On Solving Interval-valued Optimization Problems with TOPSIS Decision Model

Dong Qiu, Xiao Jin and Li Xiang

Abstract—**In this paper, we regard the optimization problems as continuous decision problems and propose a new method of using TOPSIS model in fuzzy decision-making to solve the interval valued optimization problems. Compared with the traditional interval analysis theory, this method does not need to define the differentiability of interval valued function itself. It is more extensive and has certain practical significance. In order to get the optimality conditions, we give new definitions of interval order relation based on TOPSIS method and the optimal solution of interval valued optimization problem. On this basis, we obtain the necessary conditions for the TOPSIS-optimal solution points of interval valued optimization problem.**

Index Terms—**interval-valued optimization problems, decision making, TOPSIS.**

I. Introduction

DECISION making refers to the process of selecting the best solution in a series of feasible schemes. However, due to many factors affecting the choice, decision-makers often need to consider comprehensively from multiple dimensions and finally make a decision. Therefore, most decision-making problems are multi-attribute decision-making problems. In the past research process, academia and practitioners widely accepted the use of probability methods to analyze decision-making problems. However, with the increasing complexity of the problem and the deepening of research, the traditional probability research methods can not meet the research needs, and can not quantitatively explain the uncertainty in decision-making problems. Therefore, using the concept of fuzzy number or fuzzy set to deal with uncertain information has become a more popular way. In 1965, Zadeh first proposed the concept of fuzzy number in [1]. In 1970, Bellman and Zadeh proposed the concept of fuzzy decision-making [2] which is based on [1]. Since then, fuzzy theory has been introduced into multi-attribute decision-making problems and become a powerful tool to deal with uncertain information [3], [4], [5].

In order to better study the problem of fuzzy multi-attribute decision-making, scholars analyze and study fuzzy sets from many angles, and successively put forward intuitionistic fuzzy sets [6], [7], interval intuitionistic fuzzy sets [8], [9], [10], hesitant fuzzy sets [11], [12], [13] and other models. However, the same thing is that no matter which kind of fuzzy set is studied, it is necessary to sort the fuzzy information and make the final decision according to the sorting situation. For example, Xu proposed the uncertain multi-attribute decision making (UMADM) model [7]. This method sorts the schemes according to the comprehensive weight of deterministic information and uncertain information. Hung and Chen proposed a preference ranking fuzzy technology based on entropy weight ideal solution similarity (TOPSIS) method to solve decision-making problems in intuitionistic fuzzy environment [13]. Sivaraman et al proposed a scoring function for ranking interval valued intuitionistic fuzzy numbers (IVIFNS) [14]. Garg proposed a generalized improved scoring function to sort different IVIFN [15].

At present, the research of fuzzy decision-making brings together a large number of researchers and experts from all over the world. Its results have been extended to new fields such as computer science, engineering, scientific operation management, mathematics, economic affairs and automatic control system [16], [17], [18], [19], [20], [21], [42]. Fuzzy decision-making analyzes the uncertainty in decision-making problems from a quantitative perspective, which makes it very interesting. Therefore, this field has been developing since its establishment.

The common fuzzy decision-making methods are fuzzy ranking, fuzzy optimization and fuzzy countermeasures. In the fuzzy decision problem, given the scheme set, various objective functions and constraints, the fuzzy optimization problem becomes a fuzzy optimization problem. Therefore, fuzzy optimization problem is essentially a branch of fuzzy decision problem. Fuzzy optimization problem refers to the optimization problem in which the optimization objective or constraint is fuzzy variable. In particular, when the fuzzy number in the fuzzy optimization problem is taken as the interval number, the fuzzy optimization problem becomes the interval optimization problem.

Interval analysis is a branch of mathematics that uses interval variables to replace point variables. It was originally developed from the error theory of computational mathematics [22], [23]. In 1966, R.E. Moore put forward the theory of interval operation for the first time [22]. Calculation error is always a troublesome problem in numerical analysis. It comes from data errors, truncation errors and rounding errors [24]. Interval analysis tries to make to keep the calculated results within the required accuracy [22], [23]. However, in many problems, it is often to speculate about the accuracy of calculation results or use high-precision operation as far as possible to ensure the accuracy of calculation results [25], [26]. As the accumulation of calculation errors may make the calculation results meaningless, interval mathematics provides a simple method. It takes all kinds of errors into

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account. At the same time, as the result of calculation, an interval containing accurate results can be obtained, which may realize the problem that numerical analysis hopes to solve [22], [23], [27], [28].

