Multiple Periodic Solutions for Cohen-Grossberg BAM Neural Networks with Mixed Delays and Impulses

Yongzhi Liao and Qilin Tang

Abstract—**We study Cohen-Grossberg BAM neural networks with time delays and impulses. Via inverse function technique and Leray-Schauder theorem, 2^{n+m} periodic solutions for the model are derived. Further, by constructing a suitable Lyapunov function, global exponential stability of periodic solutions of the model is achieved.**

Index Terms—**Cohen-Grossberg; mixed delays; multiple periodic solutions; impulse.**

I. Introduction

IN the past decades, since Cohen-Grossberg neural networks (CGNNs) with their various generalizations shows their potential applications in classification, associative memory, parallel computation and their ability to solve optimization problems,the studies of CGNNs have attracted considerable research interests (see [1-6]). Pro- posed by Cohen and Grossberg [1] in 1983, this class of neural networks can be described as follows:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -a_i(x_i(t))\Big[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ji}(t)g_j[x_j(t)] - I_i\Big],$$

where $i = 1, 2, \ldots, n$.

At the same time, bidirectional associative memory (BAM) neural network presented by Kosko [7] has also been applied in many fields such as pattern recognition and automatic control, image and signal processing. In recent years, many investigations about the existence and stability of equilibrium of CGNNs and BAM neural networks. Recently, some researchers begin to consider Cohen-Grossberg BAM neural networks (see [8-11]).

In addition, experimental proofs show that time delays [12-14] can affect the stability of neural networks and cause some other dynamical behaviors (such as periodic, anti-periodic and almost periodic oscillation, bifurcation, chaos, etc). Moreover, the property of periodic oscillatory solutions to neural networks is also of incredible importance and have wide applications. In recent years, scholars have studied the periodicity of neural networks, and they derive sufficient conditions for the existence and stability of periodic solutions to delayed BAM neural networks. For example, see [15-18], and the references therein.

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In [19], Chen and Cao considered the following Cohen-Grossberg BAM networks with distributed delays:

$$\begin{cases} x_i'(t) = -a_i(x_i(t))\Big[b_i(t, x_i(t)) - \sum\limits_{j=1}^{m} p_{ji}(t) \\ \qquad \times \displaystyle\int_0^{\infty} K_{ji}(s) f_j(t, \lambda_j y_j(t-s)]\mathrm{d}s - I_i(t)\Big], \\ y_j'(t) = -c_j(y_j(t))\Big[d_j(t, y_j(t)) - \sum\limits_{i=1}^{n} q_{ji}(t) \\ \qquad \times \displaystyle\int_0^{\infty} L_{ij}(s) g_i(t, \mu_i x_i(t-s)]\mathrm{d}s - J_j(t)\Big], \end{cases}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. By using the Lyapunov functional method and some analytical techniques, the authors establishes some sufficient conditions for the existence, uniqueness and global exponential stability of the periodic solution for the system above.

In [20], the authors discussed a class of Cohen-Grossberg BAM neural networks with periodic coefficients and mixed delays as follows:

$$\begin{cases} x_i'(t) = -a_i(x_i(t)) \\ \qquad \times \Big\{\alpha_i(x_i(t)) - \sum\limits_{j=1}^{m} \Big[p_{ji}(t) f_j(y_j(t - \tau_{ji}(t))) \\ \qquad + h_{ji}(t) \displaystyle\int_{-\infty}^{t} K_{ji}(t-s) f_j(y_j(s))\mathrm{d}s\Big] + I_i(t)\Big\}, \\ y_j'(t) = -b_j(y_j(t)) \\ \qquad \times \Big\{\beta_j(y_j(t)) - \sum\limits_{i=1}^{n} \Big[q_{ij}(t) g_i(x_i(t - \sigma_{ij}(t))) \\ \qquad + w_{ij}(t) \displaystyle\int_{-\infty}^{t} N_{ij}(t-s) g_i(x_i(s))\mathrm{d}s\Big] + J_j(t)\Big\}, \end{cases}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. By using M-matrix theory and some analysis techniques, they investigate the existence and exponential stability of periodic solutions for this kind of neural networks.

The systems above are all continuous systems, which don't consider the dynamics behaviors of impulse points. However, many evolution processes contain impulsive effects, where their states are changed abruptly at certain moments of time. The theory of impulsive differential systems have been developed by numerous mathematicians (see [21-25]). Impulsive differential equations with or without delays have extensive use like the application in biology, medicine, mechanics, engineering, chaos theory and so on (see [26-28]). During these years, plenty of scholars have focused their attention on the dynamics of impulsive Cohen-Grossberg BAM neural networks (see [29-32]).

For example, in [33], Li considered the following Cohen-Grossberg-type BAM neural networks with time-varying delays and impulses:

$$\begin{cases} x_i'(t) = -a_i(x_i(t))\Big[b_i(x_i(t)) - \sum_{j=1}^{m} h_{ij}(t) \\ \qquad \times f_j(\lambda_j y_j(t-\tau_{ij}(t))) - \bar{I}\Big], \quad t \neq t_k, t \geq t_0, \\ \Delta x_i(t_k) = I_k(x_i(t_k^-)), \quad k \in N \triangleq \{1,2,\dots\}, \\ y_j'(t) = -c_j(y_j(t))\Big[d_j(y_j(t)) - \sum_{i=1}^{n} w_{ji}(t) \\ \qquad \times g_i(\mu_i x_i(t-\sigma_{ji}(t))) - \bar{J}\Big], \quad t \neq t_k, t \geq t_0, \\ \Delta y_j(t_k) = J_k(y_j(t_k^-)), \quad k \in N \triangleq \{1,2,\dots\}, \end{cases}$$

where $i = 1,2,\dots,n$, $j = 1,2,\dots,m$. By using Lyapunov functionals, the analysis method and impulsive control, Li studies the existence, uniqueness and exponential stability of the equilibrium point for the Cohen-Grossberg-type BAM neural networks with time-varying delays.

In [34], Li and Zhang proposed the following impulsive Cohen-Grossberg-type BAM neural networks with distributed delays:

$$\begin{cases} x_i'(t) = -a_i(x_i(t))\Big[b_i(x_i(t)) - \sum_{j=1}^{m} c_{ij} g_j(y_j(t)) \\ \qquad - \sum_{j=1}^{m} d_{ij} \int_0^{+\infty} K_{ij}(s) g_j(y_j(t-s))\mathrm{d}s \\ \qquad - I_i\Big], \quad t \neq t_k, \\ x_i(t^+) = x_i(t^-) + P_{ik}(x_i(t^-)), \quad t = t_k, \\ \qquad k \in N \triangleq \{1,2,\dots\}, i = 1,2,\dots,n, \\ y_j'(t) = -\bar{a}_j(y_j(t))\Big[\bar{b}_j(y_j(t)) - \sum_{i=1}^{n} \bar{c}_{ji}(t) f_i(x_i(t)) \\ \qquad - \sum_{i=1}^{n} \bar{d}_{ji} \int_0^{+\infty} \bar{K}_{ji} f_i(x_i(t-s))\mathrm{d}s \\ \qquad - \bar{I}_j\Big], \ t \neq t_k, \\ y_j(t^+) = y_j(t^-) + Q_{jk}(y_j(t^-)), \quad t = t_k, \\ \qquad k \in N \triangleq \{1,2,\dots\}, j = 1,2,\dots,m. \end{cases}$$

some sufficient conditions ensuring the existence, uniqueness and global exponential stability of equilibrium point for the above system are obtained By establishing an integro-differential inequality and employing the homeomorphism theory.

Although many results on the existence of periodic solutions to impulsive Cohen-Grossberg BAM neural networks are already got, the results on the multiplicity of periodic solutions for impulsive Cohen-Grossberg BAM neural networks with mixed delays are still absent. Therefore, new sufficient conditions for the multiplicity of periodic solutions to the following impulsive Cohen-Grossberg BAM are proposed in this paper:

$$\begin{cases} x_i'(t) = -a_i(x_i(t))\Big[\alpha_i(t, x_i(t)) \\ \qquad - \sum_{j=1}^{m} P_{ji}(t) f_j(t, y_j(t-\tau_{ji}(t))) \\ \qquad - \sum_{j=1}^{m} U_{ji}(t)\lambda_j\Big(\int_0^{\infty} X_{ji}(s) y_j(t-s)\mathrm{d}s\Big) \\ \qquad + c_i(t)\Big], \quad t > 0, t \neq t_k, \\ y_j'(t) = -b_j(y_j(t))\Big[\beta_j(t, y_j(t)) \\ \qquad - \sum_{i=1}^{n} Q_{ij}(t) g_i(t, x_i(t-\sigma_{ij}(t))) \\ \qquad - \sum_{i=1}^{n} V_{ij}(t)\mu_i\Big(\int_0^{\infty} Y_{ij}(s) x_i(t-s)\mathrm{d}s\Big) \\ \qquad + d_j(t)\Big], \quad t > 0, t \neq t_k, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = I_{ik}(x_i(t_k)), \\ \Delta y_j(t_k) = y_j(t_k^+) - y_j(t_k^-) = J_{jk}(y_j(t_k)), \end{cases} \tag{1.1}$$

where $x_i(t)$ and $y_j(t)$ are the activations of the ith neuron in neural field F_{x} and the jth neuron in neural field F_{y}; $f_j, g_i, \lambda_j, \mu_i$ denote the normal and the delayed activation functions; $P_{ji}(t), U_{ji}(t)$ denote the connection strengths of the ith neuron on the jth neuron in neural field F_{x} at time $t - \tau_{ji}$ and t, respectively; $Q_{ij}(t), V_{ij}(t)$ denote the connection strengths of the jth neuron on the ith neuron in neural field F_{y} at time $t - \sigma_{ij}$, t, respectively; $c_i(t), d_j(t)$ are the external input to $P_{ji}(t)$ and $Q_{ij}(t)$ at time t, respectively; the functions a_i, b_j represent two abstract amplification functions; while the functions α_i, β_j represent the self-excitation rate functions at time t; time delays $\tau_{ji}(t)$ and $\sigma_{ij}(t)$ correspond to the finite speed of the axonal signal transmission at time t, respectively; $x_i(t_k^+), x_i(t_k^-)$, $y_j(t_k^+), y_j(t_k^-)$ represent the right and left limit of $x_i(t_k)$ and $y_j(t_k)$, respectively. $\{t_k\}$ is a sequence of real numbers such that $0 < t_1 < t_2 < \cdots < t_k \to \infty$ as $k \to \infty$, $i = 1,2,\dots,n$, $j = 1,2,\dots,m$, $t > 0$.

The initial conditions of (1.1) are of the form

$$\begin{aligned} x_i(s) &= \varphi_i(s), \quad s \in [-\tau, 0], \\ &\tau = \max_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}\{\tau_{ji}\}, i = 1,2,\dots,n, \\ y_j(\tilde{s}) &= \psi_j(\tilde{s}), \quad \tilde{s} \in [-\sigma, 0], \\ &\sigma = \max_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}\{\sigma_{ij}\}, j = 1,2,\dots,m, \end{aligned} \tag{1.2}$$

where the function $\varphi(s) = [\varphi_1(s), \varphi_2(s), \dots, \varphi_n(s)]^T \in \mathbb{R}^n$ and $\psi(\tilde{s}) = [\psi_1(\tilde{s}), \psi_2(\tilde{s}), \dots, \psi_m(\tilde{s})]^T \in \mathbb{R}^m$ are piecewise continuous and bounded with respect to the norm

$$\begin{aligned} \|\varphi\|_\infty &= \max_{1\leqslant i\leqslant n}\big\{\sup_{\theta\in[-\tau,0]}|\varphi_i(\theta)|\big\}, \\ \|\psi\|_\infty &= \max_{1\leqslant j\leqslant m}\big\{\sup_{\vartheta\in[-\sigma,0]}|\psi_j(\vartheta)|\big\}. \end{aligned}$$

The main methods used in this paper is inverse function technique, Leray-Schauder fixed point theorem [35]. Several sufficient conditions are obtained for the existence of at least 2^{n+m} periodic solutions for system (1.1).

Let $\mathbb{R}^+ = [0, +\infty)$. Throughout this paper, we proposed the assumption that:

(A_1) $U_i(t) > 0$, $V_i(t) > 0$, $P_{ji}(t) > 0$, $Q_{ij}(t) > 0$, $c_i(t)$ and $d_i(t)$ are continuous ω-periodic functions, $\omega > 0$ is a constant, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$. There are positive constants $\underline{U}$, $\bar{U}$, $\underline{V}$, $\bar{V}$, $\underline{P}$, $\bar{P}$, $\underline{Q}$ and $\bar{Q}$, such that

$$\underline{U} \le U_i(t) \le \bar{U}, \quad \underline{V} \le V_i(t) \le \bar{V},$$
$$\underline{P} \le P_{ji}(t) \le \bar{P}, \quad \underline{Q} \le Q_{ij}(t) \le \bar{Q}.$$

(A_2) The delay kernels $X_{ji}, Y_{ij} : \mathbb{R}^+ \to \mathbb{R}^+$ are bounded, piecewise continuous and satisfy

$$\int_0^\infty X_{ji}(s)\mathrm{d}s = 1 \text{ and } \exists \kappa > 0,$$
$$\text{s.t.} \int_0^\infty X_{ji}(s)e^{\kappa s}\mathrm{d}s < \infty,$$
$$\int_0^\infty Y_{ij}(s)\mathrm{d}s = 1 \text{ and } \exists \upsilon > 0,$$
$$\text{s.t.} \int_0^\infty Y_{ij}(s)e^{\upsilon s}\mathrm{d}s < \infty,$$

where $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

(A_3) The time-varying delays $\sigma_{ij}(t), \tau_{ji}(t) : R^+ \to R^+$ are continuously ω-periodic functions and there exist positive constants $\bar{\sigma}_{ij}, \bar{\tau}_{ji}$ such that

$$\sigma_{ij}(t) < \bar{\sigma}_{ij}, \ \ \tau_{ji}(t) < \bar{\tau}_{ji}, \quad \forall t > 0.$$

(A_4) The jump operators

$$I_i = [I_{i1}, I_{i2}, \dots, I_{in}]^T : PC([0, t_k], \mathbb{R}^n) \to \mathbb{R}^n,$$
$$J_j = [J_{j1}, J_{j2}, \dots, J_{jm}]^T : PC([0, t_k], \mathbb{R}^m) \to \mathbb{R}^m$$

are continuous.

