1, \tilde{z}_2$ FOR DIFFERENT VALUES OF "a"

Value of a	$\tilde{z}_1$	$\tilde{z}_2$
a=1	(1.27,1.27,1.27)	(0.568,0.568,0.568)
a=0.5	(1.02,1.27,1.52)	(0.318,0.568,0.818)
a=0.25	(0.895,1.27,1.645)	(0.193,0.568,0.943)
a=0	(0.77,1.27,1.77)	(0.068,0.568,1.068)

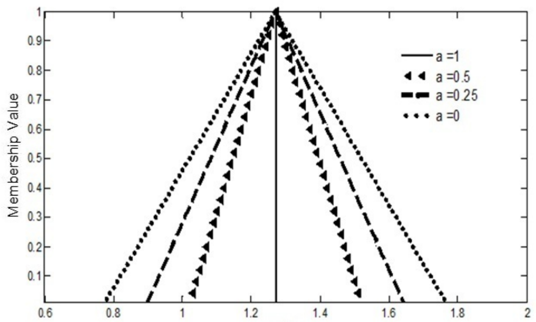

Fig. 3. Graphical representation of the optimal values of $\tilde{z}_1$ for different 'a'

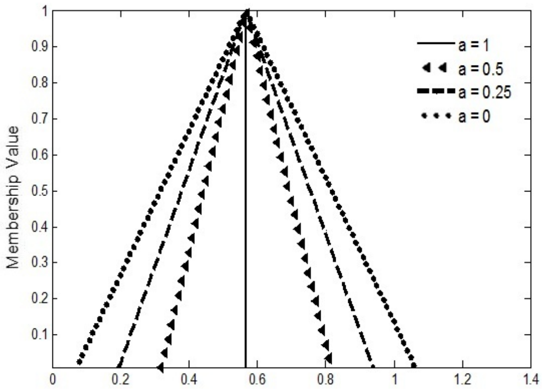

Fig. 4. Graphical representation of the optimal values of $\tilde{z}_2$ for different 'a'

V. CONCLUDING REMARKS

We have developed a new approach for solving FFMOLFP problems involving triangular fuzzy numbers. The proposed method facilitate the decision maker to solve FFMOLFP problem without transforming to an equivalent crisp problem. It also allows the decision maker the freedom of selecting his or her desired solution by choosing suitably the value of "a". Two numerical examples are provided to show the efficiency of the proposed approach. The proposed approach generates less ambiguous results without losing the fuzzy nature of the given problems.

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