Impact of Awareness on the Dynamics of Pest Control in Coconut Trees - A Mathematical Model

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Abstract—**Rugose spiraling whitefly (RSW) is one of the major pest which affects the coconut trees. A mathematical model for understanding the dynamics of this system along with agricultural awareness is analysed using saturated response functions (Holling type-II). The stability analysis of the biologically feasible equilibrium points is observed by applying Routh-Hurwitz criterion and constructing suitable Lyapunov function. We perform sensitivity analysis to determine the effect of parameters on the system. Also, an optimal control problem is proposed by introducing the control parameters to figure out the feasibility of better disease control with cost-effective control strategies. Our results evoke that public awareness on the application of various control measures can significantly reduce the contact rate thereby controlling the disease spread.**

Index Terms—**mathematical model; holling type-II; stability; sensitivity analysis; optimal control**

I. Introduction

THE coconut tree plantation occupies an area of about 12 million hectares spread over 93 countries in the world. India holds a major position contributing to the global production of coconut products. Coconut and its by-products serve multiple benefits to human life ranging from health care, skin care to bio-farming methods and construction materials. About half of our annual coconut production is consumed internally. Tamil Nadu, Assam, Kerala, West Bengal, Orissa and Karnataka are the top states in coconut tree plantation. There have been numerous challenges faced by farmers in growing up the coconut trees. One such major threat is RSW, an invasive species of whitefly which affects coconut trees by hindering the tree's growth as well as affecting the coconut yield. This particular pest is first reported in 2004 in Belize, Central America that too in coconut trees [1]. Now in India, this RSW pest has been a major catastrophe affecting coconut plantations. In Pollachi region of Tamil Nadu, this pest invasion was first spotted on August 2016 that too in coconut trees [2], [3]. This pest was quickly discovered on a variety of other plants, including mango, guava, sapota, custard apple, banana, and a variety of other economically significant decorative species. The RSW invasion will place the coconut sector in India at risk by lowering overall production rates, lowering the quality of flesh produced, and increasing production costs due to pest management [4]. The host plant is affected by this whitefly because it feeds on the leaves, removing both nutrients and water. Furthermore, it causes sooty mold to form on the leaf surface, potentially reducing photosynthesis and reducing yield and growth [5]. Population dynamics of this whitefly and its host range has been studied [6], [7].

The dynamics of population interaction can be studied with the help of mathematical models. Particularly these models provide us the biological interpretation of system by revealing the interactions as well as the impact that exist between the parameters and variables. The concept as well as the formulation of mathematical models in epidemiology has been discussed [8]. Mathematical modeling has been extensively developed to study the dynamics of epidemic diseases and to propose control strategies for the disease [9], [10]. The plant as well as the vector population dynamics has been extensively studied for African cassava mosaic virus disease (ACMD) and this study is conducted within a bounded locality [11]. The mathematical study of prey-predator models incorporating nanoparticle has been carried out [12]. The population dynamics of rugose spiraling whitefly affecting coconut trees has been analysed [13]. It resulted that, if we decrease the contact rate, we can have a better control over the infection rate. This decrease in contact rate is achieved only with the help of proper control techniques. Adopting awareness programs aimed at educating farmers leads to improved overall development, not only for cultivators but also for farmers. The importance of agricultural awareness for pest control has been studied in [14]. On the dynamics of mosaic disease, the impact of farming awareness-based interventions such as roguing, insecticide spraying, and optimal control has been studied [15]–[18]. The study in these literatures reveals that awareness campaigns can eventually decrease or even eradicate the spread of mosaic disease. It is also suggested that, the fading of awareness among farmers and delay in implementation could be avoided if the advertisement awareness campaigns are performed within short time intervals. The impact of awareness created by media campaigns on vaccination coverage in a variable population has been analysed in [19]. The analysis for the spread of infectious diseases was carried out by using stability theory of differential equations [20]. The ability of a disease to infect a population is the primary concern in any infectious disease. The basic reproduction number is a threshold parameter in epidemiological models that can determine whether a disease has the potential to infect the population or not. The models examine the reproduction number using the next generation matrix [21].

Sensitivity analysis is termed to the connection that exists between the solution that we observe and the parameters that we have considered for the mathematical model. The results define the behaviour of selected variable in the specified parameter's direction at time t [22], [23]. Since optimal control theory gives an optimal way for control procedures while minimizing adverse effects, it is becoming increasingly essential in disease control. The model dynamics

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has been studied using MATLAB software by applying a fixed control [24]–[27]. The stability study for dynamical systems have extensively discussed [28]. Motivated by the model [18], [19], we analyse the dyamics of the disease by extending the model in [13] along with awareness programs. Our research work confines specific to the Pollachi region of Tamil Nadu. Hence the parameter values were taken from the same zone. Local and global stability has been analysed at the equilibrium points. The next generation matrix is used to calculate the reproduction number. In sensitivity analysis, we study the parameter which affects the system. Further, an optimal control problem is formulated and the results are obtained using Pontryagin minimum principle.

II. Mathematical Formulation

In [13], G. Suganya and R. Senthamarai studied the dynamics of rugose spiraling whitefly affecting coconut trees. Here, we have extended the above model by incorporating awareness programs educated to farmers through media activities. To study the effect of whitefly on coconut trees, the total available tree population is subdivided as healthy trees, which is indicated by C and infected trees, which is indicated by I. P indicates the whitefly population and A indicates the number of awareness programs conducted during time t. The mathematical model is given as follows:

$$\frac{dC}{dt} = RC\left(1 - \frac{C+I}{k}\right) - \frac{\mu CP}{1+\eta P}, \tag{1}$$

$$\frac{dI}{dt} = \frac{\mu CP}{1+\eta P} - \phi I - hAI, \tag{2}$$

$$\frac{dP}{dt} = \omega I - \rho P - \xi AP, \tag{3}$$

$$\frac{dA}{dt} = r_0 + \alpha I - \beta A. \tag{4}$$

with the initial conditions as

$$C(0) = C_0, \quad I(0) = I_0, \quad P(0) = P_0, \quad A(0) = A_0. \tag{5}$$

We assume $C_0 > 0,\ I_0 > 0,\ P_0 > 0$, and $A_0 \geq 0$. Let μ be the contact rate between whitefly and healthy tree, ϕ be the mortality rate of infected trees. The healthy trees follow logistic growth with tree density k. Let R indicate the rate of replantation whereas the whitefly birth rate is denoted by ω and mortality rate is ρ. The contact rate is represented by a Holling type II functional response, which takes into account the whitefly's crowding effect and ensures the contact rate μ's boundedness for any positive constant. Let the awareness level increase at a rate r_0 through media activities. The rate of local awareness α, is proportional to the number of infected plants and β denotes fading of awareness due to falling of importance.

III. Mathematical Analysis of the model

A. Positivity

From system $(1)-(4)$ we see that

$$\left.\frac{dC}{dt}\right|_{C=0} = 0,\ \left.\frac{dI}{dt}\right|_{I=0} = \frac{\mu CP}{1+\eta P} \geq 0,\ \left.\frac{dP}{dt}\right|_{P=0} = \omega I \geq 0,$$

$$\left.\frac{dA}{dt}\right|_{A=0} = r_0 + \alpha I \geq 0.$$

Hence the solutions are positive if the system possess positive initial condition.