(A_5) $\{I_{ik}\}$, $\{J_{jk}\}$ and $\{t_k\}$ are ω-periodic sequence, i.e., there exists a positive integer l such that $[0, \omega] \cap \{t_k, k \in \mathbb{N}^*\} = \{t_1, t_2, \dots, t_l\}$, we assume that $t_{k+l} = t_k + \omega$, $I_{ik+l} = I_{ik}$, $J_{jk+l} = J_{jk}$, $k = 1, 2, \dots$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$;

(A_6) $a_i(x)$ and $b_j(y)$ are continuous and there exist positive constants $\bar{a}_i$, $\underline{a}_i$, $\bar{b}_j$ and $\underline{b}_j$ such that $0 < \underline{a}_i \leqslant a_i(x) \leqslant \bar{a}_i$, $0 < \underline{b}_j \leqslant b_j(y) \leqslant \bar{b}_j$, $x, y \in R$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$;

(A_7) $\alpha_i(t, x) \in C(\mathbb{R}^2, \mathbb{R})$ and $\beta_j(t, y) \in C(\mathbb{R}^2, \mathbb{R})$ are ω-periodic about the first argument. There is a positive constant $\bar{\alpha}_i$, $\bar{\beta}_j$, $\underline{\alpha}_i, \underline{\beta}_j$ such that

$$\underline{\alpha}_i \le \partial\alpha_i(t, x)/\partial x \le \bar{\alpha}_i \quad \text{and} \quad \underline{\beta}_j \le \partial\beta_j(t, y)/\partial y \le \bar{\beta}_j,$$

and $\alpha_i(t, 0) = 0$, $\beta_j(t, 0) = 0$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

The main contributions of this thesis are highlighted below. **(1)** Because of the derivative theorem for inverse function and the constant variation method, the original equation (1.1) can be translated into integral equation (3.1). **(2)** Through utilizing the Leray-Schauder theorem and several reasonable assumptions, at least 2^{n+m} periodic solutions for impulsive Cohen-Grossberg BAM neural networks with mixed delays are obtained. **(3)** Based on some suitable Lyapunov functions, a unique ω-periodic solution of system (2.2) is acquired, moreover, all solutions of system (2.2) can be converge exponentially to its unique ω-periodic solution.

This paper consists of four parts. The other three are stated below. In Section II, brief introduction of the basic notations and assumptions are presented. In Section III, we bear out the existence of the 2^{n+m} periodic solutions of system (1.1). In Section IV, by means of using Lyapunov function method, we obtain some sufficient conditions which ensure the globally stable of a unique periodic solution that belongs to some special set for system (1.1). Finally, in Section V, an example is given to illustrate the effectiveness of our main results. Conclusions and future works are presented in Section VI.

II. Preliminaries and notations

The transform system (1.1) and state some notations that will be used in later are briefly introduced in this section.

From (A_6), the antiderivative of $1/a_i(x_i)$ and $1/b_j(y_j)$ exists. We choose an antiderivative $h_i(x_i)$ of $1/a_i(x_i)$ and an antiderivative $z_j(y_j)$ of $1/b_j(y_j)$ that satisfies $h_i(0) = 0$ and $z_j(0) = 0$. Obviously, $(\mathrm{d}/\mathrm{d}x_i)h_i(x_i) = 1/a_i(x_i)$, $(\mathrm{d}/\mathrm{d}y_j)z_j(y_j) = 1/b_j(y_j)$. Since $a_i(x_i) > 0, b_j(y_j) > 0$, we obtain that $h_i(u)$ and $z_j(u)$ are strictly monotone increasing about $u(u \in R)$. In view of derivative theorem for inverse function, the inverse function $h_i^{-1}(x_i)$ of $h_i(x_i)$ and $z_j^{-1}(y_j)$ of $z_j(y_j)$ are differential, as well as $(\mathrm{d}/\mathrm{d}x_i)h_i^{-1}(x_i) = a_i(x_i)$ and $(\mathrm{d}/\mathrm{d}y_j)z_j^{-1}(y_j) = b_j(y_j)$. From (A_7), composition function $\alpha_i(t, h^{-1}(u))$ and $\beta_j(t, z^{-1}(v))$ are differentiable. Denote $u_i(t) = h_i(x_i(t))$, $v_j(t) = z_j(y_j(t))$. It is easy to see that $u_i'(t) = x_i'(t)/a_i(x_i(t))$, $v_j'(t) = y_j'(t)/b_j(y_j(t))$ and $x_i(t) = h_i^{-1}(u_i(t))$, $y_j(t) = z_j^{-1}(v_j(t))$, Substituting these equalities into system(1.1), we get

$$\begin{cases} u_i'(t) = -\alpha_i(t, h_i^{-1}(u_i(t))) \\ \qquad + \sum_{j=1}^m P_{ji}(t) f_j(t, z_j^{-1}(v_j(t - \tau_{ji}(t)))) \\ \qquad + \sum_{j=1}^m U_{ji}(t)\lambda_j\Big(\int_0^\infty X_{ji}(s) z_j^{-1}(v_j(t-s))\mathrm{d}s\Big) \\ \qquad - c_i(t), \quad t \ne t_k, \\ v_j'(t) = -\beta_j(t, z_j^{-1}(v_j(t))) \\ \qquad + \sum_{i=1}^n Q_{ij}(t) g_i(t, h_i^{-1}(u_i(t - \sigma_{ij}(t)))) \\ \qquad + \sum_{i=1}^n V_{ij}(t)\mu_i\Big(\int_0^\infty Y_{ij}(s) h_i^{-1}(u_i(t-s))\mathrm{d}s\Big) \\ \qquad - d_j(t), \quad t \ne t_k, \\ \Delta u_i(t_k) = h_i[h_i^{-1}(u_i(t_k)) + I_{ik}(h_i^{-1}(u_i(t_k)))] - u_i(t_k^-) \\ \qquad \triangleq r_i(u_i(t_k)), \ \ t = t_k, \\ \Delta v_j(t_k) = z_j[z_j^{-1}(v_j(t_k)) + J_{jk}(z_j^{-1}(v_j(t_k)))] - v_j(t_k^-) \\ \qquad \triangleq \delta_j(v_j(t_k)), \ \ t = t_k, \end{cases} \tag{2.1}$$

where $t > 0, \ i = 1, 2, \dots, n, \ j = 1, 2, \dots, m, \ k \in N.$

System (2.1) can be rewritten as

$$\begin{cases} u_i'(t) = -\theta_i(t, u_i(t))u_i(t) \\ \qquad + \sum_{j=1}^{m} P_{ji}(t) f_j(t, z_j^{-1}(v_j(t-\tau_{ji}(t)))) \\ \qquad + \sum_{j=1}^{m} U_{ji}(t)\lambda_j\Big(\int_0^\infty X_{ji}(s) z_j^{-1}(v_j(t-s))\mathrm{d}s\Big) \\ \qquad -c_i(t), \quad t \neq t_k, \\ v_j'(t) = -\varphi_j(t, (v_j(t))v_j(t) \\ \qquad + \sum_{i=1}^{n} Q_{ij}(t) g_i(t, h_i^{-1}(u_i(t-\sigma_{ij}(t)))) \\ \qquad + \sum_{i=1}^{n} V_{ij}(t)\mu_i\Big(\int_0^\infty Y_{ij}(s) h_i^{-1}(u_i(t-s))\mathrm{d}s\Big) \\ \qquad -d_j(t), \quad t \neq t_k, \\ \Delta u_i(t_k) = r_i(u_i(t_k)), \quad t = t_k, \\ \Delta v_j(t_k) = \delta_j(v_j(t_k)), \quad t = t_k, \end{cases} \tag{2.2}$$

where

$$\theta_i(t, u_i(t)) \triangleq \partial\alpha_i(t, h^{-1}(u))/\partial u|_{u=\xi_i},$$

$$\varphi_j(t, (v_j(t)) \triangleq \partial\beta_j(t, z^{-1}(v))/\partial v|_{v=\zeta_j},$$

$\partial\alpha_i(t, h^{-1}(u))/\partial u|_{u=\xi_i}$ denotes the partial derivative of $\alpha_i(t, h^{-1}(u))$ at point $u = \xi_i$, $\partial\beta_j(t, z^{-1}(v))/\partial v|_{v=\zeta_j}$ denotes the partial derivative of $\beta_j(t, z^{-1}(v))$ at point $v = \zeta_j$, ξ_i is between 0 and $u_i(t)$, ζ_j is between 0 and $v_j(t)$, $t > 0$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $k \in N$.

From (A_6), (A_7) and the definition of $h^{-1}(u_i)$ and $z^{-1}(v_j)$, we obtain $\alpha_i(t, h^{-1}(u_i(t)))$ and $\beta_j(t, z^{-1}(v_j(t)))$ is strictly monotone increasing about $u_i(t)$ and $v_j(t)$, respectively. Hence, $\theta_i(t, u_i(t))$ and $\varphi_j(t, (v_j(t))$ is unique for any $u_i(t)$ and $v_j(t)$, respectively, and continuous about $u_i(t)$ and $v_j(t)$, respectively, therefore, from (A_6) and (A_7), we get

$$0 < p_i \triangleq\leq \underline{a}_i\underline{\alpha}_i \leq \theta_i(t, u_i(t)) \leq \bar{a}_i\bar{\alpha}_i \triangleq \tilde{p}_i,$$

$$0 < q_j \triangleq\leq \underline{b}_j\underline{\beta}_j \leq \varphi_j(t, (v_j(t)) \leq \bar{b}_j\bar{\beta}_j \triangleq \tilde{q}_j.$$

Denoting $I(1) = (1, +\infty)$, $I(-1) = (-\infty, -1)$, $J(1) = (1, +\infty)$, $J(-1) = (-\infty, -1)$, for every $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_n; \varepsilon_1, \ldots, \varepsilon_m]^T \in \{\pm 1\}^{n+m}$, we define the Cartesian product

$$\Delta_\varepsilon = I(\varepsilon_1)\times I(\varepsilon_2)\times\cdots\times I(\varepsilon_n)\times J(\varepsilon_1)\times J((\varepsilon_2)\times\cdots\times J(\varepsilon_m).$$

The following hypothesis will be considered:

(H_1) The activation functions $\lambda_j(t)$ and $\mu_i(t)$ are globally Lipschitz continuous, and Θ_j, Φ_i denotes the Lipschitz constant corresponding to the intervals $(-\infty, -1)$ and $(1, +\infty)$, i.e.

$$|\lambda_j(u) - \lambda_j(v)| \leq \Theta_j|u - v|,$$

$$|\mu_i(u) - \mu_i(v)| \leq \Phi_i|u - v|,$$

for $\forall u, v \in (-\infty, -1)$ or $\forall u, v \in (1, +\infty)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$;

(H_2) The activation functions $f_j(t, y) \in (R^2, R)$ and $g_i(t, x) \in (R^2, R)$ are ω-periodic about the first argument. There are ω-periodic solutions $\gamma_j(t)$, $r_j(t)$, $\Pi_i(t)$ and $\pi_i(t)$ such that $\gamma_j(t) = \max_{y\in R}|f_j(t, y)|$, $r_j(t) = \inf_{y\in R}|f_j(t, y)|$, $\Pi_i(t) = \max_{x\in R}|g_i(t, x)|$, $\pi_i(t) = \inf_{x\in R}|g_i(t, x)|$ and there are positive ω-periodic solutions $A_j(t)$ and $B_i(t)$ such that

$$|f_j(t, y_1) - f_j(t, y_2)| \leq A_j(t)|y_1 - y_2|,$$

$$|g_i(t, x_1) - g_i(t, x_2)| \leq B_i(t)|x_1 - x_2|,$$

for all $x_1, x_2, y_1, y_2 \in R$. Let $\gamma_j = \max|\gamma_j(t)|$, $r_j = \inf|r_j(t)|$, $\Pi_i = \max|\Pi_i(t)|$, $\pi_i = \inf|\pi_i(t)|$, $\bar{A}_j = \max|A_j(t)|$ and $\bar{B}_i = \max|B_i(t)|$, $t > 0$.