Interval analysis has a great development and has many applications in computational mathematics. For example, the existence and uniqueness of solutions to nonlinear equations and operator equations and the convergence of interval iteration sequence can be determined by using an interval iteration method, which is not obtained by point iteration method [29], [30]. In addition, it also has applications in interval interpolation and approximation [31], linear equations [32], nonlinear programming [33], differential equations [34] and so on. Interval language, a computer language directly used to calculate interval quantities, has also appeared [35]. Similar to fuzzy numbers, interval numbers can also be used to deal with uncertain information. Therefore, interval analysis is introduced to deal with the uncertainty in many deterministic phenomena in the real world [36], [37], [38]. Solving interval-valued optimization problem is an important way to solve the uncertainty in optimization problem [22], [23], [39].

The method of solving fuzzy decision problems can be used in interval optimization problems, such as [17], [40], [41]. In this paper, we choose the interval number order relation based on TOPSIS to solve the interval valued optimization problem. Some examples are selected to prove its practical significance.

This paper consists of five parts. In Section 2, we introduce some basic concepts, and give the interval number order relationship based on TOPSIS model, as well as a new definition of the optimal solution of interval valued optimization problem. In Section 3, some theorems are given to explain the optimality conditions of interval valued optimization problems, and examples are given to illustrate the TOPSIS-optimal solution of interval valued functions. In Section 4, a practical example is given to illustrate the effectiveness of the model. In Section 5 we summarize and analyze the research content of this paper, and points out the existing problems and follow-up research directions.

II. Preliminaries

Definition 2.1: Let $\mathbb{R}$ be the family of all real numbers and $I(\mathbb{R})$ be the bounded and closed intervals of $\mathbb{R}$, that is

$$I(\mathbb{R}) = \{A = [\underline{a}, \overline{a}] | \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \leq \overline{a}\}.$$

Particularly, when $\underline{a} = \overline{a}$, A is a real number.

Considering $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$ and $\lambda \in \mathbb{R}$, then

$$(1) A + B = [\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \tag{1}$$

$$(2) \lambda \cdot A = \begin{cases} [\lambda\underline{a}, \lambda\overline{a}], & if \quad \lambda \geq 0, \\ [\lambda\overline{a}, \lambda\underline{a}], & if \quad \lambda < 0. \end{cases} \tag{2}$$

Definition 2.2: Considering $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$, the Euclidean standard distance between interval numbers A and B is defined as

$$d(A, B) = \sqrt{\frac{(\underline{a} - \underline{b})^2 + (\bar{a} - \bar{b})^2}{2}}. \tag{3}$$

Definition 2.3: For any two elements $A, B \in I(\mathbb{R})$, there is an interval $P \in I(\mathbb{R})$ which is defined as $P = [\max\{\underline{a}, \underline{b}\}, \max\{\overline{a}, \overline{b}\}]$. Here, P is called the positive ideal interval of A and B. Similarly, we define the negative ideal interval number N between interval numbers A and B, where N satisfies $N = [\min\{\underline{a}, \underline{b}\}, \min\{\overline{a}, \overline{b}\}]$.

Definition 2.4: Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$, and $P = [\max\{\underline{a}, \underline{b}\}, \max\{\overline{a}, \overline{b}\}]$ is the positive ideal interval number of A and B, $N = [\min\{\underline{a}, \underline{b}\}, \min\{\overline{a}, \overline{b}\}]$ is the negative ideal interval number of them. Then

$$C_A = \frac{d(A, N)}{d(A, N) + d(A, P)} \tag{4}$$

is defined as the nearness degree of A to P.

Note 2.1: For the convenience of subsequent calculation, the above nearness degree C can be simply expressed as C_A^2, which can be written as

$$C_A^2 = \frac{d^2(A, N)}{d^2(A, N) + d^2(A, P)}.$$

Definition 2.5: For any two elements $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}]$ in $I(\mathbb{R})$, the following order relations between A and B can be defined.

