The M/M/1 Queueing System with Bernoulli Secondary Service and Variable Service Rate

Shengli Lv, *Member, IAENG*, Ruiyu Wang, Jingbo Li and Yueying Wu

***Abstract*—In the random service system, there are often customers who need to receive secondary service, but not every customer needs it. In this paper, we analyse the Bernoulli feedback service (receive secondary service) mechanism and set each customer to receive secondary service is random with a probability after the first service, and the secondary service rate is different from the first service rate. In addition, the customer who receives feedback service has priority over the customer who receives the service for the first time. Using the method of matrix geometric solution of quasi-birth-and-death process, the performance indexes such as the mean queueing length as well as the mean waiting time and the mean sojourn time are calculated. Numerical experiments were conducted to observe the trend of each performance index as the parameters changed. The last part of the article investigates the optimal strategy for the aim of reducing costs and improving efficiency.**

***Index Terms*—feedback priority strategy, variable service rate, quasi-birth-and-death process, M/M/1 queue, matrix geometry solutions.**

I. Introduction

EARLY in the 20th century, mathematician Erlang [1] applied the knowledge of queueing theory when studying the telephone communication problem. Therefore, scholars began to study the theory of queueing. After more than 100 years of development, queueing theory has formed a mature theory, and researchers have incorporated various strategies based on the classical queueing theory in conjunction with practical problems. Lv [2] studied machine repairable system with different service strategies. Lyu et al. [3] studied the M/M/2 queueing system with variable service strategies.

In the study of queueing system, we often assume that a customer will leave the system after receiving one service. However, in real life, there are often customers who are not satisfied with the service, or do not complete the service all at once, and need to return to the service station to receive a second service. Such a mechanism is called feedback. For example, secondary packaging of products, lathe parts processing and outpatient services etc. Until now, queueing models with feedback have been studied extensively by scholars. Kiran [4] studied M/G/1 queueing model with feedback. Ghahramani et al. [5] studied two M/M/1 queues with inconsistent arrivals and queueing models with random feedback. They obtained a probability generating function for the modles and tested results numerically for validity. Barlas and Özgün [6] studied feedback strategy among three service stations. They assumed that the arrival rates and service rates for the second visit to any service station are not the same. Tsai et al. [7] proposed a general configuration strategy for an open queueing network, which consists of any number of service stations with different service rates for each station. Som and Seth [8] refer to customers who rejoin the queue due to dissatisfaction as feedback customers. They propose a multi-server wireless capacity Markov feedback queueing model with encouraged arrivals and solve its steady-state conditions as well as various performance metrics. Saravanarajan and Chandrasekaran [9] study the Bernoulli vacations and stochastic fault feedback queueing model, combining faulted vacations and feedback. Kumar et al. [10] combined impatient customers and feedback policy to construct M/M/1/N queueing model and obtained the optimal capacity of the system by continuous optimization.

For this article, based on the traditional M/M/1 queueing model, the feedback customer is given priority to receive service. It is optimized for real life. In addition, we believe that the service rate will change when the service station provides service to customers who receive service for the second time.

II. Model Description

1) We assume that the system has an infinite capacity and only one service station. The customer arrival process obeys the Poisson process with λ.

2) The customer in the system leaves the system directly after receiving the service with probability $1-\alpha$, or returns to the service station again with probability α to receive the second service. Each customer can be given feedback service at most one time.

3) The time of customers receiving service all obeys negative exponential distribution. The service rate is μ_1 when each customer receives service for the first time, the service rate is μ_2 when they return to the service station again for the second time.

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4) Customers who arrive at the system for the first time are served according to the rule of first-come, first-served. It is assumed that all processes are independent of each other.

5) When a customer who has received the first service needs a second service, the service station will immediately perform the second service for that customer, and then serve the next customer after the second service for that customer is completed.

The diagram of the queueing model is shown in Figure 1.