(H_3) The activation functions μ_i and λ_j are bounded, for any $s \in \mathbb{R}$,

$$|\mu_i(s)| \leq 1 \text{ and } |\lambda_j(s)| \leq 1,$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. There exists $\pi, r \in (0, 1)$ such that the functions μ_i and λ_j satisfy:

$$\mu_i(s) \geq \pi, \ \lambda_j(s) \geq r \text{ if } s \geq 1,$$

and

$$\mu_i(s) \leq -\pi, \ \lambda_j(s) \leq -r \text{ if } s \leq -1,$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

(H_4) The external input satisfies:

$$|c_i(t)| \leq \underline{P}_{ii} r_i + \underline{U}_{ii} r - \sum_{j\neq i}(\bar{P}_{ji}\gamma_j + \bar{U}_{ji}) - \tilde{p}_i,$$

$$|d_j(t)| \leq \underline{Q}_{jj}\pi_j + \underline{V}_{jj}\pi - \sum_{i\neq j}(\bar{Q}_{ij}\Pi_i + \bar{V}_{ij}) - \tilde{q}_j;$$

(H_5) For any $k \in \mathbb{N}$; $\varepsilon \in \{\pm 1\}^{n+m}$, there exists $\Gamma_{ik} > 0$ and $\Sigma_{jk} > 0$ such that if $\varphi(t), \psi \in \Delta_\varepsilon$

$$0 \leq \varepsilon_i I_{ik}(\varphi) \leq \Gamma_{ik}, \quad 0 \leq \varepsilon_j J_{jk}(\psi) \leq \Sigma_{jk},$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

From the definition of $h_i^{-1}(u)$ and $z_j^{-1}(v)$, using Lagrange mean-value theorem, for all $x, y \in R$, we gets

$$\begin{aligned} |h_i^{-1}(x) - h_i^{-1}(y)| &= |a_i(\xi)(x - y)| \leq \bar{a}_i|x - y|, \\ |z_j^{-1}(x) - z_j^{-1}(y)| &= |b_j(\zeta)(x - y)| \leq \bar{b}_j|x - y|, \end{aligned} \tag{2.3}$$

where ξ and ζ is between x and y. Moreover, form (H_5), we have

$$\begin{aligned} |r_i(u_i(t_k))| = |h_i[h_i^{-1}(u_i(t_k)) + I_{ik}(h_i^{-1}(u_i(t_k)))] \\ -h_i(x_i(t_k))| \leq \frac{\Gamma_{ik}}{\underline{a}_i} \end{aligned}$$

and

$$\begin{aligned} |\delta_j(v_j(t_k))| = |z_j[z_j^{-1}(v_i(t_k)) + J_{jk}(z_i^{-1}(v_j(t_k)))] \\ -z_j(y_j(t_k))| \leq \frac{\Sigma_{jk}}{\underline{b}_j}, \end{aligned}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

The initial condition of (2.2) are of the form

$$\begin{aligned} u_i(\theta) &= h_i(\varphi_i(\theta)), \quad \theta \in [-\tau, 0], i = 1, 2, \ldots, n \\ v_j(\vartheta) &= z_j(\psi_j(\vartheta)), \quad \vartheta \in [-\sigma, 0], j = 1, 2, \ldots, m, \end{aligned} \tag{2.4}$$

where $\tau = \max_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}\{\tau_{ji}\}$, $\sigma = \max_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}\{\sigma_{ij}\}$.

III. Existence of multiple solutions for system (1.1)

Using Mawhin's continuation theorem, we investigate the existence of at least 2^{n+m} periodic solution of system (1.1).

Lemma 1. *The function* $\mathbf{x}(t) = (u_1(t), u_2(t), \ldots, u_n(t), v_1(t), v_2(t), \ldots, v_m(t))^T$ *is an* ω*-periodic solution of system* (2.2) *if and only if it is an* ω*-periodic solution of the following*

$$\begin{cases} u_i(t) = \displaystyle\int_0^{\omega} G_i^1(t,s)H_i(u_i(s))\mathrm{d}s \\ \qquad + \displaystyle\sum_{k=1}^{l} G_i^1(t,t_k)r_i(u_i(t_k)), \\ v_j(t) = \displaystyle\int_0^{\omega} G_j^2(t,s)H_j(v_j(s))\mathrm{d}s \\ \qquad + \displaystyle\sum_{k=1}^{l} G_j^2(t,t_k)\delta_j(v_j(t_k)), \end{cases} \tag{3.1}$$

where

$$H_i(u_i(t)) = \sum_{j=1}^{m} P_{ji}(t)f_j(t, z_j^{-1}(v_j(t-\tau_{ji}(t)))) - c_i(t) + \sum_{j=1}^{m} U_{ji}(t)\lambda_j\Big(\int_0^{\infty} X_{ji}(s)z_j^{-1}(v_j(t-s))\mathrm{d}s\Big),$$

$$H_j(v_j(t)) = \sum_{i=1}^{n} Q_{ij}(t)g_i(t, h_i^{-1}(u_i(t-\sigma_{ij}(t)))) - d_j(t) + \sum_{i=1}^{n} V_{ij}(t)\mu_i\Big(\int_0^{\infty} Y_{ij}(s)h_i^{-1}(u_i(t-s))\mathrm{d}s\Big),$$

and

$$G_i^1(t,s) = \frac{1}{1-e^{-\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}} \times \begin{cases} e^{-\int_s^t \theta_i(r,u_i(r))\mathrm{d}r}, & 0 \leqslant s \leqslant t \leqslant \omega, \\ e^{-\left(\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r - \int_t^s \theta_i(r,u_i(r))\mathrm{d}r\right)}, & 0 \leqslant t \leqslant s \leqslant \omega, \end{cases}$$

$$G_j^2(t,s) = \frac{1}{1-e^{-\int_0^{\omega}\varphi_j(r,v_j(r))\mathrm{d}r}} \times \begin{cases} e^{-\int_s^t \varphi_j(r,v_j(r))\mathrm{d}r}, & 0 \leqslant s \leqslant t \leqslant \omega, \\ e^{-\left(\int_0^{\omega}\varphi_j(r,v_j(r))\mathrm{d}r - \int_t^s \varphi_j(r,v_j(r))\mathrm{d}r\right)}, & 0 \leqslant t \leqslant s \leqslant \omega, \end{cases}$$

$i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$

Proof: On the one hand, let $t_p \leqslant t \leqslant t_{p+1}, p \leqslant l$. From the first expression of (2.2), we have

$$\left[u_i(t)e^{\int_0^t \theta_i(r,u_i(r))\mathrm{d}r}\right]' = H_i(u_i(t))e^{\int_0^t \theta_i(r,u_i(r))\mathrm{d}r}, \tag{3.2}$$

Integrating (3.2) on intervals $(0, t_1^-), (t_1^+, t_2^-), \ldots, (t_p^+, t)$, and adding all of them, by the third expression of (2.2), we obtain

$$\begin{aligned} &\int_0^t \left[u_i(s)e^{\int_0^s \theta_i(r,u_i(r))\mathrm{d}r}\right]'\mathrm{d}s \\ &= \int_0^t H_i(u_i(s))e^{\int_0^s \theta_i(r,u_i(r))\mathrm{d}r}\mathrm{d}s + \sum_{k=1}^{p}\int_{t_k^-}^{t_k^+}\left[u_i(s)e^{\int_0^s \theta_i(r,u_i(r))\mathrm{d}r}\right]'\mathrm{d}s \\ &= \int_0^t H_i(u_i(s))e^{\int_0^s \theta_i(r,u_i(r))\mathrm{d}r}\mathrm{d}s + \sum_{k=1}^{p} r_i(u_i(t_k))e^{\int_0^{t_k} \theta_i(r,u_i(r))\mathrm{d}r}, \end{aligned}$$

that is

$$u_i(t) = u_i(0)e^{-\int_0^t \theta_i(r,u_i(r))\mathrm{d}r} + \int_0^t H_i(u_i(s))e^{-\int_s^t \theta_i(r,u_i(r))\mathrm{d}r}\mathrm{d}s + \sum_{k=1}^{p} r_i u_i(t_k)e^{-\int_{t_k}^t \theta_i(r,u_i(r))\mathrm{d}r}, \tag{3.3}$$

where $i = 1, 2, \ldots, n$.

Because $u_i(\omega) = u_i(0)$, from(3.3), we obtain

$$u_i(0) = \int_0^{\omega} \frac{e^{-\int_s^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}}{1-e^{-\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}}H_i(u_i(s))\mathrm{d}s + \sum_{k=1}^{l} r_i u_i(t_k)\frac{e^{-\int_{t_k}^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}}{1-e^{-\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}}. \tag{3.4}$$

Notice, assume there are $t_1, t_2, \ldots, t_p$ impulse points in $(0, t]$, but there are $t_1, t_2, \ldots, t_l$ impulse points in $(0, \omega]$.

Substituting (3.4) into (3.3), we have

$$\begin{aligned} u_i(t) = &\int_0^t \left[\frac{e^{-\int_0^t \theta_i(r,u_i(r))\mathrm{d}r}}{1-e^{-\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}}e^{-\int_s^{\omega}\theta_i(r,u_i(r))\mathrm{d}r} + e^{-\int_s^t \theta_i(r,u_i(r))\mathrm{d}r}\right]H_i(u(s))\mathrm{d}s \\ &+ \int_t^{\omega} \frac{e^{-\int_0^t \theta_i(r,u_i(r))\mathrm{d}r}}{1-e^{-\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}} \times e^{-\int_s^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}H_i(u(s))\mathrm{d}s \\ &+ \sum_{k=1}^{l} r_i(u_i(t_k))\left[\frac{e^{-\int_{t_k}^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}}{1-e^{-\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}} \times e^{-\int_0^t \theta_i(r,u_i(r))\mathrm{d}r}\right] \\ &+ \sum_{k=1}^{p} r_i(u_i(t_k))e^{-\int_{t_k}^t \theta_i(r,u_i(r))\mathrm{d}r} \\ = &\int_0^{\omega} G_i^1(t,s)H_i(u_i(s))\mathrm{d}s + \sum_{k=1}^{l} G_i^1(t,t_k)r_i(u_i(t_k)), \end{aligned} \tag{3.5}$$

where $i = 1, 2, \ldots, n$.

Similarly, we assume $t_q \leqslant t \leqslant t_{q+1}, q \leqslant l$. From the second and forth expression of (2.2), and form the proof of

$u_i(t)$, we obtain

$$v_j(t)=\int_0^\omega G_j^2(t,s)H_j(v_j(s))\mathrm{d}s+\sum_{k=1}^{l}G_j^2(t,t_k)\delta_j(v_j(t_k)),$$

where $j=1,2,\dots,m$.

On the other hand, let $\mathbf{x}(t)$ be an ω-periodic solution of (3.1). If $t\neq t_k, k\in N$, from (3.1), we can get

$$\begin{aligned}u_i'(t)=&\bigg[\int_0^t G_i^1(t,s)H_i(u_i(s))\mathrm{d}s\\&+\int_t^\omega G_i^1(t,s)H_i(u_i(s))\mathrm{d}s\bigg]_t'\\=&\frac{e^{-\int_t^t\theta_i(r,u_i(r))\mathrm{d}r}}{1-e^{-\int_0^\omega\theta_i(r,u_i(r))\mathrm{d}r}}H_i(u_i(t))\\&-\frac{e^{-\left(\int_0^\omega\theta_i(r,u_i(r))\mathrm{d}r-\int_t^t\theta_i(r,u_i(r))\mathrm{d}r\right)}}{1-e^{-\int_0^\omega\theta_i(r,u_i(r))\mathrm{d}r}}H_i(u_i(t))\\&+\int_0^t\frac{\partial G_i^1(t,s)}{\partial t}H_i(u_i(s))\mathrm{d}s\\&+\int_t^\omega\frac{\partial G_i^1(t,s)}{\partial t}H_i(u_i(s))\mathrm{d}s\\=&H_i(u_i(t))-\bigg[\int_0^t G_i^1(t,s)H_i(u_i(s))\mathrm{d}s\\&+\int_t^\omega G_i^1(t,s)H_i(u_i(s))\mathrm{d}s\bigg]\theta_i(t,u_i(t))\\=&H_i(u_i(t))-\theta_i(t,u_i(t))u_i(t),\end{aligned}$$

where $i=1,2,\dots,n$.

If $t=t_k, k\in N$, then by the first expression of (3.1), we obtain

$$\begin{aligned}&u_i(t_k^+)-u_i(t_k^-)\\&=\sum_{k=1}^{l}G_i^1(t_k^+,t_k)r_i(u_i(t_k))-\sum_{k=1}^{l}G_i^1(t_k^-,t_k)r_i(u_i(t_k))\\&=r_i(u_i(t_k)),\end{aligned}$$

where $i=1,2,\dots,n$.

Similarly, we can also prove the $v_j(t)$ to satisfy the second and forth expression of (2.2) and thus we get that, $\mathbf{x}(t)$ is also an ω-periodic solution of system (2.2). This completes the proof of Lemma (3.1). ■

Notice. According to $0<p_i\le\theta_i(t,u_i(t))\le\tilde{p}_i$, $0<q_j\le\varphi_j(t,(v_j(t))\le\tilde{q}_j$, we easily get

$$G_i^1(t,s)\le\frac{e^{p_i\omega}}{1-e^{-p_i\omega}},\quad G_j^2(t,s)\le\frac{e^{q_j\omega}}{1-e^{-q_j\omega}},$$

$$\frac{1}{\tilde{p}_i}\le\int_0^\omega G_i^1(t,s)\mathrm{d}s\le\frac{1}{p_i},\quad\frac{1}{\tilde{q}_j}\le\int_0^\omega G_j^2(t,s)\mathrm{d}s\le\frac{1}{q_j},$$

where $i=1,2,\dots,n$, $j=1,2,\dots,m$.