B. Boundedness

All solutions of system which originate in $\mathbb{R}^4_+$ are uniformly bounded.

$$\frac{dC}{dt}+\frac{dI}{dt} = RC\left(1-\frac{C+I}{k}\right) - \frac{\mu CP}{1+\eta P} + \frac{\mu CP}{1+\eta P} - \phi I - hAI,$$

$$\frac{dN}{dt} + \zeta N \leq -RC^2\frac{N}{k} + RC + \zeta C + \zeta I - \phi I - hAI,$$

$$\frac{dN}{dt} + \zeta N \leq l,$$

$$0 \leq N(t) \leq e^{-\zeta t}\left(N(0) - \frac{l}{\zeta}\right) + \frac{l}{\zeta}.$$

where $l = \frac{(R+\zeta)^2 k}{4R}$. As $t \to \infty$, $N(t) \to \frac{l}{\zeta}$ since $\sup_{t\to\infty} N(t) = \frac{l}{\zeta}$. Similarly we find that

$$\sup_{t\to\infty} P(t) = \frac{\omega k}{\rho}.$$

and

$$\sup_{t\to\infty} A(t) = \frac{r_0 + \alpha k}{\beta} = A_m(say).$$

Hence the system is bounded. Thus, the positive invariant set of the system is

$$\Omega = \left\{(C,I,P,A) \in \mathbb{R}^4_+ \mid 0 \leq C, I \leq \frac{l}{\zeta}, P \leq \frac{\omega k}{\rho}, A \leq \frac{r_0+\alpha k}{\beta}\right\}.$$

C. Existence of Equilibria

The system $(1)-(4)$ has the basic reproduction number

$$R_0 = \frac{\mu k \omega \beta^2}{(\beta\phi + hr_0)(\rho\beta + \xi r_0)}.$$

Further, the equilibrium points for the system are:

- Tree-Pest free equilibrium: $E_0 = (0,0,0,\frac{r_0}{\beta})$
- Pest free equilibrium: $E_1 = \left(k,0,0,\frac{r_0}{\beta}\right)$
- Coexistence equilibrium: $E_* = (C^*, I^*, P^*, A^*)$

$$where\ C^* = (\phi + hA^*)\frac{\alpha(\rho+\xi A^*) + \eta\omega(\beta A^* - r_0)}{\mu\alpha\omega},$$

$$I^* = \frac{\beta A^* - r_0}{\alpha},\ P^* = \frac{\omega(\beta A^* - r_0)}{\alpha(\rho + \xi A^*)},$$

and A^* is the positive root of the cubic equation

$$\begin{aligned}&(Rh\rho\eta\xi\alpha + Rh\eta^2\beta^2)A^3 + (R\phi\alpha\eta\beta\xi + R\phi\eta^2\omega\beta^2\\ &+Rh\alpha^2\xi\rho + Rh\rho\beta\eta\alpha - Rh\rho\eta r_0\alpha\xi + R\eta\mu\omega\beta^2\\ &+Rh\eta\beta\omega\alpha)A^2 + (R\phi\rho\eta\beta\alpha + R\phi\alpha^2\xi + R\phi\eta\omega\beta\alpha\\ &+Rh\rho\alpha^2 + Rh\eta^2\omega r_0^2 + \beta R\mu\alpha\omega - Rk\mu\omega\beta\eta\\ &-R\phi\alpha^2\xi\eta r_0 - 2R\phi\eta^2\omega\beta r_0 - Rh\rho r_0\eta\alpha - Rh\eta\omega r_0\alpha\\ &-\beta R\mu\omega\eta r_0 - Rr_0\omega\mu\eta\beta)A + (Rk\mu\alpha\omega\eta r_0 + R\phi\rho\alpha^2\\ &-R\phi\rho\eta r_0\alpha - Rk\mu\alpha^2\omega - R\phi\eta\omega r_0\alpha + R\phi\eta^2\omega r_0^2\\ &-R\phi\mu\alpha\omega + Rr_0^2\eta\mu\omega + kr_0\mu^2\alpha\omega) = 0.\end{aligned} \tag{6}$$

D. Stability Analysis

Here, we've discussed the stability results of the system $(1)-(4)$. The local stability is implemented using Routh Hurwitz (R-H) criterion and global stability by constructing suitable Lyapunov function.

Lemma: The tree-pest free equilibrium E_0 is always unstable.

Theorem 1: The pest free equilibrium E_1 is locally asymptotically stable (LAS) for $R_0 < 1$ and unstable otherwise.
Proof: The Jacobian matrix evaluated at E_1 is given by

$$J(E_1) = \begin{bmatrix} -R & -R & -\mu k & 0 \\ 0 & -\phi - h\frac{r_0}{\beta} & \mu k & 0 \\ 0 & \omega & -\rho - \xi\frac{r_0}{\beta} & 0 \\ 0 & \alpha & 0 & -\beta \end{bmatrix}.$$

The characteristic equation of the matrix is

$$(-R-\lambda)\left[(-\phi - h\frac{r_0}{\beta} - \lambda)(-\rho - \xi\frac{r_0}{\beta} - \lambda)(-\beta-\lambda) + \mu k(\omega(\beta+\lambda))\right] = 0.$$

which implies

$$\begin{aligned} &\lambda^4 + \lambda^3[R + h\frac{r_0}{\beta} + \phi + \rho + \xi r_0 + \beta] + \lambda^2[R\phi \\ &+ Rh\frac{r_0}{\beta} + R\rho + R\xi r_0 + R\beta + \rho\phi + \xi\phi r_0 + \phi\beta \\ &+ h\rho\frac{r_0}{\beta} + h\xi\frac{r_0^2}{\beta} + hr_0 + \rho\beta + \xi r_0 - \mu k\omega] + \lambda[R\rho\phi \\ &+ R\xi r_0\phi + R\phi\beta + Rh\frac{r_0\rho}{\beta} + Rh\frac{r_0^2\xi}{\beta} + Rhr_0 \\ &+ R\rho\beta + R\xi r_0 - R\mu k\omega + \rho\beta\phi + \phi\xi r_0 \\ &+ hr_0\rho + h\xi\frac{r_0^2}{\beta} - \mu k\omega\beta] + [R\rho\beta\phi + \phi\xi r_0 R \\ &+ Rhr_0\rho + Rh\xi\frac{r_0^2}{\beta} - R\mu k\omega\beta] = 0. \qquad (7) \end{aligned}$$

which is of the form $\lambda^4 + \upsilon_1\lambda^3 + \upsilon_2\lambda^2 + \upsilon_3\lambda + \upsilon_4 = 0$. The system is LAS if it satisfies the R-H criterion, $\upsilon_4 > 0$, $\upsilon_1\upsilon_2 - \upsilon_3 > 0$ and $(\upsilon_1\upsilon_2 - \upsilon_3)\upsilon_3 - \upsilon_1^2\upsilon_4 > 0$. Hence E_1 is locally asymptotically stable if $\frac{\mu k\omega\beta^2}{(\beta\phi + hr_0)(\rho\beta + \xi r_0)} < 1$ i.e., $R_0 < 1$.

Theorem 2: The coexistence equilibrium E_* is LAS if the roots of characteristic equation $\lambda^4 + q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4 = 0$ of the Jacobian matrix satisfies Routh–Hurwitz criterion i.e. $q_4 > 0$, $q_1q_2 - q_3 > 0$ and $(q_1q_2 - q_3)q_3 - q_1^2q_4 > 0$.
Proof: The Jacobian matrix of corresponding interior equilibrium E^* is given by

$$J(E^*) = \begin{bmatrix} B_{11}^* & B_{12}^* & B_{13}^* & B_{14}^* \\ B_{21}^* & B_{22}^* & B_{23}^* & B_{24}^* \\ B_{31}^* & B_{32}^* & B_{33}^* & B_{34}^* \\ B_{41}^* & B_{42}^* & B_{43}^* & B_{44}^* \end{bmatrix}.$$

The eigenvalues of the matrix is given by roots of the characteristic equation which is of the form

$$\lambda^4 + q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4 = 0. \qquad (8)$$

The roots of above equation (8) will possess negative real parts when,

$$q_4 > 0,\ q_1q_2 - q_3 > 0 \ and \ (q_1q_2 - q_3)q_3 - q_1^2q_4 > 0. \quad (9)$$

Hence the system $(1)-(4)$ around E_* is LAS.