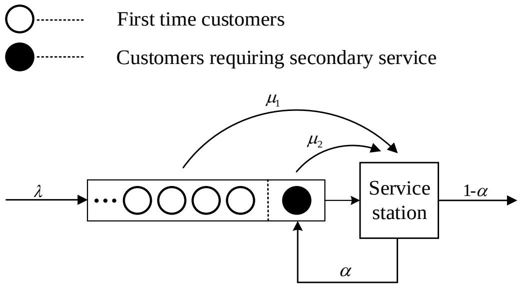

Fig. 1. Schematic diagram of the queueing model.

III. State Transfer Diagram

Let $L(t)$ be the number of customers in the system at moment t (including the one being served and those waiting in line), and $J(t)$ is defined as the service state of the service station. When $J(t) = 0$ means that there is no feedback request in the system at time t and the service station is serving the customer who is receiving the service for the first time; when $J(t) = 1$ means that there is a customer initiating feedback in the system at time t and the service station is serving the customer who is given the feedback. This means that the state of the queueing system can be divided into a feedback state and a Non-feedback state. Specifically, $\{(0,0)\}$ means that the system is in the idle state.

According to the memoryless nature of the exponential distribution, $\{L(t), J(t)\}$ constitutes a two-dimensional Markov process whose state space is

$$\Omega = \{ (0,0)\} \cup \{ (k,j), k \geq 1, j = 0, 1\}.$$

By arranging the states of the model in dictionary order, the state transfer diagram of this two-dimensional Markov chain can be obtained as Figure 2.

From the state transfer diagram, the transfer rate matrix is

$$Q = \begin{pmatrix} A_0 & C_0 & & & \\ B_1 & A & C & & \\ & B & A & C & \\ & & B & A & C \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$A_0 = -\lambda$, $C_0 = \begin{pmatrix} \lambda & 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} (1-\alpha)\mu_1 \\ \mu_2 \end{pmatrix}$,

$A = \begin{pmatrix} -(\mu_1 + \lambda) & \alpha\mu_1 \\ 0 & -(\mu_2 + \lambda) \end{pmatrix}$, $B = \begin{pmatrix} (1-\alpha)\mu_1 & 0 \\ \mu_2 & 0 \end{pmatrix}$,

$C = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$.

Therefore $\{L(t), J(t)\}$ is a Quasi-birth-and-death(QBD) [11].

IV. Steady-state conditions and steady-state probabilities

A. Stability

Theorem 1. The matrix equation $R^2B + RA + C = 0$ has a minimum non-negative solution $R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$, where

$r_{11} = \frac{\lambda(\lambda+\mu_2)}{\mu_1(\lambda-\alpha\lambda+\mu_2)}$, $r_{12} = \frac{\alpha\lambda}{\lambda-\alpha\lambda+\mu_2}$, $r_{21} = \frac{\lambda^2}{\mu_1(\lambda-\alpha\lambda+\mu_2)}$, $r_{22} = \frac{\lambda}{\lambda-\alpha\lambda+\mu_2}$.

Proof Bringing R into the system of equations $R^2B + RA + C = 0$. Then

$$\begin{cases} ({r_{11}}^2 + r_{12}r_{21})(1-\alpha)\mu_1 + (r_{11}r_{12} + r_{12}r_{22})\mu_2 \\ -(\mu_1 + \lambda)r_{11} + \lambda = 0, \\ \alpha\mu_1 r_{11} - (\mu_2 + \lambda)r_{12} = 0, \\ (r_{21}r_{11} + r_{22}r_{21})(1-\alpha)\mu_1 + (r_{21}r_{12} + {r_{22}}^2)\mu_2 \\ -(\mu_1 + \lambda)r_{21} = 0, \\ \alpha\mu_1 r_{21} - (\mu_2 + \lambda)r_{22} + \lambda = 0. \end{cases} \tag{1}$$

The minimum non-negative solution can be calculated by solving the Eq. (1)

$$R = \begin{pmatrix} \frac{\lambda(\lambda+\mu_2)}{\mu_1(\lambda-\alpha\lambda+\mu_2)} & \frac{\alpha\lambda}{\lambda-\alpha\lambda+\mu_2} \\ \frac{\lambda^2}{\mu_1(\lambda-\alpha\lambda+\mu_2)} & \frac{\lambda}{\lambda-\alpha\lambda+\mu_2} \end{pmatrix}. \tag{2}$$

Proof complete.