Consider the Banach space X of piecewise continuous ω-periodic functions

$$\begin{aligned}X=\{\ \mathbf{x}=&(u_1,\dots,u_n,v_1\dots,v_m)^T:u_i=u_i(t),\\&v_j=v_j(t)\in PC([0,\omega],\mathbb{R}),\\&u_i(t+\omega)=u_i(t),\ \ v_j(t+\omega)=v_j(t)\},\end{aligned}$$

endowed with the norm

$$\begin{aligned}\|\mathbf{x}\|&=\sum_{i=1}^{n}\|u_i\|+\sum_{j=1}^{m}\|v_j\|\\&=\sup_{t\in[0,\omega]}\left(\sum_{i=1}^{n}|u_i(t)|+\sum_{j=1}^{m}|v_j(t)|\right),\quad\forall\ \mathbf{x}\in X\end{aligned}$$

and define the operator $U:X\to X$ given by $U(\mathbf{x})=(U(u_1),\dots,U(u_n),U(v_1)\dots,U(v_m))^T$, where

$$U(u_i(t))=\int_0^\omega G_i^1(t,s)H_i(u_i(s))\mathrm{d}s+\sum_{k=1}^{l}G_i^1(t,t_k)r_i(u_i(t_k)),$$

$$U(v_j(t))=\int_0^\omega G_j^2(t,s)H_j(v_j(s))\mathrm{d}s+\sum_{k=1}^{l}G_j^2(t,t_k)\delta_j(v_j(t_k)),$$

where $i=1,2,\dots,n$, $j=1,2,\dots,m$.

Denote $\mathbf{x}=(u,v)^T$, $u=(u_1,u_2,\dots,u_n)$, $v=(v_1,v_2,\dots,v_m)$ for $\forall\ \mathbf{x}\in X$.

Clearly, from Lemma (3.1) that $\mathbf{x}(t)=(u(t),v(t))^T$ is an ω-periodic solution of (2.2) if and only if it is a fixed point of the operator U. For every set

$$\varepsilon=[\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n;\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m]^T\in\{\pm1\}^{n+m},$$

we define the closed convex set

$$K_\varepsilon=\{\mathbf{x}\in X:\mathbf{x}=(u,v)^T,\varepsilon_i u_i(t)\ge1,\varepsilon_j v_j(t)\ge1\},$$

where $\forall t\in\mathbb{R}$, $i=1,2,\dots,n$, $j=1,2,\dots,m$.

The following fixed point theorem can help prove our main results in this section.

Theorem 1. *(Leray-Schauder)*[35]. *Let X be a Banach space, $K\subset X$ a closed convex subset, $B\subset X$ a bounded subset, open in K, and $\mathbf{x}_0\in K$ a fixed element. Assume that the operator $U:\bar{B}\to K$ is completely continuous and satisfies the boundary condition*

$$\mathbf{x}\neq(1-\lambda)\mathbf{x}_0+\lambda U(\mathbf{x}),\ \ \forall\mathbf{x}\in\partial B,\ \ \lambda\in(0,1).\quad(3.6)$$

Then the operator U has at least one fixed point in $\bar{B}$.

Moreover, the following theorem, it based on from ([22]), will be used.

Theorem 2. *(Compactness criterion). The set $\mathcal{F}\subset X$ is relatively compact if and only if the following hold*:

(1) *$\mathcal{F}$ is bounded, that is, there exists $\mathcal{K}>0$ such that $||\mathbf{x}||\le\mathcal{K}$, for any $\mathbf{x}\in\mathcal{F}$;*

(2) *$\mathcal{F}$ is quasi-equicontinuous in $[0,\omega]$, i.e., for any $\varepsilon>0$, there exists $\delta>0$ such that for any $\mathbf{x}\in\mathcal{F}$, $l\in\mathbb{N}^*$, $T_1,T_2\in(t_{l-1},t_l]\cap[0,\omega]$, such that $T_1-T_2<\delta$, one has $||\mathbf{x}(T_1)-\mathbf{x}(T_2)||_l<\varepsilon$.*

Lemma 2. *Let $\varepsilon\in\{\pm1\}^{n+m}$. If $(H_1)-(H_2)$ holds, the operator U is continuous on K_ε.*

Proof: Let $\forall\ \mathbf{x},\mathbf{y}\in K_\varepsilon\subset X$, $\mathbf{x}=(u,v)^T$, $\mathbf{y}=(\tilde{u},\tilde{v})^T$, then

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|&=\sum_{i=1}^{n}\|u_i-\tilde{u}_i\|+\sum_{j=1}^{m}\|v_j-\tilde{v}_j\|\\&=\sup_{t\in[0,\omega]}\left(\sum_{i=1}^{n}|u_i(t)-\tilde{u}_i(t)|+\sum_{j=1}^{m}|v_j(t)-\tilde{v}_j(t)|\right).\end{aligned}$$

By means of $(H_1)-(H_2)$, inequality (2.3) and assumption (A_2), we first evaluate:

$$\begin{aligned}&|H_i(u_i(t))-H_i(\tilde{u}_i(t))|\\ \le&\sum_{j=1}^{m}|P_{ji}(t)||f_j(t,z_j^{-1}(v_j(t-\tau_{ji}(t))))\\ &-f_j(t,z_j^{-1}(\tilde{v}_j(t-\tau_{ji}(t))))|\\ &+\sum_{j=1}^{m}|U_{ji}(t)|\bigg|\lambda_j\bigg(\int_0^{\infty}X_{ji}(s)z_j^{-1}(v_j(t-s))\mathrm{d}s\bigg)\\ &-\lambda_j\bigg(\int_0^{\infty}X_{ji}(s)z_j^{-1}(\tilde{v}_j(t-s))\mathrm{d}s\bigg)\bigg|\\ \le&\bigg(\sum_{j=1}^{m}\bar{P}_{ji}\bar{A}_j\bar{b}_j+\sum_{j=1}^{m}\bar{U}_{ji}\Theta_j\bar{b}_j\bigg)\|v_j-\tilde{v}_j\|\\ \le&M_i(\sum_{j=1}^{m}\|v_j-\tilde{v}_j\|),\end{aligned}$$

where $i=1,2,\dots,n$, $j=1,2,\dots,m$. Similarly,

$$\begin{aligned}&|H_j(v_j(t))-H_j(\tilde{v}_j(t))|\\ \le&\bigg(\sum_{i=1}^{n}\bar{Q}_{ij}\bar{B}\bar{a}_i+\sum_{i=1}^{n}\bar{V}_{ij}\Phi_i\bar{a}_i\bigg)\|u_i-\tilde{u}_i\|\\ \le&M_j(\sum_{i=1}^{n}\|u_i-\tilde{u}_i\|),\end{aligned}$$

where $M_i=\max\{\bar{P}_{ji}\bar{A}\bar{b}_j+\bar{U}_{ji}\Theta_j\bar{b}_j\}$, $M_j=\max\{\bar{Q}_{ij}\bar{B}\bar{a}_i+\bar{V}_{ij}\Phi_i\bar{a}_i\}$, $i=1,2,\dots,n$, $j=1,2,\dots,m$.

Next, for $t>0$, we evaluate:

$$\begin{aligned}&||U((u_i(t))-U((\tilde{u}_i(t))||\\ \le&\int_0^{\omega}G_i^1(t,s)|H_i(u_i(s))-H_i(\tilde{u}_i(s))|\mathrm{d}s\\ &+\sum_{k=1}^{l}G_i^1(t,t_k)|(r_i(u_i(t_k))-r_i(\tilde{u}_i(t_k)))|\\ \le&\frac{M_i}{p_i}(\sum_{j=1}^{m}\|v_j-\tilde{v}_j\|)\\ &+\frac{e^{p_i\omega}}{1-e^{-p_i\omega}}\sum_{k=1}^{l}\bigg[\bigg(\frac{\bar{a}_i}{\underline{a}_i}+1\bigg)|u_i(t_k)-\tilde{u}_i(t_k)|\\ &+\frac{1}{\underline{a}_i}|I_{ik}(h_i^{-1}(u_i(t_k)))-I_{ik}(h_i^{-1}(\tilde{u}_i(t_k)))|\bigg]\end{aligned}$$

and

$$\begin{aligned}&||U((v_j(t))-U((\tilde{v}_j(t))||\\ \le&\int_0^{\omega}G_j^2(t,s)|H_j(v_j(s))-H_j(\tilde{v}_j(s))|\mathrm{d}s\\ &+\sum_{k=1}^{l}G_j^2(t,t_k)|\delta_j(v_j(t_k))-\delta_j(\tilde{v}_j(t_k))|\\ \le&\frac{M_j}{q_j}(\sum_{i=1}^{n}\|u_i-\tilde{u}_i\|)\\ &+\frac{e^{q_j\omega}}{1-e^{-q_j\omega}}\sum_{k=1}^{l}\bigg[\bigg(\frac{\bar{b}_j}{\underline{b}_j}+1\bigg)|v_j(t_k)-\tilde{v}_j(t_k)|\\ &+\frac{1}{\underline{b}_j}|J_{jk}(z_j^{-1}(v_j(t_k)))-J_{jk}(h_j^{-1}(\tilde{v}_j(t_k)))|\bigg],\end{aligned}$$

where $i=1,2,\dots,n$, $j=1,2,\dots,m$.

Then

$$\begin{aligned}&\|U(\mathbf{x})-U(\mathbf{y})\|\\ \le&\sum_{i=1}^{n}|U((u_i(t))-U((\tilde{u}_i(t))|\\ &+\sum_{j=1}^{m}|U((v_j(t))-U((\tilde{v}_j(t))|\\ \le&M(1+l)\|\mathbf{x}-\mathbf{y}\|+\sum_{k=1}^{l}\bigg[\sum_{i=1}^{n}\frac{e^{p_i\omega}}{\underline{a}_i(1-e^{-p_i\omega})}\\ &\times|I_{ik}(h_i^{-1}(u_i(t_k)))-I_{ik}(h_i^{-1}(\tilde{u}_i(t_k)))|\\ &+\sum_{j=1}^{m}\frac{e^{q_j\omega}}{\underline{b}_j(1-e^{-q_j\omega})}\\ &\times|J_{jk}(z_j^{-1}(v_j(t_k)))-J_{jk}(z_j^{-1}(\tilde{v}_j(t_k)))|\bigg],\end{aligned}$$

where $M=\max\bigg\{\sum_{i=1}^{n}\frac{M_i}{p_i},\sum_{j=1}^{m}\frac{M_j}{q_i},\frac{e^{p_i\omega}}{1-e^{-p_i\omega}}(\frac{\bar{a}_i}{\underline{a}_i}+1),\frac{e^{q_j\omega}}{1-e^{-q_j\omega}}(\frac{\bar{b}_j}{\underline{b}_j}+1)\bigg\}$.

Based on the continuity of the operators I_{ik} and J_{jk}, we get that the operator U is continuous on K_ε. This completes the proof. ■

Lemma 3. *If (A_1) and $(H_2)-(H_5)$ holds, for every $\varepsilon\in\{\pm\}^{n+m}$, the operator U maps K_ε into itself, that is $U(K_\varepsilon)\subset K_\varepsilon$.*

Proof: Let $\varepsilon\in\{\pm1\}^{n+m}$, $\mathbf{x}=(u,v)^T\in K_\varepsilon$. Form $(H_2)-(H_4)$, the strictly monotonicity of $h_i(u)$ and $z_j(u)$ about $u(u\in R)$, we obtain

$$\begin{aligned}\varepsilon_iH_i(u_i(t))=&\varepsilon_iP_{ii}(t)f_i(t,z_i^{-1}(v_i(t-\tau_{ji}(t)))\\ &+\varepsilon_iU_{ii}(t)\lambda_i\bigg(\int_0^{\infty}X_{ii}(s)z_i^{-1}(v_i(t-s))\mathrm{d}s\bigg)\\ &+\sum_{j\neq i}P_{ji}(t)\varepsilon_if_j(t,z_j^{-1}(v_j(t-\tau_{ji}(t))))\\ &+\sum_{j\neq i}U_{ji}(t)\varepsilon_i\lambda_j\\ &\times\bigg(\int_0^{\infty}X_{ji}(s)z_j^{-1}(v_j(t-s))\mathrm{d}s\bigg)-\varepsilon_ic_i(t)\\ \ge&\underline{P}_{ii}r_i+\underline{U}_{ii}r-\sum_{j\neq i}(\bar{P}_{ji}\gamma_j+\bar{U}_{ji})-|c_i(t)|\\ \ge&\tilde{p}_i,\quad i=1,2,\dots,n.\end{aligned}$$

Similarly, we get

$$\varepsilon_jH_j(v_j(t))\ge\underline{Q}_{jj}\pi_j+\underline{V}_{jj}\pi-\sum_{i\neq j}(\bar{Q}_{ij}\Pi_i+\bar{V}_{ij})-|d_j(t)|$$

$$\geq \tilde{q}_j, \quad j=1,2,\dots,m.$$

Thus, by means of (H_5), it follows that

$$\begin{aligned}\varepsilon_i U(u_i(t)) &= \int_0^\omega G_i^1(t,s)\varepsilon_i H_i(u_i(s))\mathrm{d}s \\ &\quad + \sum_{k=1}^{l} G_i^1(t,t_k)\varepsilon_j r_i(u_i(t_k)) \\ &\geq \frac{1}{\tilde{p}_i}\tilde{p}_i = 1, \quad i=1,2,\dots,n\end{aligned}$$

and

$$\begin{aligned}\varepsilon_j U(v_j(t)) &= \int_0^\omega G_j^2(t,s)\varepsilon_i H_j(v_j(s))\mathrm{d}s \\ &\quad + \sum_{k=1}^{l} G_j^2(t,t_k)\varepsilon_j \delta_j(v_j(t_k)) \\ &\geq \frac{1}{\tilde{q}_j}\tilde{q}_j = 1, \quad j=1,2,\dots,m.\end{aligned}$$