Theorem 3: The coexistence equilibrium E_* is globally asymptotically stable (GAS) if it holds the below inequalities:

$$\begin{aligned} k &< C_m + C^* + I^*, \\ m_1 &< \frac{2}{\omega^2}\left(\frac{RI_m}{k} + \phi\right)(\rho + \xi A_m), \\ 0 &< m_2 < \frac{2hI^*}{\alpha}. \end{aligned}$$

Proof: A Lyapunov function $V^*(C; I; P; A)$ in Ω is constructed as follows,

$$V_*(C,I,P,A) = \frac{1}{2}(C - C^* + I - I^*)^2 + \frac{m_1}{2}(P - P^*)^2 + \frac{m_2}{2}(A - A^*)^2, \quad (10)$$

$$\frac{dV_*}{dt} = (C - C^* + I - I^*)\left(\frac{dC}{dt} + \frac{dI}{dt}\right) + m_1(P - P^*)\frac{dP}{dt} + m_2(A - A^*)\frac{dA}{dt},$$

where $m_1,\ m_2$ are positive constants. $\frac{dV_*}{dt}$ is calculated along the solution of the system. After simplification, we get

$$\begin{aligned} \frac{dV_*}{dt} =& -\left(\frac{R}{k}(I^* + C + C^*) - R\right)(C - C^*)^2 \\ &- \left(\frac{R}{k}(2C + C^* + I^*) - R + \phi + hA\right)(C - C^*)(I - I^*) \\ &- \left(\frac{RI}{k} + \phi\right)(I - I^*)^2 - m_1(\rho + \xi A)(P - P^*)^2 \\ &+ m_2\beta(A - A^*) - m_1\xi P^*(P - P^*)(A - A^*) \\ &+ m_1\omega(I - I^*)(P - P^*) - (2hI^* - m_2\alpha)(I - I^*)(A - A^*). \end{aligned}$$

Thus, inside the region of attraction, $\frac{dV^*}{dt}$ stands as negative-definite on condition that:

$$\begin{aligned} k &< C_m + C^* + I^*, \\ m_1 &< \frac{2}{\omega^2}\left(\frac{RI_m}{k} + \phi\right)(\rho + \xi A_m), \\ 0 &< m_2 < \frac{2hI^*}{\alpha}. \end{aligned}$$

It is seen that $\frac{dV_*}{dt} < 0$ and $\frac{dV_*}{dt} = 0$ iff, $C = C^*,\ I = I^*,\ P = P^* \ and \ A = A^*$ in Ω. Using the Lyapunov LaSalle theorem [28], we conclude that E_* is GAS whenever $R_0 > 1$.

IV. Sensitivity Analysis

To assess the sensitive parameters of the system, we've performed sensitivity analysis with the help of MATLAB software. In this task, we calculate the sensitivities by formulating differential equations which is done by differentiation of the state variables corresponding to the sensitive parameters. To perform the analysis, we choose $\mu,\ \omega,\ \rho,\ \alpha$ as sensitive parameters. In Figure 9, we portray the sensitivity characteristic of the parameters $\mu,\ \omega,\ \rho,\ \alpha$ in the proposed model. The graph indicates the partial derivative of the state variables with respect to the selected parameters. It is clear that contact rate μ plays a crucial role since it decreases the healthy tree population and rises the infection. Furthermore it is to be noted that the birth rate and death rate of whitefly is able to make slight changes in the state variables. The parameter α increases healthy tree population, decreases infected tree, whitefly population and increases the awareness campaigns. The logarithmic sensitivity analysis is the ratio of the relative change in the variable to the relative change in the parameter. ie., the normalized forward sensitivity index of a variable X that depends differentiably on a parameter a is defined as: $\frac{\partial logX(t)}{\partial loga} = \frac{a}{X(t,a)}X_a(t,a)$ and the anticipated change due to the result of doubling the parameter can be noticed from this analysis. Figure 10 describes the logarithmic sensitivity analysis of the state variables. It is noted that, 0.35% decline in the healthy tree population, 3.4% rise in the infected tree population, 1.1% rise in whitefly population and 0.2% in awareness by the effect of doubling the value of μ. On doubling the parameter ω, there is 0.04% decrease in healthy tree population, 0.3% increase in infected tree population, 58.2% increase in whitefly population and a slight change in awareness. The result of doubling the parameter ρ rises the healthy tree population slightly, reduces the infected tree population by 0.04%, decreases whitefly population by 6.8% and a decrease in awareness is seen. The effect of doubling the parameter α shows a increase in healthy tree, decrease in infected tree population of about 0.07%, decreases whitefly population by 2.3% and increases awareness campaigns by 11.7%.

V. Optimal Control Problem

This study is performed to figure out the cost-effective solution such as insecticide spraying and awareness campaigns in order to have better disease control. We have formulated an optimal control problem with control $U(t)$ and $U_1(t)$. The assumption is made in such a way that insecticide spraying should cover the entire pest population of a confined area. The reframed system with the control $0 \leq U(t) \leq 1$ and $0 \leq U_1(t) \leq 1$ is given by:

$$\frac{dC}{dt} = RC\left(1 - \frac{C+I}{k}\right) - (1-U)\frac{\mu CP}{1+\eta P}, \tag{11}$$

$$\frac{dI}{dt} = (1-U)\frac{\mu CP}{1+\eta P} - \phi I - hAI, \tag{12}$$

$$\frac{dP}{dt} = (1-U)\omega I - \rho P - \xi AP, \tag{13}$$

$$\frac{dA}{dt} = U_1 r_0 + \alpha I - \beta A, \tag{14}$$

with the initial conditions as

$$C(0) = C_0, \quad I(0) = I_0, \quad P(0) = P_0, \quad A(0) = A_0. \tag{15}$$

It is assumed that $C(0) > 0,\ I(0) > 0,\ P(0) > 0,\ A(0) \geq 0$. The control parameters U and U_1 indicates the decline in the rate of infection due to the result of insecticide spraying and awareness campaigns. The cost function including the implementation of optimal spraying and awareness programs is considered in the following form:

$$J(U(t), U_1(t)) = \int_{t_0}^{t_f} \left(P_1 U^2(t) + QU_1^2(t) + TP^2 - SA^2 - R_1 C^2\right) dt \tag{16}$$

Here $P_1,\ Q,\ R,\ S,\ T$ are positive constants. The objective functional is chosen so that optimal cost is involved in spraying insecticide as well as implementing awareness campaigns. Our aim is to find optimal $U(t)$ and $U_1(t)$ which involves minimum cost.
We have constructed the Hamiltonian $\hbar$ in order to evaluate the optimal control problem as given below:

$$\begin{aligned} \hbar &= P_1 U^2(t) + QU_1^2(t) + TP^2 - SA^2 - R_1 C^2 \\ &+ \delta_1\left[RC\left(1 - \frac{C+I}{k}\right) - (1-U)\frac{\mu CP}{1+\eta P}\right] \\ &+ \delta_2\left[(1-U)\frac{\mu CP}{1+\eta P} - \phi I - hAI\right] + \delta_3\left[(1-U)\right. \\ &\left.\omega I - \rho P - \xi AP\right] + \delta_4\left[U_1 r_0 + \alpha I - \beta A\right]. \end{aligned} \tag{17}$$

where $\delta_i,\ i = 1,2,3$ are the adjoint variables. We've used "Pontryagin Minimum Principle" in order to resolve optimal control problem and the following results were obtained.