Lemma If $H = A + B + C$ is a finite generating element and x is a steady-state probability vector of H. Then a sufficient necessary condition for the spectral radius $SP(R) < 1$ of the state transfer matrix Q (where R is the rate array of Q) is $xBe > xCe$ (where e is a two-dimensional column vector which all elements are equal to 1) [11].

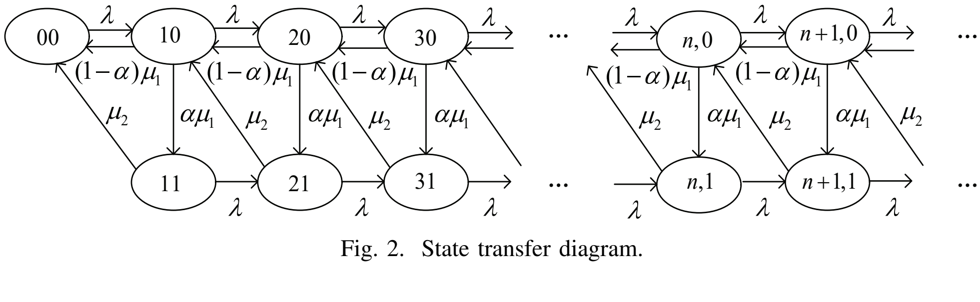

Fig. 2. State transfer diagram.

Theorem 2. The M/M/1 queueing system with feedback priority and variable service rate is steady-state when and only when

$$\lambda < \frac{\mu_1\mu_2}{\mu_2 + \alpha\mu_1}. \tag{3}$$

Proof Prerequisites: $H = \begin{pmatrix} -\alpha\mu_1 & \alpha\mu_1 \\ \mu_2 & -\mu_2 \end{pmatrix}$. Let x be a two-dimensional row vector, $x = (x_1, x_2)$. From $xH = 0$, we can get $x = c(\mu_2, \alpha\mu_1)$, c as any positive real number. And because $xBe > xCe$, then,

$$c(\mu_2, \alpha\mu_1)\begin{pmatrix} (1-\alpha)\mu_1 & 0 \\ \mu_2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} > c(\mu_2, \alpha\mu_1)\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

simplify to get: $\lambda < \frac{\mu_1\mu_2}{\mu_2+\alpha\mu_1}$.

Its adequacy is clearly evident. Proof complete.

In summary, under steady-state conditions, the mean input λt of the system in time period $[0, t]$ should be smaller than the mean output $\frac{\mu_1\mu_2}{\mu_2+\alpha\mu_1}t$.

B. Steady-state probability

The steady-state probability is defined as

$$\pi_{ij} = \lim_{t\to\infty} P\left\{L(t) = i, J(t) = j\right\}, (i, j) \in \Omega.$$

Corresponding to the block structure of Q, the steady-state probability vector π is labeled

$$\pi = (\pi_0, \pi_1, \pi_2, \ldots),$$

where $\pi_0 = \pi_{00}$, $\pi_i = (\pi_{i0}, \pi_{i1})$, $i \geq 1$.

When $\lambda < \frac{\mu_1\mu_2}{\mu_2+\alpha\mu_1}$, the steady-state probability vector π exists, in order to solve the probability vector $\pi_0, \pi_1, \pi_2, \ldots$, the following square matrix is introduced

$$B[R] = \begin{pmatrix} A_0 & C_0 \\ B_1 & RB + A \end{pmatrix}. \tag{4}$$

Then the steady-state distribution of the QBD process $\{L(t), J(t)\}$ satisfies the following relation

$$\begin{cases} (\pi_0, \pi_1)B[R] = 0, \\ \pi_0 e + \pi_1 (I - R)^{-1} e = 1, \\ \pi_k = \pi_1 R^{k-1}, k \geq 2. \end{cases} \tag{5}$$