Hence, the proof is now finished. ■

Lemma 4. *Suppose (A_1) and $(H_2)-(H_5)$ holds. Let $\varepsilon \in \{\pm\}^{n+m}$ and $\mathbf{x}_\varepsilon \in K_\varepsilon$ the constant function defined by $\mathbf{x}_\varepsilon(t)=\varepsilon$, for any $t \in \mathbb{R}$. If there exists $\mathbf{x}=(u,v)^T \in K_\varepsilon$ and $\lambda \in (0,1)$ such that*

$$\mathbf{x} = (1-\lambda)\mathbf{x}_\varepsilon + \lambda U(\mathbf{x}), \tag{3.7}$$

then $\|\mathbf{x}-\mathbf{x}_\varepsilon\| < R$, where $\mathbf{x}=(u,v)^T \in X$ and

$$\begin{aligned}R &= \sum_{i=1}^{n}\Bigg(\frac{\bar{P}_{ii}(r_i+\gamma_i)+\bar{U}_{ii}(r+1)}{p_i} - 2 \\ &\quad + \frac{e^{p_i\omega}}{1-e^{-p_i\omega}}\sum_{k=1}^{l}\Gamma_{ik}\Bigg) \\ &\quad + \sum_{j=1}^{m}\Bigg(\frac{\bar{Q}_{jj}(\pi_j+\Pi_j)+\bar{V}_{jj}(\pi+1)}{q_j} - 2 \\ &\quad + \frac{e^{q_j\omega}}{1-e^{-q_j\omega}}\sum_{k=1}^{l}\Sigma_{jk}\Bigg).\end{aligned} \tag{3.8}$$

Proof: Assume that $\mathbf{x} \in K_\varepsilon$ and $\lambda \in (0,1)$ satisfy Eq. (3.9). Then

$$\begin{aligned}u_i(t)-\varepsilon_i &= \lambda[U(u_i(t))-\varepsilon_i], \\ v_j(t)-\varepsilon_j &= \lambda[U(v_j(t))-\varepsilon_j],\end{aligned}$$

where $\forall t \in [0,\omega]$, $i=1,2,\dots,n$, $j=1,2,\dots,m$.

Since $\mathbf{x} \in K_\varepsilon$, for any $t \in [0,\omega]$, it follows that

$$\varepsilon_i u_i(t) \geq 1, \quad \varepsilon_j v_j(t) \geq 1,$$

$$\varepsilon_i(u_i(t)-\varepsilon_i) \geq 0, \quad \varepsilon_j(v_j(t)-\varepsilon_j) \geq 0,$$

where $i=1,2,\dots,n$, $j=1,2,\dots,m$. Therefore, for any $t \in [0,\omega]$, we get:

$$\begin{aligned}|u_i(t)-\varepsilon_i| &= |\varepsilon_i(u_i(t)-\varepsilon_i)| = \varepsilon_i(u_i(t)-\varepsilon_i) \\ &= \varepsilon_i\lambda[U(u_i)(t)-\varepsilon_i] = \lambda[\varepsilon_i U(u_i)(t)-1]\end{aligned}$$

and

$$\begin{aligned}|v_j(t)-\varepsilon_j| &= |\varepsilon_j(v_j(t)-\varepsilon_j)| = \varepsilon_j(v_j(t)-\varepsilon_j) \\ &= \varepsilon_j\lambda[U(v_j)(t)-\varepsilon_j] = \lambda[\varepsilon_j U(v_j)(t)-1],\end{aligned}$$

where $i=1,2,\dots,n$, $j=1,2,\dots,m$.

Form (H_2)-(H_4), we evaluate

$$\begin{aligned}\varepsilon_i H_i(u_i(t)) &\leq \sum_{j=1}^{m}(\bar{P}_{ji}\gamma_j + \bar{U}_{ji}) + |c_i(t)| \\ &\leq \bar{P}_{ii}(r_i+\gamma_i)+\bar{U}_{ii}(r+1)-p_i\end{aligned}$$

and

$$\varepsilon_j H_j(v_j(t)) \leq \bar{Q}_{jj}(\pi_j+\Pi_j)+\bar{V}_{jj}(\pi+1)-q_j,$$

where $i=1,2,\dots,n$, $j=1,2,\dots,m$.

By using (H_5), for any $t \in [0,\omega]$, we can get

$$\begin{cases}\varepsilon_i U(u_i(t)) \leq \dfrac{\bar{P}_{ii}(r_i+\gamma_i)+\bar{U}_{ii}(r+1)}{p_i} - 1 \\ \qquad + \dfrac{e^{p_i\omega}}{1-e^{-p_i\omega}}\displaystyle\sum_{k=1}^{l}\Gamma_{ik}, \\ \varepsilon_j U(v_j(t)) \leq \dfrac{\bar{Q}_{jj}(\pi_j+\Pi_j)+\bar{V}_{jj}(\pi+1)}{q_j} - 1 \\ \qquad + \dfrac{e^{q_j\omega}}{1-e^{-q_j\omega}}\displaystyle\sum_{k=1}^{l}\Sigma_{jk},\end{cases}$$

hence, for any $t \in [0,\omega]$, we have

$$\begin{cases}|u_i(t)-\varepsilon_i| \leq \dfrac{\bar{P}_{ii}(r_i+\gamma_i)+\bar{U}_{ii}(r+1)}{p_i} - 2 \\ \qquad + \dfrac{e^{p_i\omega}}{1-e^{-p_i\omega}}\displaystyle\sum_{k=1}^{l}\Gamma_{ik}, \\ |v_j(t)-\varepsilon_j| \leq \dfrac{\bar{Q}_{jj}(\pi_j+\Pi_j)+\bar{V}_{jj}(\pi+1)}{q_j} - 2 \\ \qquad + \dfrac{e^{q_j\omega}}{1-e^{-q_j\omega}}\displaystyle\sum_{k=1}^{l}\Sigma_{jk},\end{cases}$$

where $i=1,2,\dots,n$, $j=1,2,\dots,m$.

That is

$$\begin{aligned}\|x-x_\varepsilon\| &= \sup_{t\in[0,\omega]}\left(\sum_{i=1}^{n}|u_i(t)-\varepsilon_i| + \sum_{j=1}^{m}|v_j(t)-\varepsilon_j|\right) \\ &< \sum_{i=1}^{n}\Bigg(\frac{\bar{P}_{ii}(r_i+\gamma_i)+\bar{U}_{ii}(r+1)}{p_i} - 2 \\ &\quad + \frac{e^{p_i\omega}}{1-e^{-p_i\omega}}\sum_{k=1}^{l}\Gamma_{ik}\Bigg) \\ &\quad + \sum_{j=1}^{m}\Bigg(\frac{\bar{Q}_{jj}(\pi_j+\Pi_j)+\bar{V}_{jj}(\pi+1)}{q_j} - 2 \\ &\quad + \frac{e^{q_j\omega}}{1-e^{-q_j\omega}}\sum_{k=1}^{l}\Sigma_{jk}\Bigg).\end{aligned}$$

Hence, the proof is now finished. ■

Theorem 3. *Let $\varepsilon \in \{-1\}^{n+m}$, the operator $U: \overline{B}_\varepsilon \subset K_\varepsilon \to K_\varepsilon$ has at least one fixed point in $\overline{B}_\varepsilon$, if hypotheses (H_1)-(H_5) hold, where $B_\varepsilon = \{\mathbf{x} \in K_\varepsilon : \|\mathbf{x}-\mathbf{x}_\varepsilon\| < R\}$ with $\mathbf{x}_\varepsilon$ and R given by Lemma 4.*

Proof: According to Leray-Schauder theorem, we only need to show that the operator $U: \overline{B}_\varepsilon \subset K_\varepsilon \to K_\varepsilon$ is completely continuous.

Assume $\Omega \subset \overline{B}_\varepsilon$ a bounded set. By means of Theorem 2, we will show that $U(\Omega)$ is relatively compact.

For any $\mathbf{x} \in \overline{B}_\varepsilon$, we get

$$\begin{aligned}\|U(\mathbf{x})\| &= \sup_{t\in[0,\omega]}\left(\sum_{i=1}^{n}|U(u_i(t))| + \sum_{j=1}^{m}|U(v_j(t))|\right)\\ &\le \sum_{i=1}^{n}\bigg(\frac{\bar{P}_{ii}(r_i+\gamma_i)+\bar{U}_{ii}(r+1)}{p_i} - 1\\ &\quad + \frac{e^{p_i\omega}}{1-e^{-p_i\omega}}\sum_{k=1}^{l}\Gamma_{ik}\bigg)\\ &\quad + \sum_{j=1}^{m}\bigg(\frac{\bar{Q}_{jj}(\pi_j+\Pi_j)+\bar{V}_{jj}(\pi+1)}{q_j} - 1\\ &\quad + \frac{e^{q_j\omega}}{1-e^{-q_j\omega}}\sum_{k=1}^{l}\Sigma_{jk}\bigg).\end{aligned}$$

Therefore, the set $U(\Omega)$ is bounded.

In the following, we will show that $U(X)$ is quasi-equicontinuous in $[0,\omega]$. Let $\mathbf{x} \in \Omega$, $k \in \{1,2,\ldots,l\}$ and $T_1, T_2 \in (t_k, t_{k+1}] \cap [0,\omega]$. Let's assume $T_1 < T_2$, and evaluate:

$$\begin{aligned}&|U(u_i(T_1)) - U(u_i(T_2))|\\ &\le \left[\bar{P}_{ii}(r_i+\gamma_i)+\bar{U}_{ii}(r+1)-p_i\right]\\ &\quad \times \int_0^{\omega}|G_i^1(T_1,s)-G_i^1(T_2,s)|\mathrm{d}s\\ &\quad + \sum_{k=1}^{l}\frac{\Gamma_{ik}}{\underline{a}_i}\left|G_i^1(T_1,t_k)-G_i^1(T_2,t_k)\right|,\end{aligned}$$

where $i = 1,2,\ldots,n$. First, form (3.7) and $0 < p_i \le \theta_i(r,u_i(r)) \le \tilde{p}_i$, we have

$$\begin{aligned}&\int_0^{\omega}\left|G_i^1(T_1,s)-G_i^1(T_2,s)\right|\mathrm{d}s\\ &= \frac{1}{1-e^{-\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}}\\ &\quad \times\bigg[\int_0^{T_1}\left|e^{-\int_s^{T_1}\theta_i(r,u_i(r))\mathrm{d}r} - e^{-\int_s^{T_2}\theta_i(r,u_i(r))\mathrm{d}r}\right|\mathrm{d}s\\ &\quad + \int_{T_1}^{T_2}\bigg|e^{-\int_s^{T_2}\theta_i(r,u_i(r))\mathrm{d}r}\\ &\qquad - e^{-\int_0^{T_1}\theta_i(r,u_i(r))\mathrm{d}r - \int_s^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}\bigg|\mathrm{d}s\\ &\quad + \int_{T_2}^{\omega}\bigg|e^{-\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r + \int_{T_1}^{s}\theta_i(r,u_i(r))\mathrm{d}r}\\ &\qquad - e^{-\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r + \int_{T_2}^{s}\theta_i(r,u_i(r))\mathrm{d}r}\bigg|\mathrm{d}s\bigg]\\ &\le \frac{2e^{p_i\omega}}{p_i}\left(e^{\tilde{p}_i(T_2-T_1)}-1\right) + \frac{1}{p_i}\left(1-e^{-p_i(T_2-T_1)}\right),\end{aligned}$$

where $e^{p_i x} < e^{\tilde{p}_i x} < e^{p_i\omega}(x \in (0,\omega))$, $e^{-x} < 1(x > 0)$, $i = 1,2,\ldots,n$. Consequently,

$$\int_0^{\omega}\left|G_i^1(T_1,s)-G_i^1(T_2,s)\right|\mathrm{d}s \to 0, \quad \text{as} \quad T_1 \to T_2,$$

where $i = 1,2,\ldots,n$.

Next, we calculate

$$\begin{aligned}&\sum_{k=1}^{l}\frac{\Gamma_{ik}}{\underline{a}_i}|G_i^1(T_1,t_k)-G_i^1(T_2,t_k)|\\ &\le \sum_{k=1}^{p}\frac{\Gamma_{ik}}{\underline{a}_i(1-e^{-\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r})}\left|e^{\int_{T_1}^{T_2}\theta_i(r,u_i(r))\mathrm{d}r}-1\right|\\ &\quad + \sum_{k=p+1}^{l}\frac{\Gamma_{ik}}{\underline{a}_i}\left|\frac{e^{-\int_0^{T_1}\theta_i(r,u_i(r))\mathrm{d}r} - e^{-\int_0^{T_2}\theta_i(r,u_i(r))\mathrm{d}r}}{1-e^{-\int_0^{\omega}\theta_i(r,u_i(r))\mathrm{d}r}}\right|\\ &\le \sum_{k=1}^{l}\frac{\Gamma_{ik}}{\underline{a}_i(1-e^{-p_i\omega})}\left(e^{\tilde{p}_i(T_2-T_1)}-1\right),\end{aligned}$$

that is

$$\sum_{k=1}^{l}\frac{\Gamma_{ik}}{\underline{a}_i}|G_i^1(T_1,t_k)-G_i^1(T_2,t_k)| \to 0, \quad \text{as} \quad T_1 \to T_2,$$

where $i = 1,2,\ldots,n$.