B is said to be dominated by A if

$$C_A^2 \leq C_B^2,$$

and then we write $A \leq_{TOPSIS} B$.

Definition 2.6: [34] We suppose that X is an open and nonempty subset of $\mathbb{R}^n$, and the function F: $X \to I(\mathbb{R})$ is called an interval-valued function. Simply, we can write

$$F(x) = \left[\underline{F}(x), \overline{F}(x)\right],$$

where $\underline{F}(x) \leq \overline{F}(x)$ for all $x \in X$.

Definition 2.7: Let X be a nonempty subset of $\mathbb{R}^n$ and F: $X \to I(\mathbb{R})$ be an interval-valued function. The point $x^* \in X$ is said to be a TOPSIS-optimal solution point of an interval-valued optimization problem:

$$\begin{array}{ll} \min & C^2_{F(x)} \\ s.t. & x \in X, \end{array} \tag{5}$$

if $C^2_{F(x^*)} \leq C^2_{F(x)}$ for all $x \in X$, where $C^2_{F(x)} = \frac{d^2(F(x),N)}{d^2(F(x),N)+d^2(F(x),P)}$.

III. Optimality conditions of interval-valued optimization problems

Now, we introduce some corresponding properties of interval-valued functions based on TOPSIS.

Theorem 3.1: If there exists $x^* \in X$ such that $d^2(F(x), N) = 0$, then x^* must be a TOPSIS-optimal solution point of $F(x)$.

Proof. From the Definition 2.2, we get that

$$d^2(F(x), N) \geq 0, \quad d^2(F(x), P) \geq 0$$

for all $x \in X$.

Then owing to the Definition 2.4, it is easy to get

$$C^2_{F(x)} = \frac{d^2(F(x), N)}{d^2(F(x), N) + d^2(F(x), P)} \geq 0.$$

Obviously, x^* is a TOPSIS-optimal solution point of $F(x)$, if $x^* \in X$ such that $d^2(F(x), N) = 0$.□

Example 3.1: Suppose

$$F(x) = [2x - x^2, 5 - x^2],$$

where $\underline{f}(x) = 2x - x^2$, $\overline{f}(x) = 5 - x^2$, and $x \in [0, 2]$.
It is easy to get that

$$P = [\max \underline{f}(x), \max \overline{f}(x)] = [\underline{f}(1), \overline{f}(0)] = [1, 5],$$

and

$$N = [\min \underline{f}(x), \min \overline{f}(x)] = [\underline{f}(0), \overline{f}(2)] = [0, 1].$$

Then we have

$$\begin{aligned} d^2(F(x), P) &= \frac{(\underline{f}(x) - \max \underline{f}(x))^2 + (\overline{f}(x) - \max \overline{f}(x))^2}{2} \\ &= \frac{(\underline{f}(x) - \underline{f}(1))^2 + (\overline{f}(x) - \overline{f}(0))^2}{2} \\ &= \frac{(2x - x^2 - 1)^2 + (5 - x^2 - 5)^2}{2} \\ &= x^4 - 2x^3 + 3x^2 - 2x + \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} d^2(F(x), N) &= \frac{(\underline{f}(x) - \min \underline{f}(x))^2 + (\overline{f}(x) - \min \overline{f}(x))^2}{2} \\ &= \frac{(\underline{f}(x) - \underline{f}(0))^2 + (\overline{f}(x) - \overline{f}(2))^2}{2} \\ &= \frac{(2x - x^2 - 0)^2 + (5 - x^2 - 1)^2}{2} \\ &= x^4 - 2x^3 - 2x^2 + 8. \end{aligned}$$

Thus

$$\begin{aligned} C^2_{F(x)} &= \frac{d^2(F(x), N)}{d^2(F(x), N) + d^2(F(x), P)} \\ &= \frac{x^4 - 2x^3 - 2x^2 + 8}{(x^4 - 2x^3 - 2x^2 + 8) + (x^4 - 2x^3 + 3x^2 - 2x + \frac{1}{2})} \\ &= \frac{x^4 - 2x^3 - 2x^2 + 8}{2x^4 - 4x^3 + x^2 - 2x + \frac{17}{2}} \\ &= \frac{2x^4 - 4x^3 - 4x^2 + 16}{2x^4 - 8x^3 + 2x^2 - 4x + 17}. \end{aligned}$$

Let $C^2_{F(x)} = 0$, we obtain that $x^* = 2$. Owing to Theorem 3.1, it is easy to know the point x^* is the TOPSIS-optimal solution point of $F(x)$

Theorem 3.2: If

$$\begin{cases} d^2(F(x), N) > 0, \\ d^2(F(x^*), N) \le d^2(F(x), N), \\ d^2(F(x^*), P) \ge d^2(F(x), P). \end{cases} \tag{6}$$

for each $x \in X$, then x^* is a TOPSIS-optimal solution point of $F(x)$.