Theorem 4 If the objective function $J(U, U_1)$ is minimum for the optimal control $U^*(t),\ U_1^*(t)$, then there exists adjoint variables $\delta_i,\ i = 1,2,3,4$, which satisfy the equations below:

$$\frac{d\delta_1}{dt} = 2R_1C^2 + \delta_1 R\left(1 - \frac{C+I}{k}\right) - \frac{RC\delta_1}{k} - (1-U)(\delta_1 - \delta_2)\frac{\mu P}{1+\eta P}, \tag{18}$$

$$\frac{d\delta_2}{dt} = -\delta_1\frac{RC}{k} - \delta_2(\phi + hA) + \delta_3(1-U)\omega + \delta_4\alpha, \tag{19}$$

$$\frac{d\delta_3}{dt} = (1-U)(\delta_2 - \delta_1)\frac{(1+\eta P)\mu C - \mu\eta CP}{(1+\eta P)^2} - \delta_3(\rho + \xi A) - 2TP, \tag{20}$$

$$\frac{d\delta_4}{dt} = 2SA - \delta_2 hI - \delta_3\xi P - \delta_4\beta. \tag{21}$$

with the transversality condition satisfying $\delta_i(t_f) = 0,\ i = 1,2,3,4$. The optimal control is written as

$$U^*(t) = max\left\{0, min\left\{1, \frac{\frac{\mu CP}{(1+\eta P)}(\delta_2 - \delta_1) + \delta_3\omega I}{2P_1}\right\}\right\}. \tag{22}$$

$$U_1^*(t) = max\left\{0, min\left\{1, \frac{-r_0\delta_4}{2Q}\right\}\right\}. \tag{23}$$

Proof

The optimal control variable U^* and U_1^* satisfies the following equation by applying "Pontryagin Minimum Principle" [25]

$$\frac{\partial\hbar}{\partial U_i^*} = 0. \tag{24}$$

From (17) and (24), we get,

$$U^* = \frac{\frac{\mu CP}{(1+\eta P)}(\delta_2 - \delta_1) + \delta_3\omega I}{2P_1}.$$

$$U_1^* = \frac{-r_0\delta_4}{2Q}.$$

The optimal control has the following form

$$U^*(t) = \begin{cases} 0 \ for \ \dfrac{\frac{\mu CP}{(1+\eta P)}(\delta_2-\delta_1)+\delta_3\omega I}{2P_1} \le 0, \\ \dfrac{\frac{\mu CP}{(1+\eta P)}(\delta_2-\delta_1)+\delta_3\omega I}{2P_1} for \\ 0 < \dfrac{\frac{\mu CP}{(1+\eta P)}(\delta_2-\delta_1)+\delta_3\omega I}{2P_1} < 1, \\ 1 \ for \ \dfrac{\frac{\mu CP}{(1+\eta P)}(\delta_2-\delta_1)+\delta_3\omega I}{2P_1} \ge 1. \end{cases}$$

$$U_1^*(t) = \begin{cases} 0 \ for \ \dfrac{-r_0\delta_4}{2Q} \le 0, \\ \dfrac{-r_0\delta_4}{2Q} for \ 0 < \dfrac{-r_0\delta_4}{2Q} < 1, \\ 1 \ for \ \dfrac{-r_0\delta_4}{2Q} \ge 1. \end{cases}$$

Thus $U^*(t),\ U_1^*(t)$ takes the compact form as,

$$U^*(t) = max\left\{0, min\left\{1, \frac{\frac{\mu CP}{(1+\eta P)}(\delta_2-\delta_1)+\delta_3\omega I}{2P_1}\right\}\right\}.$$

$$U_1^*(t) = max\left\{0, min\left\{1, \frac{-r_0\delta_4}{2Q}\right\}\right\}.$$

The aforementioned equations are the necessary conditions that satisfy the optimal control $U(t),\ U_1(t)$ as well as the system's state variables. According to [25], the existence conditions are confirmed by the corresponding adjoint equations.

$$\frac{d\delta_1}{dt} = -\frac{\partial\hbar}{\partial C},\ \frac{d\delta_2}{dt} = -\frac{\partial\hbar}{\partial I},\ \frac{d\delta_3}{dt} = -\frac{\partial\hbar}{\partial P},\ \frac{d\delta_4}{dt} = -\frac{\partial\hbar}{\partial A}. \quad (25)$$

From the set of equation (25), we get $(18)-(21)$.

VI. Numerical Simulation

We've chosen the values in such a way that it provides a reference point for every parameter to perform numerical analysis. We assume, the initial population to be 50 healthy trees, 5 infected trees, 10 whitefly per tree and the number of awareness programs conducted as 10. Hence the initial conditions are considered as $C(0) = 50$, $I(0) = 5$, $P(0) = 10$, $A(0) = 10$. All parameter values used for analysis are given in Table 1. The number of coconut trees planted per acre ranges from 60-70. Here, we consider the carrying capacity as 70 acre^{-1}. The maximum number of trees which requires replanting is assumed to be 3 and hence the replanting rate is 0-0.003. The maximum number of mortality is assumed to be 2 trees and hence the mortality rate is given by 0-0.002. We've used the values of whitefly population same as that of ACMD [11] since it is much relevant to our case. The rate at which the adult whitefly infects the trees is considered to be contact rate. (i.e) 0.5-1 tree in 25 days [11]. The parametric values related to awareness and saturation constant are referred from [16], [18].

Figure 1 displays the numerical simulation of healthy tree as a function of time (days) with parameter values mentioned in Table 1. It is seen that, due to contact rate, healthy trees are infected and replanting of intensely infected tree corresponds to increase in healthy tree population. Figure 2 indicates the numerical simulation of infected tree as a function of time (days) with parameter values given in Table 1. The result infers that, more trees get infected due to contact rate and death rate of whitefly corresponds to decrease in the infected tree population. Figure 3 interprets the variation of whitefly population with respect to its birth and death rate. Figure 4 infers the number of awareness programs which can be increased through local awareness rate α. The number of campaigns decreases due to increase in fading rate of awareness among farmers. The surface plots of state variables are shown in Figure 5-8. The population of healthy tree with respect to time and contact rate is shown in Figure 5. The infected trees in regard to time and contact rate is shown in Figure 6. From Figure 7 we see the variation in whitefly population with respect to its birth rate and time. Figure 8 implies the awareness programs with respect to r_0 and time. The description of parameter effect on reproduction number R_0 is shown in Figure 10-14. These figures shows the range of reproduction number based on the selected parameters. i.e., we can visualize when R_0 will be less than 1 and greater

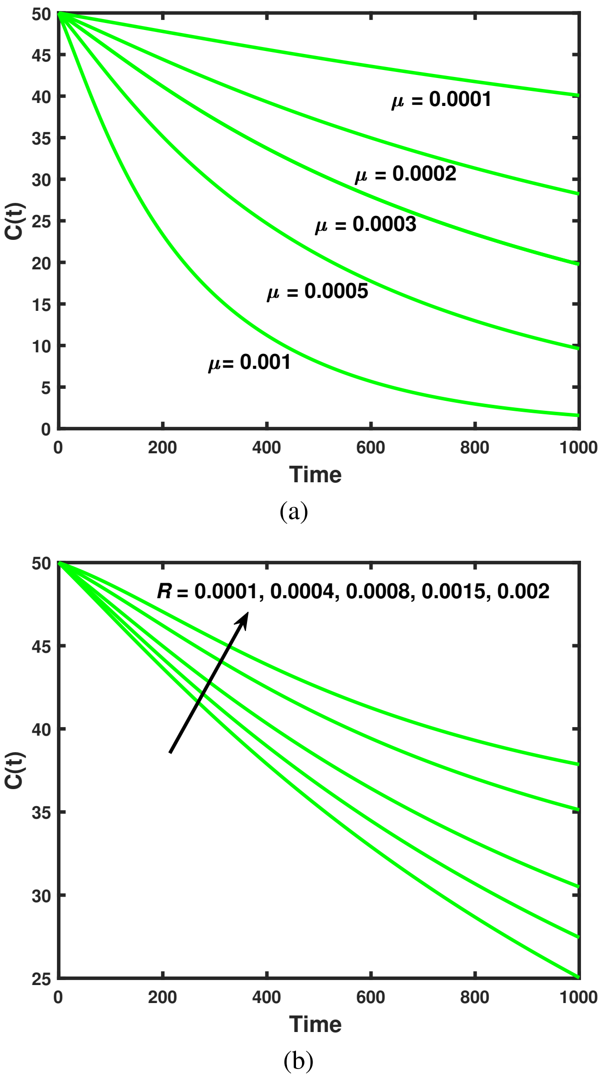


Fig. 1. Profiles of Healthy tree $C(t)$ versus time (1000 days) (a) for different values of contact rate μ (b) for different values of replanting rate R with other parameters as given in Table 1

TABLE I
THE PARAMETER VALUES WHICH ARE CALCULATED BASED ON [11], [16], [18] AND USED FOR ANALYSIS.