Therefore, we have

$$\begin{aligned}&|U(u_i(T_1)) - U(u_i(T_2))|\\ &\le \left[\bar{P}_{ii}(r_i+\gamma_i)+\bar{U}_{ii}(r+1)-p_i\right]\\ &\quad \times\frac{2e^{p_i\omega}+1}{p_i}\left(e^{\tilde{p}_i(T_2-T_1)}-1\right)\\ &\quad + \sum_{k=1}^{l}\frac{\Gamma_{ik}}{\underline{a}_i(1-e^{-p_i\omega})}\left(e^{\tilde{p}_i(T_2-T_1)}-1\right),\end{aligned}$$

that is

$$|U(u_i(T_1)) - U(u_i(T_2))| \to 0, \quad \text{as} \quad T_1 \to T_2,$$

where $i = 1,2,\ldots,n$. Similarly,

$$\begin{aligned}&|U(v_j(T_1)) - U(v_j(T_2))|\\ &\le \left[\bar{Q}_{jj}(\pi_j+\Pi_j)+\bar{V}_{jj}(\pi+1)-q_j\right]\\ &\quad \times\frac{2e^{p_i\omega}}{p_i}\left(e^{\tilde{p}_i(T_2-T_1)}-1\right) + \frac{1}{p_i}\left(1-e^{-p_i(T_2-T_1)}\right)\\ &\quad + \sum_{k=1}^{l}\frac{\Sigma_{jk}}{\underline{b}_j(1-e^{-q_j\omega})}\left(e^{\tilde{q}_j(T_2-T_1)}-1\right),\end{aligned}$$

that is

$$|U(v_j(T_1)) - U(v_j(T_2))| \to 0, \quad \text{as} \quad T_1 \to T_2,$$

where $i = 1,2,\ldots,n$.

Hence, for any $\mathbf{x} \in \Omega$, $k \in \{1,2,\ldots,l\}$ and $T_1, T_2 \in (t_k, t_{k+1}] \cap [0,\omega]$, the following estimate holds

$$\begin{aligned}&\|U(\mathbf{x})(T_1) - U(\mathbf{x})(T_2)\|\\ &= \sup_{t\in[0,\omega]}\bigg(\sum_{i=1}^{n}|U(u_i(T_1)) - U(u_i(T_2))|\\ &\quad + \sum_{j=1}^{m}|U(v_j(T_1)) - U(v_j(T_2))|\bigg),\end{aligned}$$

that is

$$\|U(\mathbf{x})(T_1) - U(\mathbf{x})(T_2)\| \to 0, \quad \text{as} \quad T_1 \to T_2,$$

which suggests that $U(X)$ is quasi-equicontinuous in $[0,\omega]$. Further, $U(X)$ is relatively compact and the proof is now complete. ■

Corollary 1. *If hypotheses $(H_1)-(H_5)$ hold, then there exists at least 2^{n+m} periodic solutions of system (1.3), that is, at least one periodic solution in every set $\overline{B}_\varepsilon$ for every $\varepsilon \in \{\pm 1\}^{n+m}$.*

IV. Global exponential stability of the periodic solution

Some suitable Lyapunov functionals to derive sufficient conditions ensuring that system (2.2) has a unique ω-periodic solution and all solutions of system (2.2) exponentially converge to its unique ω-periodic solution are constructed in this section.

Lemma 5. *Let $\varepsilon \in \{\pm 1\}^{n+m}$. If hypotheses (H_1)-(H_5) are fulfilled, then the set Δ_ε is invariant.*

Proof: Let $\varepsilon \in \{\pm\}^{n+m}$, consider an initial function satisfying $u(\theta), v(\vartheta) \in \Delta_\varepsilon$, for any $\theta \in [-\sigma, 0]$ and $\vartheta \in [-\tau, 0]$. Let $\mathbf{x}(t) = (u(t), v(t))^T = \mathbf{x}(t; (u(\theta), v(\vartheta))^T)$ is a solution of system (2.2).

Assume that there exists $\eta \in (0, t_1]$ such that $\mathbf{x}(t) \in \Delta_\varepsilon$, for any $t \in (0,\eta)$ and $\mathbf{x}(\eta) \in \partial\Delta_\varepsilon$. Hence, there exists $i \in \{1,2,\ldots,n\}$, $j \in \{1,2,\ldots,m\}$ such that $u_i(\eta) = \varepsilon_i$, $v_j(\eta) = \varepsilon_j$. According to (H_3)-(H_4) we get

$$\begin{aligned}
\varepsilon_i \dot{u}_i(t) = \varepsilon_i \bigg[& -\theta_i(t, u_i(t))u_i(t) \\
& + \sum_{j=1}^{m} P_{ji}(t) f_j(t, z_j^{-1}(v_j(t-\tau_{ji}(t)))) - c_i(t) \\
& + \sum_{j=1}^{m} U_{ji}(t)\lambda_j \Big(\int_0^\infty X_{ji}(s) z_j^{-1}(v_j(t-s))\mathrm{d}s \Big) \bigg] \\
\geq & -\theta_i(t, u_i(t)) + \underline{P}_{ii} r_i + \underline{U}_{ii} r \\
& - \sum_{j\neq i} (\bar{P}_{ji}\gamma_j + \bar{U}_{ji}) - |c_i(t)| \\
\geq & -\tilde{p}_i + \underline{P}_{ii} r_i + \underline{U}_{ii} r \\
& - \sum_{j\neq i} (\bar{P}_{ji}\gamma_j + \bar{U}_{ji}) - |c_i(t)| > 0
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_j \dot{v}_j(t) = \varepsilon_j \bigg[& -\varphi_j(t, v_j(t))v_j(t) \\
& + \sum_{i=1}^{n} Q_{ij}(t) g_i(t, h_i^{-1}(u_i(t-\sigma_{ij}(t)))) - d_j(t) \\
& + \sum_{i=1}^{n} V_{ij}(t)\mu_i \Big(\int_0^\infty Y_{ij}(s) h_i^{-1}(u_i(t-s))\mathrm{d}s \Big) \bigg] \\
\geq & -\tilde{q}_j + \underline{Q}_{jj}\pi_j + \underline{V}_{jj}\pi \\
& - \sum_{i\neq j} (\bar{Q}_{ij}\Pi_i + \bar{V}_{ij}) - |d_j(t)| > 0,
\end{aligned}$$

where $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$.

Therefore, the function $\varepsilon_i u_i$ and $\varepsilon_j v_j$ are strictly increasing on some small interval $(\eta-\delta, \eta] \subset (0,\eta]$. Hence $\varepsilon_i u_i(t) < \varepsilon_i u_i(\eta) = \varepsilon_i^2 = 1$, $\varepsilon_j v_j(t) < 1$ for any $t \in (\eta-\delta, \eta]$. This is absurd, since $\mathbf{x}(t) \in \Delta_\varepsilon$, for any $t \in (0,\eta)$.

It follows that $\mathbf{x}(t) \in \Delta_\varepsilon$, for any $t \in (t_0, t_1]$ (where $t_0 = 0$). Hypothesis (H_5) guarantees that $\mathbf{x}(t_1^+)$ as well.

By mathematical induction, it can be easily shown that $\mathbf{x}(t) \in \Delta_\varepsilon$ for any $t \in (t_{k-1}, t_k]$ and $x(t_k^+) \in \Delta_\varepsilon$ for any $k \in N$.

Therefore, the solution $\mathbf{x}(t; (u(\theta), v(\vartheta))^T)$ with the initial condition $\theta \in [-\sigma, 0]$, $\vartheta \in [-\tau, 0]$ will remain in Δ_ε for any $t \geq 0$. This completes the proof. ■

Theorem 4. *Assume that all hypotheses $(H_1)-(H_5)$ hold. Suppose further that*

(H_6) $p_i - \bar{a}_i \sum_{j=1}^{m} \Big(\bar{Q}_{ij}\bar{B}_i + \bar{V}_{ij}\Phi_i \Big) > 0$, $q_j - \bar{b}_j \sum_{i=1}^{n} \Big(\bar{P}_{ji}\bar{A}_j + \bar{U}_{ji}\Theta_j \Big) > 0$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$.

(H_7) *Impulsive operators $I_{ik}(x_i(t_k))$, $J_{jk}(y_j(t_k))$ satisfy*

$$I_{ik}(x_i(t_k)) = -\varpi_{ik} x_i(t_k), \quad 1 - \frac{a_i}{\bar{a}_i} \leq \varpi_{ik} \leq 1 + \frac{a_i}{\bar{a}_i},$$

$$J_{jk}(y_j(t_k)) = -\varrho_{jk} y_j(t_k), \quad 1 - \frac{b_j}{\bar{b}_j} \leq \varrho_{jk} \leq 1 + \frac{b_j}{\bar{b}_j},$$

where $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, $k \in N$.

Then for every $\varepsilon \in \{\pm 1\}^{n+m}$, there exists a unique exponentially stable periodic solution in $\overline{B}_\varepsilon$ in and its region of attraction includes Δ_ε.

Proof: Let $\varepsilon \in \{\pm 1\}^{n+m}$, $\mathbf{x}(t) = (u(t), v(t))^T = \mathbf{x}(t; (u(\theta), v(\vartheta))^T)$ and $\mathbf{y}(t) = (\tilde{u}(t), \tilde{v}(t))^T = \mathbf{y}(t; (\tilde{u}(\theta), \tilde{v}(\vartheta))^T)$ are two solution of the system (2.2) with the initial functions

$$(u(\theta), v(\vartheta))^T, (\tilde{u}(\theta), \tilde{v}(\vartheta))^T \in \Delta_\varepsilon, \quad \theta \in [-\tau, 0], \vartheta \in [-\sigma, 0].$$

From Lemma 4.1, we get that $\mathbf{x}(t; (u(\theta), v(\vartheta))^T, \mathbf{y}(t; (\tilde{u}(\theta), \tilde{v}(\vartheta))^T) \in \Delta_\varepsilon$ for $\forall t > 0$. When $t > 0$, $t \neq t_k$, from (A_1), H_1-H_2 and (2.3), we can get

$$\begin{aligned}
& D^+|u_i(t) - \tilde{u}_i(t)| \\
= & \operatorname{sgn}(u_i(t) - \tilde{u}_i(t))(\dot{u}_i(t) - \dot{\tilde{u}}_i(t)) \\
= & \operatorname{sgn}(u_i(t) - \tilde{u}_i(t)) \Big\{ -\theta_i(t, u_i(t))(u_i(t) - \tilde{u}_i(t)) \\
& + \sum_{j=1}^{m} P_{ji}(t) \\
& \times \big[f_j(t, z_j^{-1}(v_j(t-\tau_{ji}(t)))) - f_j(t, z_j^{-1}(\tilde{v}_j(t-\tau_{ji}(t)))) \big] \\
& + \sum_{j=1}^{m} U_{ji}(t) \Big[\lambda_j \Big(\int_0^\infty X_{ji}(s) z_j^{-1}(v_j(t-s))\mathrm{d}s \Big) \\
& - \lambda_j \Big(\int_0^\infty X_{ji}(s) z_j^{-1}(\tilde{v}_j(t-s))\mathrm{d}s \Big) \Big] \Big\} \\
\leq & -p_i|u_i(t) - \tilde{u}_i(t)| \\
& + \sum_{j=1}^{m} \bar{b}_j \Big[\bar{P}_{ji}\bar{A}_j |v_j(t-\tau_{ji}(t)) - \tilde{v}_j(t-\tau_{ji}(t))|
\end{aligned}$$

$$+\bar{U}_{ji}\Theta_j\int_0^{\infty}\Big(X_{ji}(s)|v_j(t-s))-\tilde{v}_j(t-s)|\Big)\mathrm{d}s\Big],$$

where $i=1,2,\ldots,n,\ j=1,2,\ldots,m.$

Similarly,

$$\begin{aligned}&D^+|v_j(t)-\tilde{v}_j(t)|\\&=\mathrm{sgn}(v_j(t)-\tilde{v}_j(t))(\dot{v}_j(t)-\dot{\tilde{v}}_j(t))\\&\le -q_j|v_j(t)-\tilde{v}_j(t)|\\&\quad+\sum_{i=1}^{n}\bar{a}_i\Big[\bar{Q}_{ij}\bar{B}_i|u_i(t-\sigma_{ij}(t))-\tilde{u}_i(t-\sigma_{ij}(t))|\\&\quad+\bar{V}_{ij}\Phi_i\int_0^{\infty}\Big(Y_{ij}(s)|u_i(t-s))-\tilde{u}_i(t-s)|\Big)\mathrm{d}s\Big],\end{aligned}$$

where $i=1,2,\ldots,n,\ j=1,2,\ldots,m.$

Let

$$\begin{cases}\mathcal{F}_i(x)=p_i-x\\\qquad-\bar{a}_i\sum\limits_{j=1}^{m}\Big[\bar{Q}_{ij}\bar{B}_ie^{x\sigma_{ij}}+\bar{V}_{ij}\Phi_i\int_0^{\infty}Y_{ij}(s)e^{xs}\mathrm{d}s\Big],\\\mathcal{G}_j(y)=q_j-y\\\qquad-\bar{b}_j\sum\limits_{i=1}^{n}\Big[\bar{P}_{ji}\bar{A}_je^{y\tau_{ji}}+\bar{U}_{ji}\Theta_j\int_0^{\infty}X_{ji}(s)e^{ys}\mathrm{d}s\Big],\end{cases}$$