Bringing Eq. (2) into Eq. (4) and Eq. (5) gives

$$\begin{cases} \pi_{00} = -\dfrac{\alpha\mu_1\lambda - \mu_1\mu_2 + \lambda\mu_2}{\mu_1\mu_2}, \\ \pi_{10} = \dfrac{\lambda(\lambda+\mu_2)(\alpha\mu_1\lambda - \mu_1\mu_2 + \lambda\mu_2)}{\mu_1^2(-\lambda + \alpha\lambda - \mu_2)\mu_2}, \\ \pi_{11} = \dfrac{\alpha\lambda(\alpha\mu_1\lambda - \mu_1\mu_2 + \lambda\mu_2)}{\mu_1(-\lambda + \alpha\lambda - \mu_2)\mu_2}. \end{cases} \tag{6}$$

Then,

$$\begin{cases} \pi_0 = \pi_{00} = -\dfrac{\alpha\mu_1\lambda - \mu_1\mu_2 + \lambda\mu_2}{\mu_1\mu_2}, \\ \pi_1 = \begin{pmatrix} \dfrac{\lambda(\lambda+\mu_2)(\alpha\mu_1\lambda - \mu_1\mu_2 + \lambda\mu_2)}{\mu_1^2(-\lambda + \alpha\lambda - \mu_2)\mu_2} \\ \dfrac{\alpha\lambda(\alpha\mu_1\lambda - \mu_1\mu_2 + \lambda\mu_2)}{\mu_1(-\lambda + \alpha\lambda - \mu_2)\mu_2} \end{pmatrix}^\top, \\ \pi_k = \pi_1 + R^{k-1} (k \geq 2). \end{cases} \tag{7}$$

V. System steady-state performance indicators

1) Mean Queueing length in steady-state system

$$E(L) = \frac{\lambda\left(-\alpha^2\mu_1\lambda - \alpha\lambda\mu_2 + {\mu_2}^2 + \alpha\mu_1(\lambda+\mu_2)\right)}{\mu_2\left(-\alpha\mu_1\lambda + (\mu_1 - \lambda)\mu_2\right)}. \tag{8}$$

Proof

$$\begin{aligned} E(L) &= \sum_{n=1}^{\infty} n\pi_n e \\ &= \pi_1 e + 2\pi_2 e + 3\pi_3 e + \ldots \\ &= \sum_{n=1}^{\infty} n\pi_1 R^{n-1} e = \pi_1 \frac{1}{(1-R)^2} e \\ &= \frac{\lambda\left(-\alpha^2\mu_1\lambda - \alpha\lambda\mu_2 + {\mu_2}^2 + \alpha\mu_1(\lambda+\mu_2)\right)}{\mu_2\left(-\alpha\mu_1\lambda + (\mu_1 - \lambda)\mu_2\right)}. \end{aligned}$$

Exceptionally, when $\alpha = 0$, that is, the customer's feedback probability is 0. In this case, the queueing model is the primordial M/M/1 queueing model. The mean queueing length is $E(L) = \frac{\lambda}{\mu_1-\lambda}$.

2) Generalized mean service time (the length of time from the service beginning to the secondary service is over)

$$T = (1-\alpha)\mu_1^{-1} + \alpha(\mu_1^{-1} + \mu_2^{-1}). \tag{9}$$

3) Mean waiting time (the length of time from a customer enters the system to he/she starts receiving service)

$$W = L \cdot T. \tag{10}$$

4) Mean sojourn time (the length of time from a customer enters the system to he/she leaves the system)

$$S = (L+1)T. \tag{11}$$

5) The probability that the system is in Non-feedback state (the probability of the service station is serving customers who is receiving service for the first time)

$$\begin{aligned} P(J(t) = 0) &= \sum_{i=1}^{\infty} \pi_i h_1 \\ &= \pi_1 h_1 + \pi_2 h_1 + \pi_3 h_1 + \cdots \\ &= \pi_1 h_1 + \pi_1 R h_1 + \pi_1 R^2 h_1 + \cdots \\ &= \pi_1 (I - R)^{-1} h_1 \\ &= \frac{\lambda}{\mu_1}, \end{aligned} \tag{12}$$

where $h_1 = (1, 0)^\top$.