Proof. From the Definition 2.4, we get

$$C^2_{F(x)} = \frac{d^2(F(x), N)}{d^2(F(x), N) + d^2(F(x), P)}.$$

Hence,

$$\begin{aligned} &\frac{1}{C^2_{F(x)}} \\ &= \frac{1}{\frac{d^2(F(x),N)}{d^2(F(x),N)+d^2(F(x),P)}} \\ &= \frac{d^2(F(x), N) + d^2(F(x), P)}{d^2(F(x), N)} \\ &= 1 + \frac{d^2(F(x), P)}{d^2(F(x), N)}. \end{aligned}$$

When the point x^* satisfies (6), it is easy to get that $\frac{1}{C^2_{F(x)}} \le \frac{1}{C^2_{F(x^*)}}$, for all $x \in X$. Hence, for each $x \in X$, it is satisfied $C^2_{F(x^*)} \le C^2_{F(x)}$. Thus, due to the Definition 2.7, we get that x^* is a TOPSIS-optimal solution point of $F(x)$.□

Definition 3.1: Let $h(x)$ be a function defined on X, if for any two points x_1 and x_2 on X and any real number $\lambda \in (0, 1)$,

$$h(\lambda x_1 + (1 - \lambda)x_2) \le \lambda h(x_1) + (1 - \lambda)h(x_2), \tag{7}$$

then $h(x)$ is called a convex function on X.

Note 3.1: By replacing "$\le$" with "$<$" in Definition 3.1, the function $h(x)$ is called a strictly convex function on X.

Theorem 3.3: [24] Let $h(x)$ be a twice differentiable function. If $h''(x) \ge 0$ is satisfied for any $x \in X$, then $h(x)$ is a convex function defined on X.

Note 3.2: It is also true that by replacing "$\ge$" with "$>$" in Theorem 3.3, the function $h(x)$ is a strictly convex function on X. Similarly, if $h''(x) \le 0$ is satisfied for any $x \in X$, then $h(x)$ is a concave function defined on X.

Theorem 3.4: Consider the fractional function $h(x) = \frac{f(x)}{g(x)}$. If the nonnegative functions $f(x)$ is convex and $g(x)$ is concave, and $f'(x)g'(x) \le 0$ is satisfied for any $x \in X$, then we can get that $h(x)$ is convex.

Proof. Since the functions $f(x)$ and $g(x)$ are differentiable, we have

$$h'(x) = (\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}.$$

Consequently, we get

$$\begin{aligned} &h''(x) \\ &= (h'(x))' \\ &= (\frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)})' \\ &= \frac{(f'(x)g(x) - g'(x)f(x))' g^2(x)}{g^4(x)} \\ &\quad - \frac{(g^2(x))' (f'(x)g(x) - g'(x)f(x))}{g^4(x)} \\ &= \frac{f''(x)g(x) - g''(x)f(x) - 2f'(x)g'(x)}{g^2(x)} + \frac{2f(x)(g'(x))^2}{g^2(x)}. \end{aligned}$$

Due to the convexity of $f(x)$ and $g(x)$ on X, we have

$$\begin{cases} f''(x) \ge 0, \\ g''(x) \le 0. \end{cases}$$

Owing to Theorem 3.4, we get

$$\frac{f''(x)g(x) - g''(x)f(x) - 2f'(x)g'(x)}{g^2(x)} \ge 0$$

and

$$\frac{2f(x)(g'(x))^2}{g^2(x)} \ge 0.$$

Hence, $h''(x) \ge 0$. From Theorem 3.3, it is easy to get that the fractional function $h(x)$ is convex. □

Theorem 3.5: If there exists a point $x^* \in X$ such that

$$\begin{aligned} &(d^2(F(x^*), P))' d^2(F(x^*), N) \\ &-(d^2(F(x^*), N))' d^2(F(x^*), P) = 0, \end{aligned} \tag{8}$$

and for each $x \in X$, $(\frac{1}{C^2_{F(x)}})'' \le 0$, then x^* is a TOPSIS-optimal solution point of $F(x)$.