where $x,y\in[0,+\infty),\ i=1,2,\ldots,n,\ j=1,2,\ldots,m.$ Together (H_6), it implies that

$$\begin{cases}\mathcal{F}_i(0)=p_i-\bar{a}_i\sum\limits_{j=1}^{m}\Big(\bar{Q}_{ij}\bar{B}_i+\bar{V}_{ij}\Phi_i\Big)>0,\\\mathcal{G}_j(0)=q_j-\bar{b}_j\sum\limits_{i=1}^{n}\Big(\bar{P}_{ji}\bar{A}_j+\bar{U}_{ji}\Theta_j\Big)>0,\end{cases}$$

where $i=1,2,\ldots,n,\ j=1,2,\ldots,m.$

It is clear that $\mathcal{F}_i,\mathcal{G}_j$ are continuous and strictly decreasing on $[0,+\infty)$ and $\mathcal{F}_i\to-\infty,\mathcal{G}_j\to-\infty$, as $x\to+\infty,y\to+\infty$, there exist x_0,y_0 such that $\mathcal{F}_i(x_0)=0$, $\mathcal{G}_j(y_0)=0$. Thus, we choose $0<\mu_0<\min\{\frac{x_0}{2},\frac{y_0}{2}\}$, then

$$\begin{cases}\mathcal{F}_i(\mu_0)=p_i-\mu_0-\bar{a}_i\sum\limits_{j=1}^{m}\Big[\bar{Q}_{ij}\bar{B}_ie^{\mu_0\sigma_{ij}}\\\qquad+\bar{V}_{ij}\Phi_i\int_0^{\infty}Y_{ij}(s)e^{\mu_0 s}\mathrm{d}s\Big]>0,\\\mathcal{G}_j(\mu_0)=q_j-\mu_0-\bar{b}_j\sum\limits_{i=1}^{n}\Big[\bar{P}_{ji}\bar{A}_je^{\mu_0\tau_{ji}}\\\qquad+\bar{U}_{ji}\Theta_j\int_0^{\infty}X_{ji}(s)e^{\mu_0 s}\mathrm{d}s\Big]>0,\end{cases}$$

where $i=1,2,\ldots,n,\ j=1,2,\ldots,m.$

Denote $C_i(t)=e^{\mu_0 t}|u_i(t)-\tilde{u}_i(t)|$, $\mathcal{C}_j(t)=e^{\mu_0 t}|v_j(t)-\tilde{v}_j(t)|$ for $t>0$, $t\neq t_k$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$, we obtain

$$\begin{aligned}D^+C_i(t)&=\mu_0C_i(t)+e^{\mu_0 t}D^+|u_i(t)-\tilde{u}_i(t)|\\&\le(\mu_0-p_i)C_i(t)+\sum_{j=1}^{m}\bar{b}_j\\&\quad\times\Big[\bar{P}_{ji}\bar{A}_je^{\mu_0 t}|v_j(t-\tau_{ji}(t))-\tilde{v}_j(t-\tau_{ji}(t))|\\&\quad+\bar{U}_{ji}\Theta_j\\&\quad\times\int_0^{\infty}\Big(X_{ji}(s)e^{\mu_0 t}|v_j(t-s)-\tilde{v}_j(t-s)|\Big)\mathrm{d}s\Big]\\&\le(\mu_0-p_i)C_i(t)\\&\quad+\sum_{j=1}^{m}\bar{b}_j\Big[\bar{P}_{ji}\bar{A}_je^{\mu_0\tau_{ji}}\mathcal{C}_j(t-\tau_{ji}(t))\\&\quad+\bar{U}_{ji}\Theta_j\int_0^{\infty}\Big(X_{ji}(s)e^{\mu_0 s}\mathcal{C}_j(t-s)\Big)\mathrm{d}s\Big]\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}D^+\mathcal{C}_j(t)&=\mu_0\mathcal{C}_j(t)+e^{\mu_0 t}D^+|v_j(t)-\tilde{v}_j(t)|\\&\le(\mu_0-q_j)\mathcal{C}_j(t)\\&\quad+\sum_{i=1}^{n}\bar{a}_i\Big[\bar{Q}_{ij}\bar{B}_ie^{\mu_0\sigma_{ij}}C_i(t-\sigma_{ij}(t))\\&\quad+\bar{V}_{ij}\Phi_i\int_0^{\infty}\Big(Y_{ij}(s)e^{\mu_0 s}C_i(t-s)\Big)\mathrm{d}s\Big],\end{aligned}\tag{4.2}$$

where $i=1,2,\ldots,n,\ j=1,2,\ldots,m.$

Consider the following Lyapunov function:

$$\begin{aligned}V(t)&=\sum_{i=1}^{n}\Big\{C_i(t)+\sum_{j=1}^{m}\bar{b}_j\\&\quad\times\Big[\bar{P}_{ji}\bar{A}_je^{\mu_0\tau_{ji}}\int_{t-\tau_{ji}(t)}^{t}\mathcal{C}_j(r)\mathrm{d}r\\&\quad+\bar{U}_{ji}\Theta_j\int_0^{\infty}\Big(X_{ji}(s)e^{\mu_0 s}\int_{t-s}^{t}\mathcal{C}_j(r)\mathrm{d}r\Big)\mathrm{d}s\Big]\Big\}\\&\quad+\sum_{j=1}^{m}\Big\{\mathcal{C}_j(t)+\sum_{i=1}^{n}\bar{a}_i\\&\quad\times\Big[\bar{Q}_{ij}\bar{B}_ie^{\mu_0\sigma_{ij}}\int_{t-\sigma_{ij}(t)}^{t}C_i(r)\mathrm{d}r\\&\quad+\bar{V}_{ij}\Phi_i\int_0^{\infty}\Big(Y_{ij}(s)e^{\mu_0 s}\int_{t-s}^{t}C_i(r)\mathrm{d}r\Big)\mathrm{d}s\Big]\Big\}\end{aligned}\tag{4.3}$$

and we note that $V(t)>0$ for $t>0$ and $V(0)$ is positive and finite. For $t>0,t\neq t_k$, calculating the derivatives of V along (4.1) and (4.2), we have

$$\begin{aligned}D^+V(t)&=\sum_{i=1}^{n}\Big\{D^+C_i(t)+\sum_{j=1}^{m}\bar{b}_j\\&\quad\times\Big[\bar{P}_{ji}\bar{A}_je^{\mu_0\tau_{ji}}\big(\mathcal{C}_j(t)-\mathcal{C}_j(t-\tau_{ji}(t))\big)\\&\quad+\bar{U}_{ji}\Theta_j\\&\quad\times\int_0^{\infty}\Big(X_{ji}(s)e^{\mu_0 s}\big(\mathcal{C}_j(t)-\mathcal{C}_j(t-s)\big)\Big)\mathrm{d}s\Big]\Big\}\\&\quad+\sum_{j=1}^{m}\Big\{D^+\mathcal{C}_j(t)+\sum_{i=1}^{n}\bar{a}_i\\&\quad\times\Big[\bar{Q}_{ij}\bar{B}_ie^{\mu_0\sigma_{ij}}\big(C_i(t)-C_i(t-\sigma_{ij}(t))\big)\\&\quad+\bar{V}_{ij}\Phi_i\\&\quad\times\int_0^{\infty}\Big(Y_{ij}(s)e^{\mu_0 s}\big(C_i(t)-C_i(t-s)\big)\Big)\mathrm{d}s\Big]\Big\}\end{aligned}$$

$$\le \sum_{i=1}^{n}\Big[(\mu_0-p_i)C_i(t)$$
$$+\sum_{j=1}^{m}\bar{b}_j\Big(\bar{P}_{ji}\bar{A}_j e^{\mu_0\tau_{ji}}\mathcal{C}_j(t)$$
$$+\bar{U}_{ji}\Theta_j\int_0^{\infty}X_{ji}(s)e^{\mu_0 s}\mathcal{C}_j(t)\mathrm{d}s\Big)\Big]$$
$$+\sum_{j=1}^{m}\Big[(\mu_0-q_j)\mathcal{C}_j(t)$$
$$+\sum_{i=1}^{n}\bar{a}_i\Big(\bar{Q}_{ij}\bar{B}_i e^{\mu_0\sigma_{ij}}C_i(t)$$
$$+\bar{V}_{ij}\Phi_i\int_0^{\infty}Y_{ij}(s)e^{\mu_0 s}C_i(t)\mathrm{d}s\Big)\Big]$$
$$=-\sum_{i=1}^{n}\mathcal{F}(\mu_0)C_i(t)-\sum_{j=1}^{m}\mathcal{G}(\mu_0)\mathcal{C}_j(t)<0. \quad (4.4)$$

Therefore, form (4.4), the function V is strictly decreasing on every interval (t_k, t_{k+1}). Hence $V(t) < V(t_k^+)$ for any $t\in(t_k,t_{k+1}]$.

Moreover, $h_i(u)$ and $z_j(u)$ are strictly monotone increasing about $u(u\in R)$, we obtain

$$|h_i(x)-h_i(y)|=\frac{1}{a_i(\xi)}|(x-y)|\le\frac{1}{\underline{a}_i}|x-y|,$$
$$|z_j(x)-z_j(y)|=\frac{1}{b_j(\zeta)}|(x-y)|\le\frac{1}{\underline{b}_j}|x-y|, \quad (4.5)$$

where $\forall x,y\in R$.

Therefore, when $t=t_k$, from (4.5) and (H_7) we have

$$\begin{cases} x_i(t_k^+)-\tilde{x}_i(t_k^+)=(1-\varpi_{ik})(x_i(t_k)-\tilde{x}_i(t_k)),\\ y_j(t_k^+)-\tilde{y}_j(t_k^+)=(1-\varpi_{ik})(y_j(t_k)-\tilde{y}_j(t_k)),\end{cases}$$

where $i=1,2,\ldots,n,,\ j=1,2,\ldots,m$. That is

$$\begin{cases} |u_i(t_k^+)-\tilde{u}_i(t_k^+)|=|h_i(x_i(t_k^+))-h_i(\tilde{x}_i(t_k^+))|\\ \qquad\qquad\le\frac{\bar{a}_i}{\underline{a}_i}|1-\varpi_{ik}||u_i(t_k)-\tilde{u}_i(t_k)|,\\ |v_j(t_k^+)-\tilde{v}_j(t_k^+)|=|z_j(y_i(t_k^+))-z_j(\tilde{y}_j(t_k^+))|\\ \qquad\qquad\le\frac{\bar{b}_j}{\underline{b}_j}|1-\varrho_{jk}||v_j(t_k)-\tilde{v}_j(t_k)|,\end{cases}$$

where $i=1,2,\ldots,n,,\ j=1,2,\ldots,m$.

Therefore, we have

$$\begin{cases} C_i(t_k^+)=e^{-\mu_0 t_k^+}|u_i(t_k^+)-\tilde{u}_i(t_k^+)|\\ \qquad\le e^{-\mu_0 t_k}\frac{\bar{a}_i}{\underline{a}_i}|1-\varpi_{ik}||u_i(t_k)-\tilde{u}_i(t_k)|\\ \qquad\le C_i(t_k),\\ \mathcal{C}_j(t_k^+)=e^{-\mu_0 t_k^+}|v_j(t_k^+)-\tilde{v}_j(t_k^+)|\\ \qquad\le e^{-\mu_0 t_k}\frac{\bar{b}_i}{\underline{b}_i}|1-\varrho_{jk}||v_j(t_k)-\tilde{v}_j(t_k)|\\ \qquad\le \mathcal{C}_j(t_k),\end{cases}$$

where $i=1,2,\ldots,n,,\ j=1,2,\ldots,m$.