6) The probability that the system is in feedback state (the probability of the service station is serving customers who needs secondary service)

$$\begin{aligned} P(J(t) = 1) &= \sum_{i=1}^{\infty} \pi_i h_2 \\ &= \pi_1 h_2 + \pi_2 h_2 + \pi_3 h_2 + \cdots \\ &= \pi_1 h_2 + \pi_1 R h_2 + \pi_1 R^2 h_2 + \cdots \\ &= \pi_1 (I - R)^{-1} h_2 \\ &= \frac{\alpha\lambda}{\mu_2}, \end{aligned} \tag{13}$$

where $h_2 = (0, 1)^\top$.

7) The probability that the system is being idle

$$P(\text{Idle state}) = \pi_{00} = -\frac{\alpha\mu_1\lambda - \mu_1\mu_2 + \lambda\mu_2}{\mu_1\mu_2}. \tag{14}$$

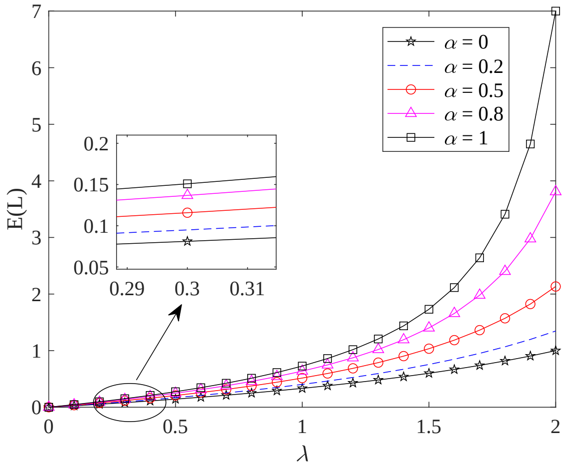

Fig. 3. The trend of $E(L)$ versus λ for α takes different values ($\mu_1 = 4$ and $\mu_2 = 5$).

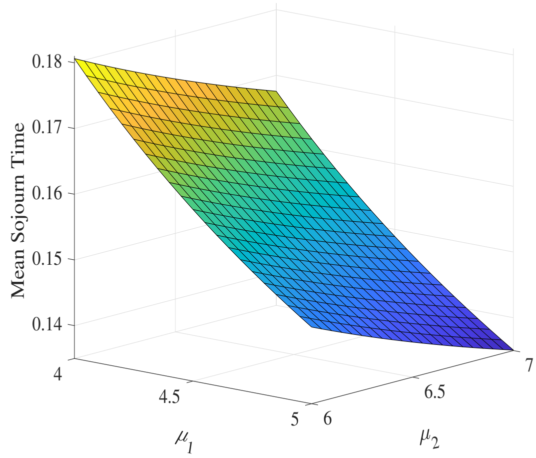

Fig. 5. The trend of mean sojourn time S versus μ_1 and μ_2 ($\alpha = 0.8$ and $\lambda = 1$).

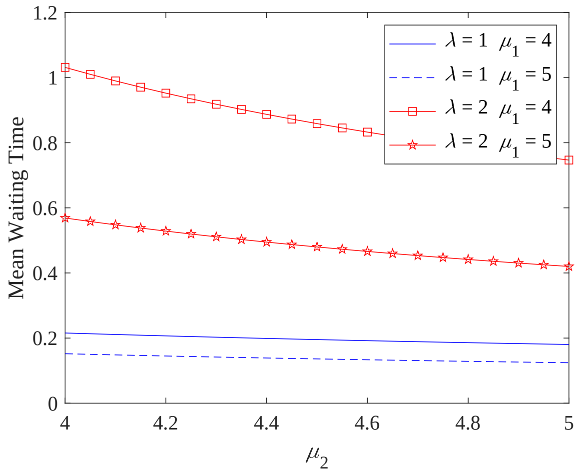

Fig. 4. The trend of mean waiting time W versus μ_2 ($\alpha = 0.5$).