Proof. From the Definition 2.4, we get

$$C^2_{F(x)} = \frac{d^2(F(x),N)}{d^2(F(x),N)+d^2(F(x),P)}.$$

Hence,

$$\begin{aligned}&\frac{1}{C^2_{F(x)}}\\&=\frac{1}{\frac{d^2(F(x),N)}{d^2(F(x),N)+d^2(F(x),P)}}\\&=\frac{d^2(F(x),N)+d^2(F(x),P)}{d^2(F(x),N)}\\&=1+\frac{d^2(F(x),P)}{d^2(F(x),N)}.\end{aligned}$$

Thus,we get

$$\begin{aligned}&(\frac{1}{C^2_{F(x)}})'\\&=(1+\frac{d^2(F(x),P)}{d^2(F(x),N)})'\\&=\frac{(d^2(F(x),P))'d^2(F(x),N)-(d^2(F(x),N))'d^2(F(x),P)}{(d^2(F(x),N))^2}.\end{aligned}$$

Owing to the theorem 3.5, we know that there exists a point $x^* \in X$ such that

$$\begin{aligned}&(d^2(F(x^*),P))'d^2(F(x^*)N)\\&-(d^2(F(x^*),N))'d^2(F(x^*),P)=0.\end{aligned} \tag{9}$$

Then, we get $(\frac{1}{C^2_{F(x^*)}})' = 0$. Since $(\frac{1}{C^2_{F(x)}})'' \leq 0$ for any $x \in X$, we obtain that $(\frac{1}{C^2_{F(x)}})$ is concave, which means $\frac{1}{C^2_{F(x^*)}} \geq \frac{1}{C^2_{F(x)}}$, for all $x \in X$. Then, we get $C^2_{F(x^*)} \leq C^2_{F(x)}$, for all $x \in X$. From the Definition 2.7, it is easy to know that x^* is a TOPSIS-optimal solution point of $F(x)$. □

IV. Application

It is known that the profit obtained by selling a commodity is affected by production cost, sales price and output, and there is a certain functional relationship between production cost, sales price and output. In addition, considering the influence of multiple conditions such as changes in the market and different sales sites, here we consider expressing the production cost and sales price as interval values. Obviously, the profit obtained from selling the commodity is also interval values. If the following functional relationship is satisfied between the profit obtained from selling a commodity and the output:

$$L(x) = [\underline{l}(x), \bar{l}(x)],$$

where $\underline{l}(x) \leq \bar{l}(x)$, $x \in X$.
Q: What is the amount of the output when we get the maximum profit? What is the maximum profit?

Example 4.1: Suppose

$$L(x) = [\underline{l}(x), \bar{l}(x)],$$

where $\underline{l}(x) = -\frac{1}{8}x^2 + 21x - 182$, $\bar{l}(x) = -\frac{1}{4}x^2 + 40x + 100$, and $x \in [40, 100]$.

From Definition 2.7, it is obvious that the interval-value optimization problem is as follows:

$$\begin{aligned}&\max \quad C^2_{L(x)}\\&s.t. \quad x \in X,\end{aligned}$$

where $C^2_{L(x)} = \frac{d^2(L(x),N)}{d^2(L(x),N)+d^2(L(x),P)}$, $X = [40, 100]$.