S.No	Symbol	Meaning	Unit	Value taken for analysis	Range
1	k	Tree density	acre^{-1}	70	60-70
2	R	Replanting rate	day^{-1}	0.0005	0-0.003
3	ϕ	Mortality rate of tree	day^{-1}	0.0002	0-0.002
4	μ	Contact rate	pest^{-1} day^{-1}	0.0002	0-0.002
5	ω	Whitefly birth rate	day^{-1}	0.2	0.1-0.3
6	ρ	Death rate of whitefly	day^{-1}	0.06	0.06-0.1
7	η	Saturation constant		0.2	0.2
8	h	Maximum activity rate	day^{-1}	0.0001	0-0.0001
9	ξ	Death rate of whitefly due to awareness	day^{-1}	0.005	0.005
10	r_0	Rate of awareness programs	day^{-1}	0.03	0.03
11	α	Rate of local awareness	day^{-1}	0.025	0.025
12	β	Fading rate of awareness	day^{-1}	0.015	0.015

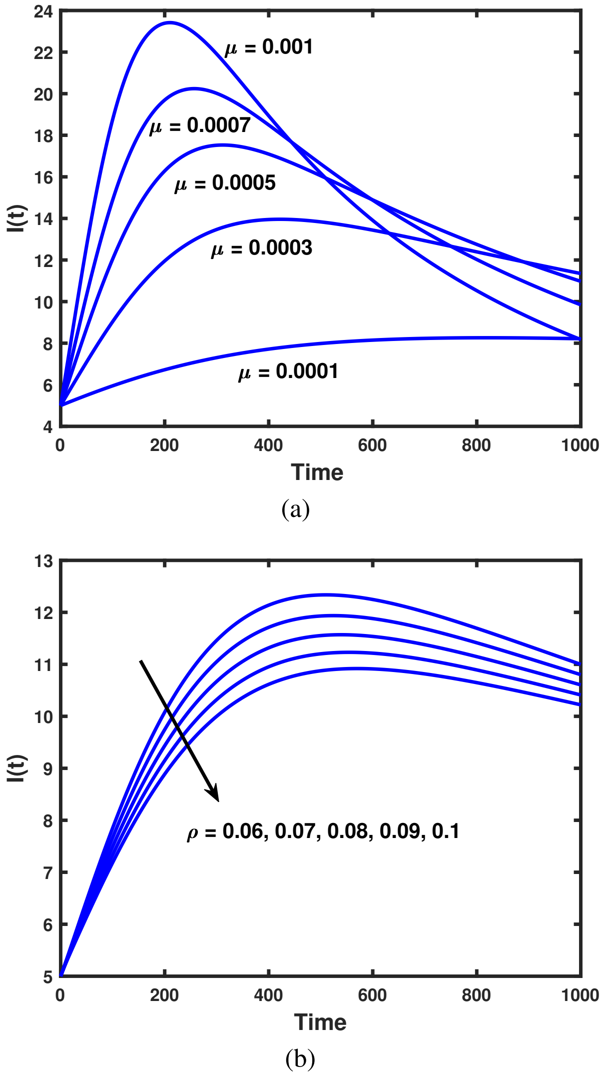


Fig. 2. Infected tree profile $I(t)$ versus time (a) for different values of contact rate μ, (b) for different values of death rate of whitefly ρ and other fixed parameters as given in Table 1

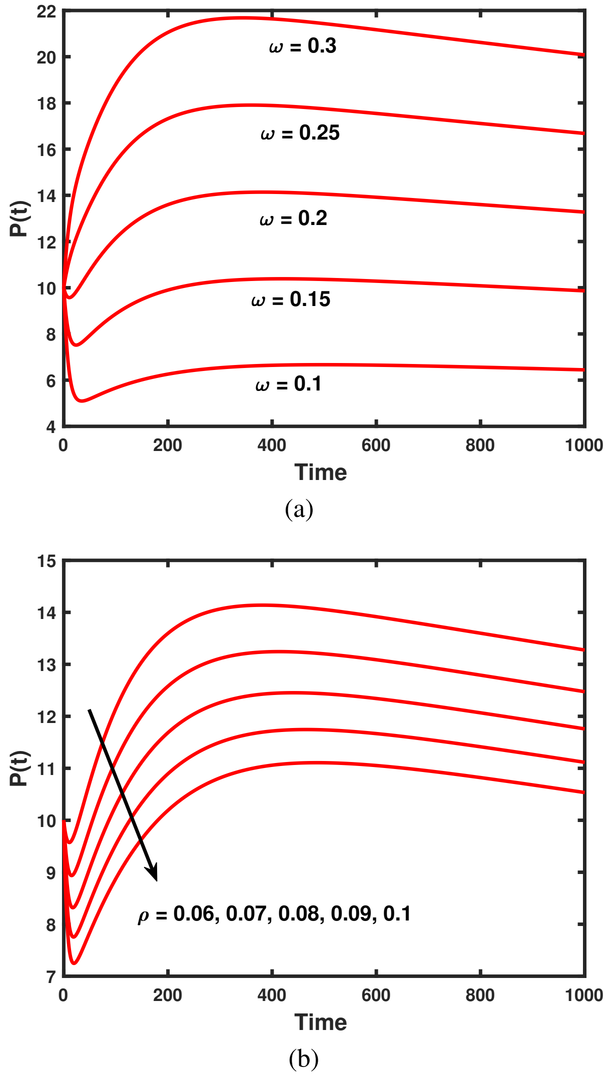
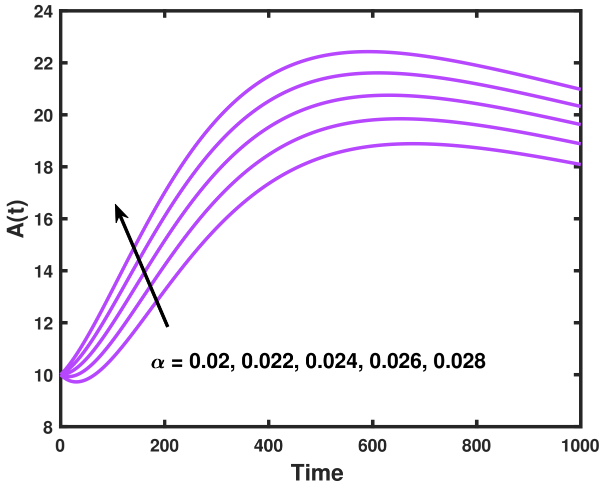


Fig. 3. Profiles of whitefly population $P(t)$ versus time (1000 days) (a) for different values of its birth rate ω, (b) for different values of death rate ρ and other fixed parameters as given in Table 1

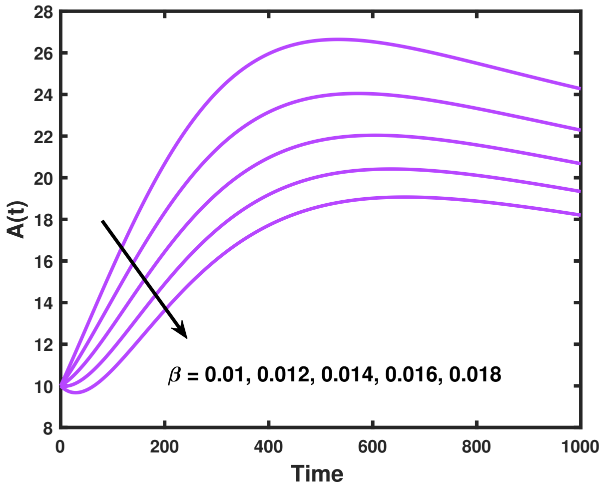
than 1.