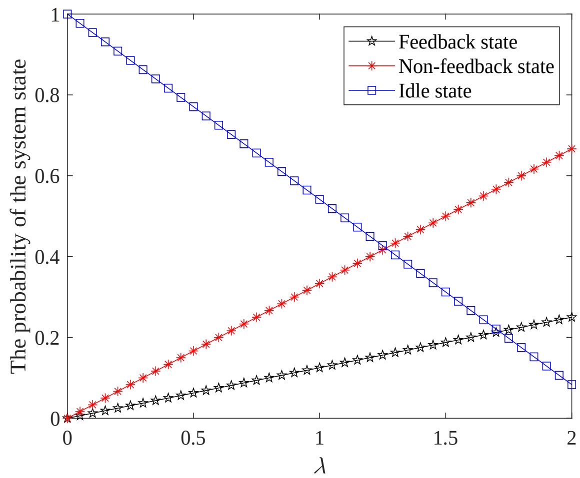

Fig. 6. The trend of the probability of the state system versus λ ($\mu_1 = 3$, $\mu_2 = 4$ and $\alpha = 0.5$).

VI. Numerical simulation

We illustrate the trend of $E(L)$ with the change of λ in this section. We let $\mu_1 = 4$, $\mu_2 = 5$, $\alpha = 0, 0.2, 0.5, 0.8$ and 1, respectively. The range of variation of λ is $0 \leq \lambda \leq 2$. It can be verified that the above settings satisfy the steady-state condition. In particular, the queueing system is a classical M/M/1 queueing system if the feedback probability $\alpha = 0$, and the customer will definitely accept the twice service if the feedback probability $\alpha = 1$. In addition, 0.2, 0.5 and 0.8 are taken as the value of α to illustrate the trend of $E(L)$ versus λ for the feedback probability is small, equal, and larger than the non-feedback probability, respectively.

Figure 3 shows that $E(L)$ increases with the increase of λ. In other words, λ and $E(L)$ are positively correlated. $E(L)$ increases with the increase of feedback probability α, and the growth trend is more and more significant, which is consistent with our actual cognition.

In Figure 4, we illustrate the trend of the mean waiting time with the change of μ_2. Let the feedback probability $\alpha = 0.5$, the arrival rate λ equal to 1 and 2, the first service rate μ_1 equal to 4 and 5, respectively. And the range of values of μ_2 is the interval value of $4 \leq \mu_2 \leq 5$. It is verified that the above values are consistent with the steady-state condition.

Figure 4 shows that when μ_1 and λ are constant values, the mean waiting time decreases with the increase of μ_2. When μ_1 and μ_2 are constant, the mean waiting time increases with the increase of λ. When μ_2 and λ are constant, the mean waiting time decreases with the increase of μ_1. These characters are consistent with our intuition.

We study the trend of the mean sojourn time with the change of μ_1 and μ_2 in Figure 5. Assuming $\alpha = 0.8$ and $\lambda = 1$, the range of variation of μ_1 is $4 \leq \mu_1 \leq 5$, and the range of variation of μ_2 is $6 \leq \mu_2 \leq 7$. The above assumptions are calculated to be consistent with the steady-state conditions.

Figure 5 depicts the effect of service rates μ_1 and μ_2 on the mean sojourn time. From Figure 5, we can observe that the service rate and the mean sojourn time are negatively related. When μ_2 is invariant, the mean sojourn time is gradually decreasing as μ_1 increases; when μ_1 is invariant, the mean sojourn time is also gradually decreasing as μ_2 increases, but the effect of μ_1 on the mean sojourn time is more obvious. These characters are consistent with the actual ones.