From the Definition 2.3, we get the positive and negative ideal interval numbers of $L(x)$ on X,

$$\begin{aligned}P &= [\max \underline{l}(x), \max \bar{l}(x)],\\&= [\underline{l}(84), \bar{l}(80)],\\&= [700, 1700].\end{aligned}$$

and

$$\begin{aligned}N &= [\min \underline{l}(x), \min \bar{l}(x)],\\&= [\underline{l}(40), \bar{l}(40)],\\&= [458, 1300].\end{aligned}$$

Hence,

$$\begin{aligned}&d^2(L(x),P)\\&=\frac{(\underline{l}(x)-\max\underline{l}(x))^2+(\bar{l}(x)-\max\bar{l}(x))^2}{2}\\&=\frac{(\underline{l}(x)-\underline{l}(84))^2+(\bar{l}(x)-\bar{l}(80))^2}{2}\\&=\frac{(\frac{1}{8}x^2-21x+882)^2+(\frac{1}{4}x^2-40x+1600)^2}{2},\end{aligned}$$

and

$$\begin{aligned}&d^2(L(x),N)\\&=\frac{(\underline{l}(x)-\min\underline{l}(x))^2+(\bar{l}(x)-\min\bar{l}(x))^2}{2}\\&=\frac{(\underline{l}(x)-\underline{l}(40))^2+(\bar{l}(x)-\bar{l}(40))^2}{2}\\&=\frac{(\frac{1}{8}x^2-21x+640)^2+(\frac{1}{4}x^2-40x+1200)^2}{2}.\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}\frac{1}{C^2_{L(x)}} &= \frac{d^2(L(x),N)+d^2(L(x),P)}{d^2(L(x),N)}\\&=1+\frac{d^2(L(x),P)}{d^2(L(x),N)}\\&=1+\frac{(\frac{1}{8}x^2-21x+882)^2+(\frac{1}{4}x^2-40x+1600)^2}{(\frac{1}{8}x^2-21x+640)^2+(\frac{1}{4}x^2-40x+1200)^2}\\&=1+\frac{(x^2-168x+7056)^2+(2x^2-320x+12800)^2}{(x^2-168x+5120)^2+(2x^2-320x+9600)^2}.\end{aligned}$$

It is easy to get that

$$\frac{1}{C^2_{L(x)}} \geq 0,$$

for all $x \in X$.
Then, owing to Theorem 3.3, we get $\frac{1}{C^2_{L(x)}}$ is convex.

Furthermore, it is easy to get

$$\begin{aligned}
&(d^2(L(x),P))' \\
&=[\frac{(\underline{l}(x)-\max\underline{l}(x))^2+(\bar{l}(x)-\max\bar{l}(x))^2}{2}]' \\
&=[\frac{(\underline{l}(x)-\underline{l}(84))^2+(\bar{l}(x)-\bar{l}(80))^2}{2}]' \\
&=\frac{[(\frac{1}{8}x^2-21x+882))^2]'+[(\frac{1}{4}x^2-40x+1600)^2]'}{2} \\
&=[(\frac{1}{8}x^2-21x+882)(\frac{1}{4}x-21)+ \\
&(\frac{1}{4}x^2-40x+1600)(\frac{1}{2}x-40)],
\end{aligned}$$

and

$$\begin{aligned}
&(d^2(L(x),N))' \\
&=[\frac{(\underline{l}(x)-\min\underline{l}(x))^2+(\bar{l}(x)-\min\bar{l}(x))^2}{2}]' \\
&=[\frac{(\underline{l}(x)-\underline{l}(40))^2+(\bar{l}(x)-\bar{l}(40))^2}{2}]' \\
&=\frac{[(\frac{1}{8}x^2-21x+640)^2]'+[(\frac{1}{4}x^2-40x+1200)^2]'}{2} \\
&=(\frac{1}{8}x^2-21x+640)(\frac{1}{4}x-21) \\
&+(\frac{1}{4}x^2-40x+1200)(\frac{1}{2}x-40).
\end{aligned}$$

Then, let

$$\begin{aligned}
&(d^2(F(x^*),P))'d^2(F(x^*)N)-(d^2(F(x^*),N))'d^2(F(x^*),P) \\
&=0,
\end{aligned}$$

we obtain that $x^*=78.48$. Hence, due to Theorem 3.5, it is easy to know that x^* is a TOPSIS-optimal solution point of $L(x)$, which means that when the amount of the output is 78.48, we can get the maximum profit, and the maximum profit is [696.1912,1699.4224].

V. Conclusions

Based on TOPSIS model, we first develop the order relation of interval numbers and the definition of optimal solution of interval optimization problem. Compared with other optimization methods based on analytical mathematics, this method has a wider range of applications. We also obtain the optimization conditions for interval valued optimality problems. These optimization conditions make it convenient to find the TOPSIS-optimal solution point of interval-valued functions.

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