In Figure 12 and 13, we portray the sensitivity characteristic of the parameters $\mu,\ \omega,\ \rho,\ \alpha$ in the proposed model. The graph clearly infers, the contact rate μ plays a crucial role in the study since it reduces the count of healthy trees and tends to increase the infection. The rate of local awareness decreases the whitefly population as well as the infected tree population.

Figure 14 infers the optimal control effect on the population dynamics of the model. Our main aim of using control term is to minimize the infected tree population as well as whitefly population so that the yield remains unaffected. It is witnessed that there is a significant difference in populations with and without the effect of control. Figure 15 denotes



(a)



(b)

Fig. 4. Profiles of awareness programs $A(t)$ versus time (1000 days) (a) for different values of its local awareness rate α, (b) for different values of fading away rate β and other fixed parameters as given in Table 1

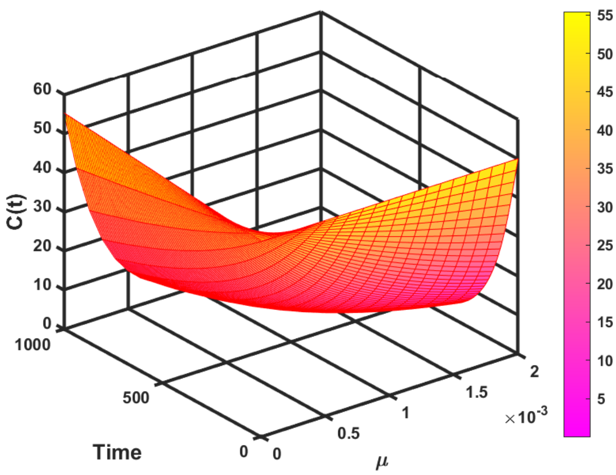


Fig. 5. Surface plot of healthy tree with respect to time and μ

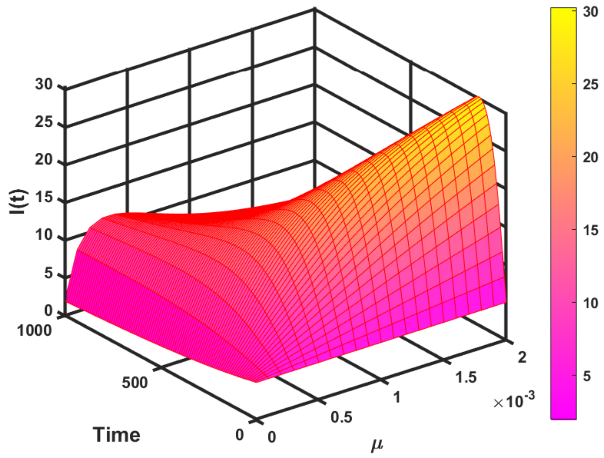


Fig. 6. Surface plot of infected tree with respect to time and μ

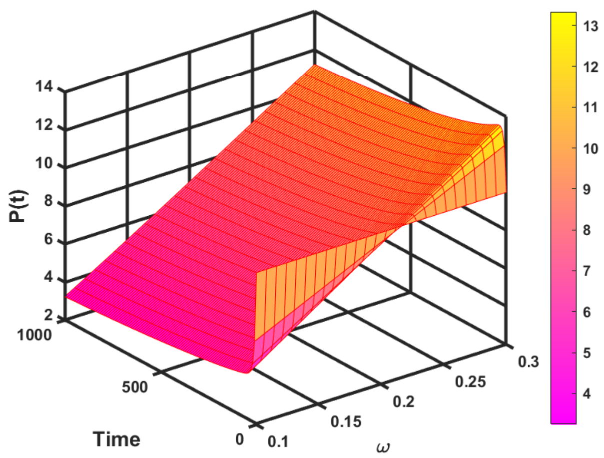


Fig. 7. Surface plot of whitefly population with respect to time and ω

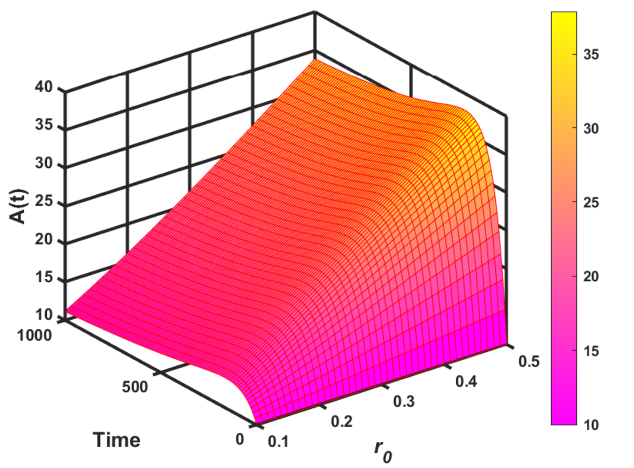


Fig. 8. Surface plot of awareness programs with respect to time and r_0

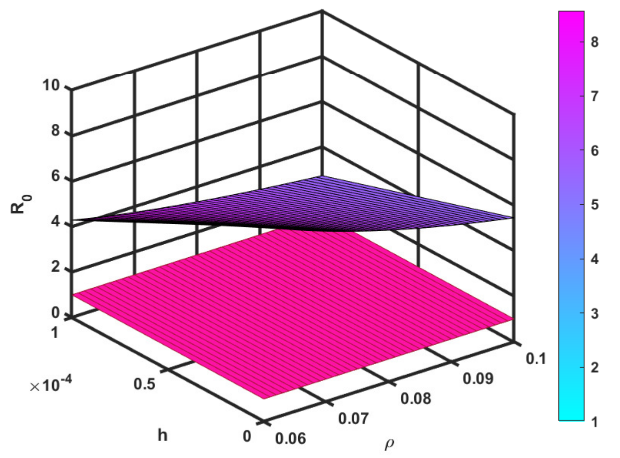
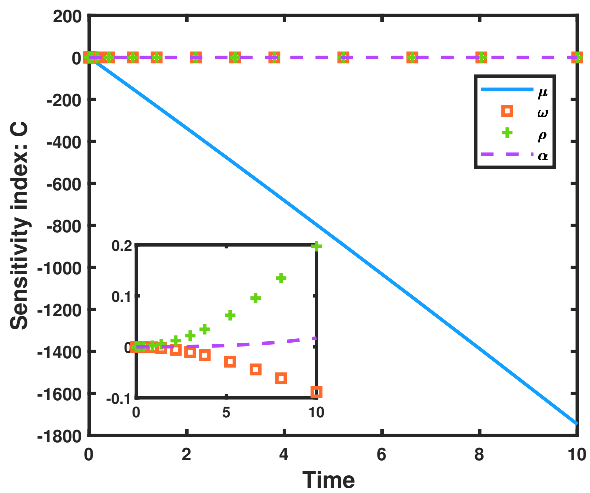


Fig. 9. Surface plot of reproduction number R_0 with respect to h and ρ



(a)