Figure 6 depicts the trend of three state probabilities of

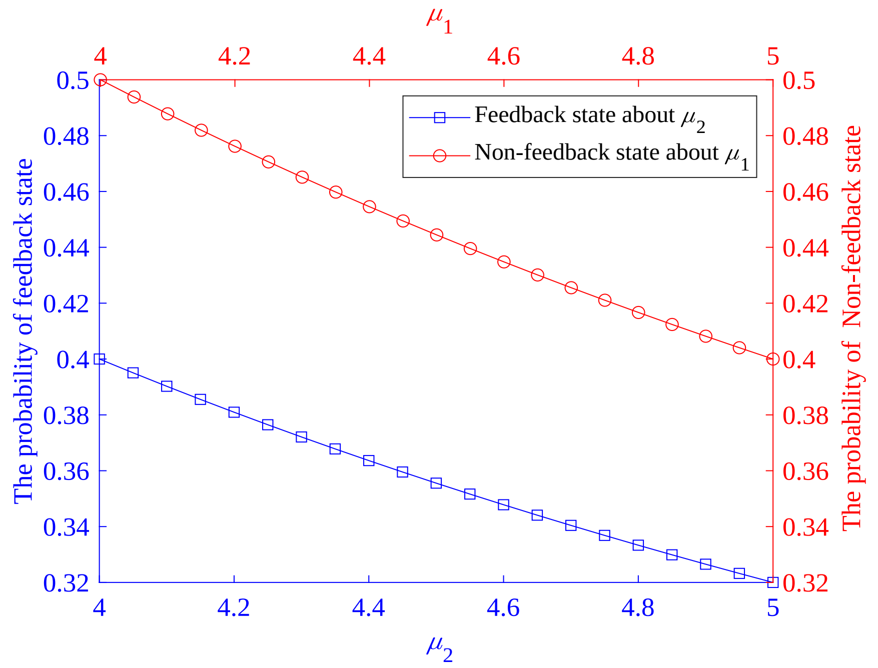

Fig. 7. The trend of the probability of the state system versus service rate ($\lambda = 2$ and $\alpha = 0.8$).

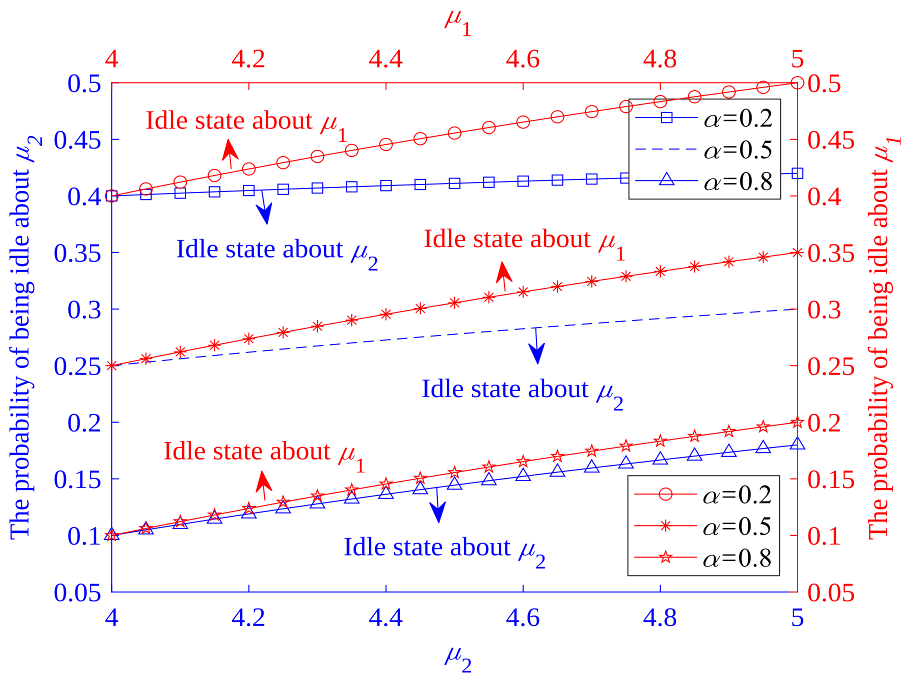

Fig. 8. The trend of the probability of the system being idle versus service rate ($\lambda = 2$, $\alpha = 0.2, 0.5$ and 0.8).

the system with the change of λ. Let's assume $\mu_1 = 3, \mu_2 = 4, \alpha = 0.5$, and the arrival rate varies in the range $0 \leq \lambda \leq 2$. After calculating, the above setting is to satisfy the steady-state condition.

From Figure 6, it can be seen that as λ increases, the probability of the system being in idle state decreases, the probability of being in feedback state and Non-feedback state increases. Moreover, the probability of being in Non-feedback state is greater than the probability of being in the feedback state. This is in line with the actual situation. For example, when patients are queueing for registration in a hospital, as the arrival rate of patients increases, i.e. the number of customers arriving per unit of time increases, the pressure on the system increases, and the probability of the service station being idle becomes smaller and smaller. At the same time, customers will not necessarily receive the second service, but each customer will receive the first service. So, the probability of the system being in Non-feedback state is greater than the probability of feedback state.