Also,

$$V(t_k^+)=\sum_{i=1}^{n}\Big\{C_i(t_k^+)$$
$$+\sum_{j=1}^{m}\bar{b}_j\Big[\bar{P}_{ji}\bar{A}_j e^{\mu_0\tau_{ji}}\int_{t_k^+-\tau_{ji}(t_k^+)}^{t_k^+}\mathcal{C}_j(r)\mathrm{d}r$$
$$+\bar{U}_{ji}\Theta_j\int_0^{\infty}\Big(X_{ji}(s)e^{\mu_0 s}\int_{t_k^+-s}^{t_k^+}\mathcal{C}_j(r)\mathrm{d}r\Big)\mathrm{d}s\Big]\Big\}$$
$$+\sum_{j=1}^{m}\Big\{\mathcal{C}_j(t_k^+)$$
$$+\sum_{i=1}^{n}\bar{a}_i\Big[\bar{Q}_{ij}\bar{B}_i e^{\mu_0\sigma_{ij}}\int_{t_k^+-\sigma_{ij}(t_k^+)}^{t_k^+}C_i(r)\mathrm{d}r$$
$$+\bar{V}_{ij}\Phi_i\int_0^{\infty}\Big(Y_{ij}(s)e^{\mu_0 s}\int_{t_k^+-s}^{t_k^+}C_i(r)\mathrm{d}r\Big)\mathrm{d}s\Big]\Big\}$$
$$\le V(t_k),\ \ k\in\mathbb{Z}^+. \quad (4.6)$$

Hence, form (4.4) and (4.6), the function V is strictly decreasing for $t\ge 0$, which demonstrates that $V(t)\le V(0)$ for $t\ge 0$. By (4.3), we have

$$\sum_{i=1}^{n}C_i(t)+\sum_{j=1}^{m}\mathcal{C}_j(t)\le V(t)\le V(0)$$
$$\le\sum_{i=1}^{n}\Big[1+\bar{a}_i\sum_{j=1}^{m}\Big(\bar{Q}_{ij}\bar{B}_i\frac{(e^{\mu_0\sigma_{ij}}-1)}{\mu_0}$$
$$+\frac{\bar{V}_{ij}\Phi_i}{\mu_0}(\int_0^{\infty}Y_{ij}(s)e^{\mu_0 s}\mathrm{d}s-1)\Big)\Big]\max_{\theta\in[-\tau,0]}|u(\theta)-\tilde{u}(\theta)|$$
$$+\sum_{j=1}^{m}\Big[1+\bar{b}_j\sum_{i=1}^{n}\Big(\bar{P}_{ji}\bar{A}_j\frac{(e^{\mu_0\tau_{ji}}-1)}{\mu_0}$$
$$+\frac{\bar{U}_{ji}\Theta_j}{\mu_0}(\int_0^{\infty}X_{ji}(s)e^{\mu_0 s}\mathrm{d}s-1)\Big)\Big]\max_{\vartheta\in[-\sigma,0]}|v(\vartheta)-\tilde{v}(\vartheta)|$$
$$\le\frac{1}{\underline{a}_i}\sum_{i=1}^{n}\Big[1+\frac{\bar{a}_i}{\mu_0}\sum_{j=1}^{m}\Big(\bar{Q}_{ij}\bar{B}_i e^{\mu_0\sigma_{ij}}$$
$$+\bar{V}_{ij}\Phi_i\int_0^{\infty}Y_{ij}(s)e^{\mu_0 s}\mathrm{d}s\Big)\Big]\|\varphi_i-\tilde{\varphi}_i\|_{\infty}$$
$$+\frac{1}{\underline{b}_j}\sum_{j=1}^{m}\Big[1+\frac{\bar{b}_j}{\mu_0}\sum_{i=1}^{n}\Big(\bar{P}_{ji}\bar{A}_j e^{\mu_0\tau_{ji}}$$
$$+\bar{U}_{ji}\Theta_j\int_0^{\infty}X_{ji}(s)e^{\mu_0 s}\mathrm{d}s\Big)\Big]\|\psi_j-\tilde{\psi}_j(\psi)\|_{\infty}.$$

In view of the definiens of $C_i(t)$, $\mathcal{C}_j(t)$ and the inequality above, we get

$$\sum_{i=1}^{n}|u_i(t)-\tilde{u}_i(t)|+\sum_{j=1}^{m}|v_j(t)-\tilde{v}_j(t)|$$
$$\le\mathcal{A}e^{-\mu_0 t}\Big(\sum_{i=1}^{n}\|\varphi_i-\tilde{\varphi}_i\|_{\infty}+\sum_{j=1}^{m}\|\psi_j-\tilde{\psi}(\psi)_j\|_{\infty}\Big),$$

where

$$\mathcal{A}=\max\Big\{\frac{1}{\underline{a}_i}\sum_{i=1}^{n}\Big[1+\frac{\bar{a}_i}{\mu_0}\sum_{j=1}^{m}\Big(\bar{Q}_{ij}\bar{B}_i e^{\mu_0\sigma_{ij}}$$
$$+\bar{V}_{ij}\Phi_i\int_0^{\infty}Y_{ij}(s)e^{\mu_0 s}\mathrm{d}s\Big)\Big],$$

$$\frac{1}{\underline{b}_j}\sum_{j=1}^{m}\Bigg[1+\frac{\bar{b}_j}{\mu_0}\sum_{i=1}^{n}\Bigg(\bar{P}_{ji}\bar{A}_j e^{\mu_0\tau_{ji}} + \bar{U}_{ji}\Theta_j\int_0^\infty X_{ji}(s)e^{\mu_0 s}\mathrm{d}s\Bigg)\Bigg]\Bigg\} > 0.$$

Finally, it follows that

$$\|\mathbf{x}(t)-\mathbf{y}(t)\| = \sup_{t\in[0,\omega]}\Bigg(\sum_{i=1}^{n}|u_i-\tilde{u}_i| + \sum_{j=1}^{m}|v_j-\tilde{v}_j|\Bigg) \le \mathcal{A}e^{-\mu_0 t}\Bigg(\sum_{i=1}^{n}\|\varphi_i-\tilde{\varphi}_i\|_\infty + \sum_{j=1}^{m}\|\psi_j-\tilde{\psi}(\psi)_j\|_\infty\Bigg).$$

Thanks to (H_4)-(H_7), it easily follows that a unique periodic solution $\mathbf{x}^*_\varepsilon(t)\le\Delta_\varepsilon$, for any $t\in\mathbb{R}$ are exists, which is globally exponentially stable and its region of attraction includes Δ_ε. This completes the proof. ■

Conclusion 4.1. From Theorem 4, it is easy to obtain that the existence of unique exponentially stable periodic solution for system (1.1) in every $\overline{B}_\varepsilon$.

V. An example

Giving the following Cohen-Grossberg BAM neural networks system with mixed delays and impulses

$$\begin{cases} x_i'(t) = -a_i(x_i(t))\Big[\alpha_i(t,x_i(t)) \\ \quad -P_{1i}(t)f_1(t,y_1(t-\tau_{1i}(t))) \\ \quad -P_{2i}(t)f_2(t,y_2(t-\tau_{2i}(t))) \\ \quad -U_{1i}(t)\lambda_1\big(\int_0^\infty X_{1i}(s)y_1(t-s)\mathrm{d}s\big) \\ \quad -U_{2i}(t)\lambda_2\big(\int_0^\infty X_{2i}(s)y_2(t-s)\mathrm{d}s\big) \\ \quad +0.5\sin t\Big], \\ y_j'(t) = -b_j(y_j(t))\Big[\beta_j(t,y_j(t)) \\ \quad -Q_{1j}(t)g_1(t,x_1(t-\sigma_{1j}(t))) \\ \quad -Q_{2j}(t)g_2(t,x_2(t-\sigma_{2j}(t))) \\ \quad -V_{1j}(t)\mu_1\big(\int_0^\infty Y_{1j}(s)x_1(t-s)\mathrm{d}s\big) \\ \quad -V_{2j}(t)\mu_2\big(\int_0^\infty Y_{2j}(s)x_2(t-s)\mathrm{d}s\big) \\ \quad +0.5\cos t\Big], \\ \qquad t>0, t\ne t_k=2k, \\ \Delta x_i(t_k) = -0.1x_i(t_k), \quad t=t_k=2k, \\ \Delta y_j(t_k) = -0.2y_j(t_k), \quad t=t_k=2k, \end{cases} \tag{5.1}$$

where $i=1,2,\ j=1,2,\ k\in N$.

Let

$$a_1(u)=2+\cos u,\quad a_2(u)=2-\cos u,\quad b_1(u)=1.5+\sin u$$

$$b_2(u)=3-\sin u,\quad \alpha_1(t,u)=\alpha_2(t,u)=(8-\sin t)+u,$$

$$\beta_1(t,v)=\beta_2(t,v)=(16-\cos t)+v.$$

The activation functions

$$f_1(t,y_1(t-e^{2\sin t})) = 0.05\sin\frac{\pi t}{2}y_1(t-e^{2\sin t}),$$

$$f_2(t,y_2(t-e^{2\sin t})) = 0.01\sin\frac{\pi t}{2}y_2(t-e^{2\sin t}),$$

$$g_1(t,x_1(t-e^{\cos t})) = 0.1\cos\frac{\pi t}{2}\sin(x_1(t-e^{\cos t})),$$

$$g_2(t,x_2(t-e^{\cos t})) = 0.1\cos\frac{\pi t}{2}\sin(x_2(t-e^{\cos t}),$$

$$\tau_{ij}(t)=e^{2\sin t},\quad \sigma_{ji}(t)=e^{\cos t},\quad \lambda_j(u)=\tanh(4u),$$

$$\mu_i(u)=\tanh(5u)\tanh(10u^2-1).$$

$$\begin{bmatrix} P_{11}(t) & P_{12}(t) & P_{21}(t) & P_{22}(t) \\ Q_{11}(t) & Q_{12}(t) & Q_{21}(t) & Q_{22}(t) \\ U_{11}(t) & U_{12}(t) & U_{21}(t) & U_{22}(t) \\ V_{11}(t) & V_{12}(t) & V_{21}(t) & V_{22}(t) \\ X_{11}(s) & X_{12}(s) & X_{21}(s) & X_{22}(s) \\ Y_{11}(s) & Y_{12}(s) & Y_{21}(s) & Y_{22}(s) \end{bmatrix} = \begin{bmatrix} 6-\cos t & 2-\sin t & 2-\cos t & 6+\sin t \\ 6-\sin t & 2-\cos t & 1-0.5\cos t & 5+\sin t \\ 5+\cos 2t & 1.2-\cos t & 0 & 5 \\ 2-\cos t & 0 & 1 & 4+\sin t \\ e^{-s} & 2e^{-2s} & 3e^{-3s} & e^{-s} \\ 2e^{-2s} & e^{-s} & 3e^{-3s} & e^{-s} \end{bmatrix}.$$

Through simple computation, we get

$$\begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} & \bar{P}_{21} & \bar{P}_{22} \\ \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{21} & \bar{Q}_{22} \\ \bar{U}_{11} & \bar{U}_{12} & \bar{U}_{21} & \bar{U}_{22} \\ \bar{V}_{11} & \bar{V}_{12} & \bar{V}_{21} & \bar{V}_{22} \end{bmatrix} = \begin{bmatrix} 7 & 3 & 3 & 7 \\ 7 & 3 & 1.5 & 6 \\ 6 & 2.2 & 0 & 5 \\ 2 & 0 & 1 & 5 \end{bmatrix},$$

$$\begin{bmatrix} \underline{P}_{11} & \underline{P}_{12} & \underline{P}_{21} & \underline{P}_{22} \\ \underline{Q}_{11} & \underline{Q}_{12} & \underline{Q}_{21} & \underline{Q}_{22} \\ \underline{U}_{11} & \underline{U}_{12} & \underline{U}_{21} & \underline{U}_{22} \\ \underline{V}_{11} & \underline{V}_{12} & \underline{V}_{21} & \underline{V}_{22} \end{bmatrix} = \begin{bmatrix} 5 & 1 & 1 & 5 \\ 5 & 1 & 0.5 & 4 \\ 4 & 0.2 & 0 & 5 \\ 1 & 0 & 1 & 3 \end{bmatrix},$$

$$\begin{bmatrix} \bar{\sigma}_{ij} & \bar{a}_i & \bar{b}_j & \bar{\alpha}_i & \bar{\beta}_j & \tilde{p}_i & \tilde{q}_j \\ \bar{\tau}_{ji} & \underline{a}_i & \underline{b}_j & \underline{\alpha}_i & \underline{\beta}_j & p_i & q_j \end{bmatrix} = \begin{bmatrix} e^2 & 3 & 2.5 & 1 & 1 & 3 & 2.5 \\ e & 2 & 0.5 & 1 & 1 & 2 & 0.5 \end{bmatrix},$$

$$\begin{bmatrix} \gamma_j & \Pi_i & \bar{A}_j \\ r_j & \pi_i & \bar{B}_i \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 & 0.05 \\ 0.1 & 0.1 & 0.01 \end{bmatrix}.$$

It is easy to illustrate that hypothesis $(A_1)-(A_7)$, (H_2), (H_4) and (H_5) hold. The activation function λ_j and μ_i satisfies hypothesis (H_1) and (H_3) with $\Theta_j = 0.0008$, $r\simeq 0.9001$, $\Phi_i\simeq 0.0009$, $\pi\simeq 0.9999$. Therefore, there exist at least 4 periodic solutions of system (5.1).

Moreover, we calculate $p_i - \bar{a}_i\sum_{j=1}^{2}\Big(\bar{Q}_{ij}\bar{B}_i+\bar{V}_{ij}\Phi_i\Big)\simeq 0.499>0$, $q_j-\bar{b}_j\sum_{i=1}^{2}\Big(\bar{P}_{ji}\bar{A}_j+\bar{U}_{ji}\Theta_j\Big)\simeq 0.24>0$. Let $\varpi_{ik}=\varrho_{jk}=1$. Therefore, $(H_6),(H_7)$ are satisfied, and from Theorem 4, system (5.1) has 4 exponentially stable periodic solutions.

VI. Conclusions and Future Works

This paper considers a class of impulsive Cohen-Grossberg BAM neural networks with mixed delays. First of all, the differential system is changed into integral system by using the derivative theorem for inverse function and the constant variation method. Then, under some suitable hypotheses and the Leray-Schauder theorem, at least 2^{n+m} periodic solutions for impulsive Cohen-Grossberg BAM neural networks with mixed delays are obtained. By some suitable Lyapunov functions, this article investigates a unique ω-periodic solution of system (2.2) and demonstrates that all solutions of system (2.2) converge exponentially to its unique ω-periodic solution. An example is given to illustrate the validity of the main conclusions in this paper.

In the future, the following aspects can be explored further:

(1) The fractional order models could be considered, see [36], [37].
(2) Some other dynamic behaviors could be learned.
(3) The dynamic behaviours for discrete Cohen-Grossberg neural networks could be investigated.

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