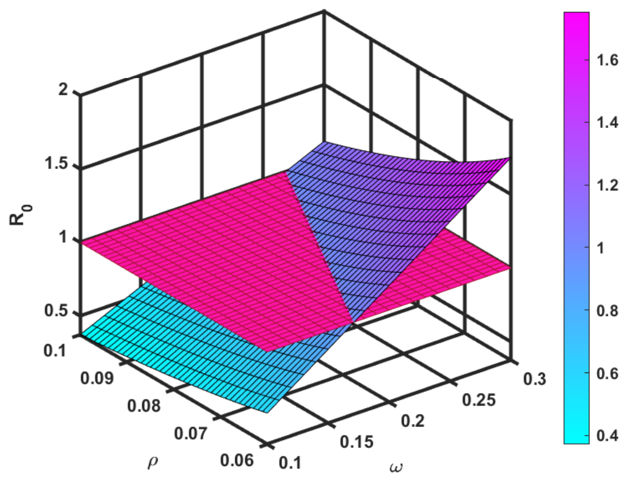
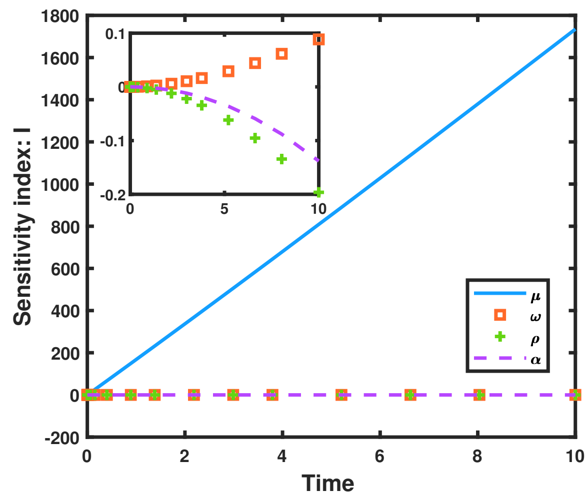


Fig. 10. Surface plot of reproduction number R_0 with respect to ω and ρ



(b)

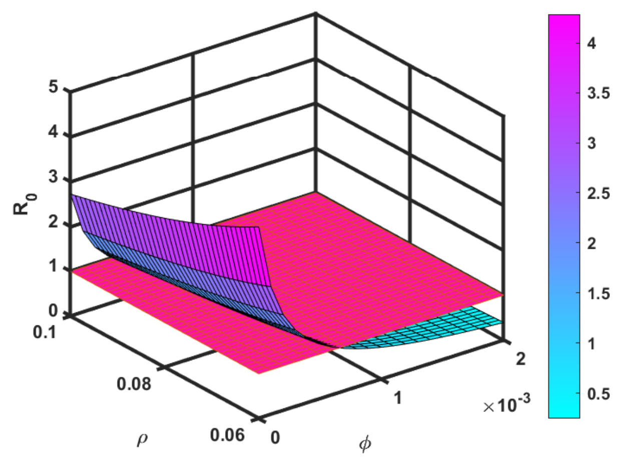
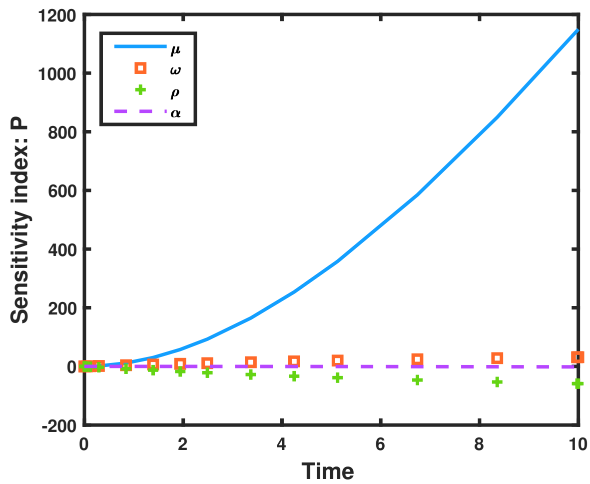
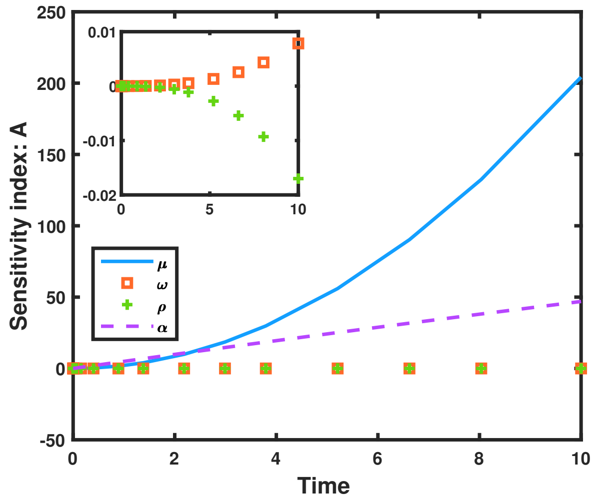


Fig. 11. Surface plot of reproduction number R_0 with respect to ϕ and ρ

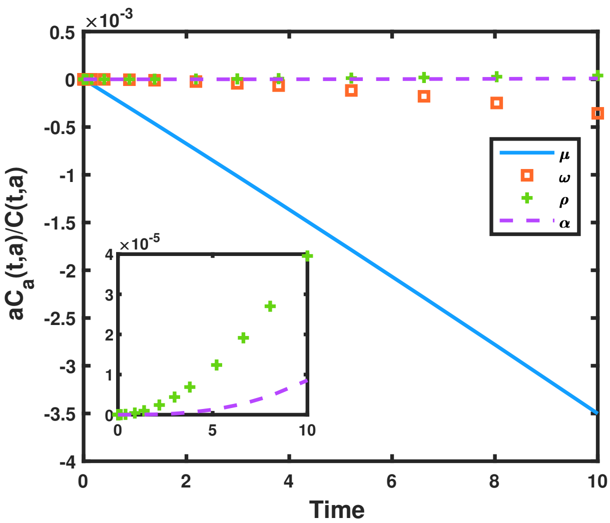


(c)

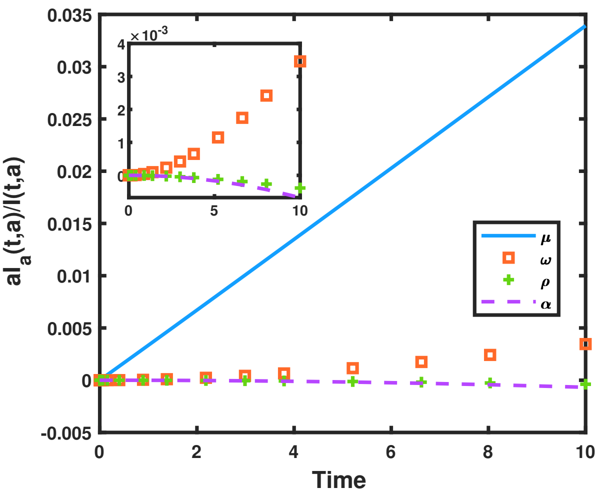


(d)

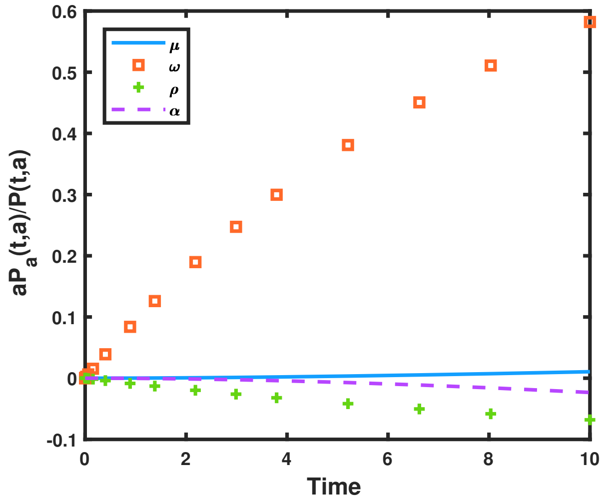
Fig. 12. Plot of sensitivity index of the state variables (a) healthy trees $C(t)$ (b) infected trees $I(t)$ (c) whitefly population $P(t)$ (d) awareness programs $A(t)$ corresponding to the sensitive parameters μ, ω, ρ, α.



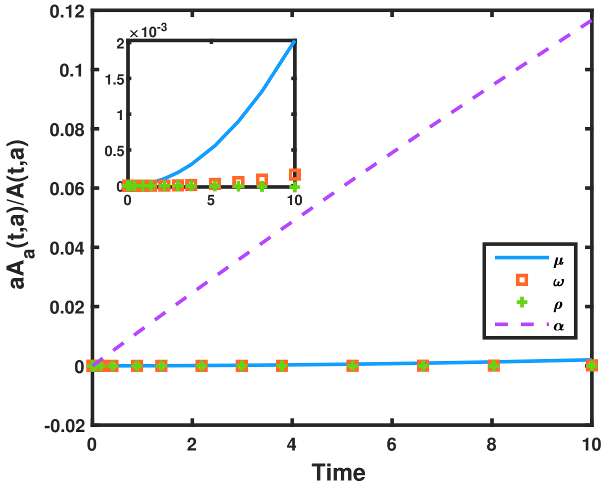
(a)



(b)

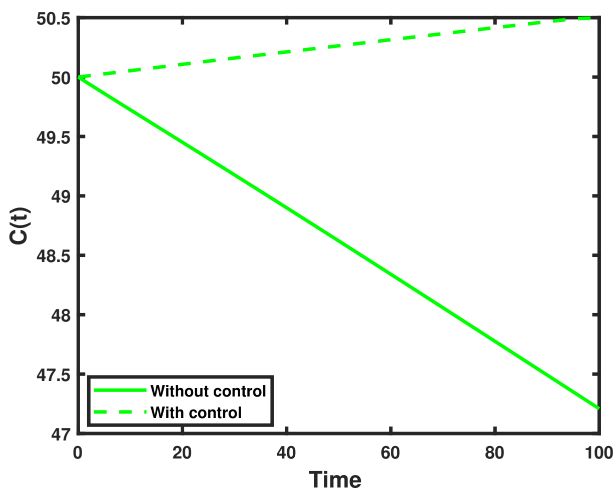


(c)

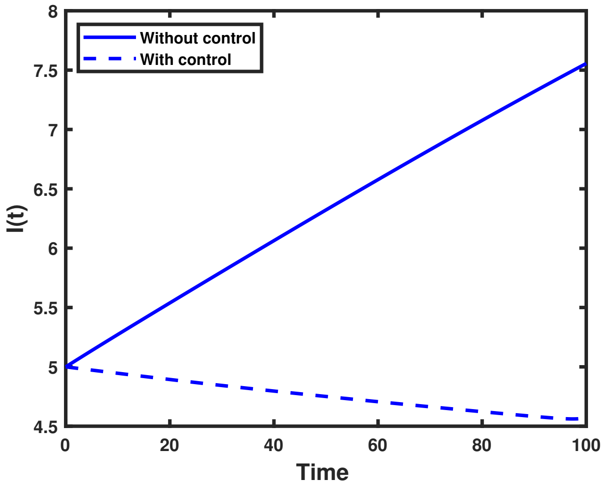


(d)

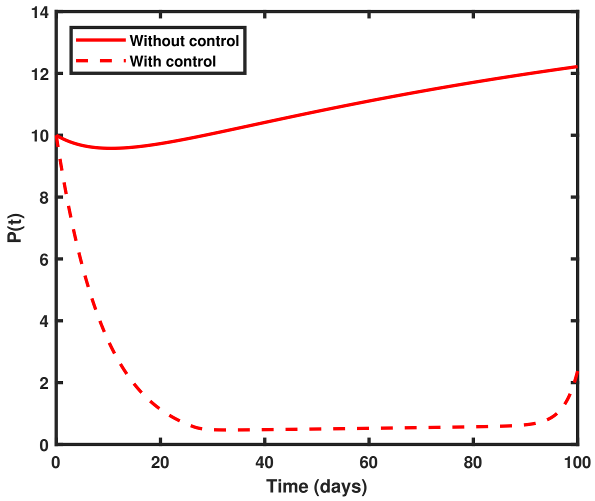
Fig. 13. Plot of logarithmic sensitivity analysis for state variables (a) healthy trees $C(t)$ (b) infected trees $I(t)$ (c) whitefly population $P(t)$ (d) awareness programs $A(t)$. We use a common term a to indicate the selected parameters.



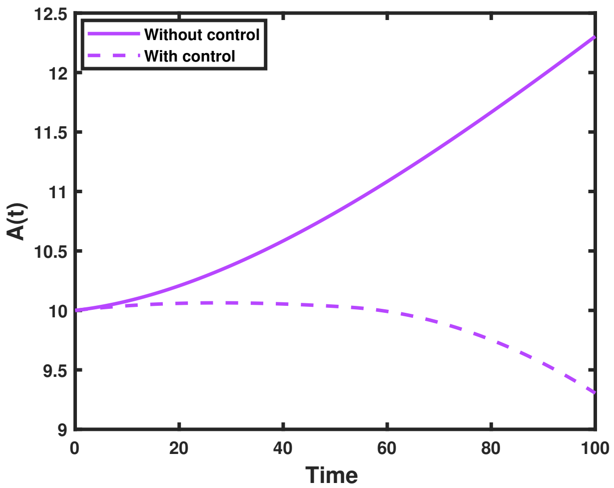
(a)



(b)

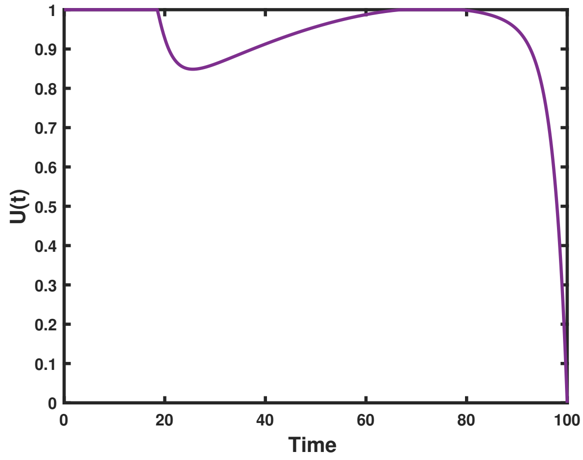


(c)

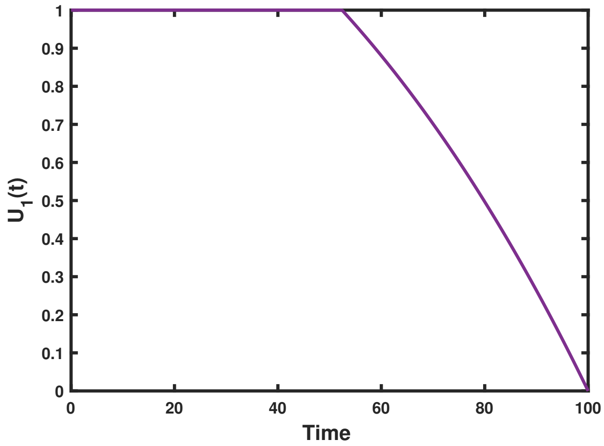


(d)

Fig. 14. Comparison of state variables (a) healthy trees $C(t)$ (b) infected trees $I(t)$ (c) whitefly population $P(t)$ (d) awareness programs $A(t)$ of system $(1)-(4)$ with system $(18)-(21)$. It is clear that, use of control decreases the infection and whitefly population.



(a)



(b)

Fig. 15. Plot of control function (a) $U(t)$ and (b) $U_1(t)$ versus time with parameter values as given in Table 1.

the control effect $U(t)$ and $U_1(t)$ versus function of time. The control can be operated through insecticide spraying and implementing awareness programs. Thus conducting several awareness programs which will educate the farmers in better way to go with cost effective optimum level of spraying techniques which is much needed to control the disease spread.

VII. Conclusion

A mathematical model has been projected to study the population dynamics of the system. It deals with the impact of awareness on the interaction of rugose spiraling whitefly with coconut trees. The equilibrium points as well as the condition to be LAS and GAS have been analysed. To witness the system dynamics and to study the species behaviour with respect to sensitive parameters, we have carried out the numerical simulation and sensitivity analysis. Furthermore, we have applied optimal control theory to develop a cost-effective insecticide usage and awareness programs. Thus, by educating the farmers in better way to understand the whitefly population control through various awareness programs can help in better control or even eradication of the disease.

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