In Figure 7, we illustrate the trend of feedback and Non-feedback state probabilities with respect to μ_2 and μ_1 respectively. In Figure 8, we study the trend of the idle state probability with respect to μ_1 and μ_2.

From Figure 7, it can be seen that the probability of the system being in feedback state decreases as μ_2 grows and Non-feedback probability decreases as μ_1 grows. Figure 8 depicts the probability of the system being in idle state increases with the growth of μ_1 and μ_2. Moreover, as the probability of a customer initiating feedback increases, the probability of the system being in idle state decreases, which is realistic.

VII. Socially optimal strategies

It is clear from the numerical analysis that a change in each parameter in the model will have an impact on the queueing system. From a strategy point of view, however, the only controllable parameter is the service rate. Therefore, we will study the relationship between the following policy functions and the service rate. Let $\lambda = 2$, $\alpha = 0.5$, the value range of μ_1 is $4 \le \mu_1 \le 5$, and the value range of μ_2 is $4 \le \mu_2 \le 5$. After verification, the above settings are satisfied with the steady-state condition.

A. *Mean cost function*

The mean cost function per unit of time is established according to the above performance indicators, and the impact of changes of parameters on the cost is observed to minimize the cost as much as possible. The expected value of the total cost per unit of time is equal to the sum of the operating cost of the service station and the waiting cost of the customers.

Let Z be the expected value of all costs per unit of time, C_1 be the mean cost per unit time of the service station, C_2 be the cost per unit time of one customer spends in the system, it is the waiting cost, and $E(L)$ be the mean number of customers in the system, it is the mean queueing length. Then we have

$$Z = C_1 + C_2 \frac{\lambda\left(-\alpha^2\mu_1\lambda - \alpha\lambda\mu_2 + {\mu_2}^2 + \alpha\mu_1\left(\lambda + \mu_2\right)\right)}{\mu_2\left(-\alpha\mu_1\lambda + \left(\mu_1 - \lambda\right)\mu_2\right)}. \tag{15}$$

Let $C_1 = 10$, $C_2 = 5$. In Figure 9, we can see that the mean cost per unit of time is negatively related to the service rate. When μ_2 is constant, the mean cost per unit of time decreases with the increase of μ_1. When μ_1 is constant, the mean cost per unit of time decreases as μ_2 increases. From Figure 9, we can also observe that the mean cost per unit of time decreases more significantly as μ_1 increases, which means that μ_1 has a more significant impact on the mean cost per unit of time. Therefore, to keep the cost at the lowest, the service rate μ_1 must be increased as much as possible.

B. *Mean revenue function of the service station*

If the waiting cost of customers is not taken into account, the mean revenue of service station per unit of time can be expressed as $M = K\lambda - e\frac{\lambda}{\mu}$, where K is the revenue of service station serving one customer, and e is the cost of per unit time of the service station serving for customers. Supposing that when a customer receives the service for the first time, the cost of per unit time of the service station is e_1, when a customer receives the secondary service, the cost of per unit time of the service station is e_2. Then the cost per unit time of the service station can be expressed as $e_1(1-\alpha)\lambda\frac{1}{\mu_1} + e_1\alpha\lambda\frac{1}{\mu_1} + e_2\alpha\lambda\frac{1}{\mu_2}$. According to the revenue expression, the mean revenue of the service station is

$$M = K\lambda - [e_1(1-\alpha)\lambda\frac{1}{\mu_1} + e_1\alpha\lambda\frac{1}{\mu_1} + e_2\alpha\lambda\frac{1}{\mu_2}]. \tag{16}$$

In practice, the magnitude of e_1 and e_1 is uncertain. Here we let $K = 50$, $e_1 = 10$, $e_2 = 12$. From Figure 10, we find that the greater of the values of μ_1 and μ_2, the higher the mean revenue value, and the service rate μ_1 has a greater impact on the mean revenue of the service station.

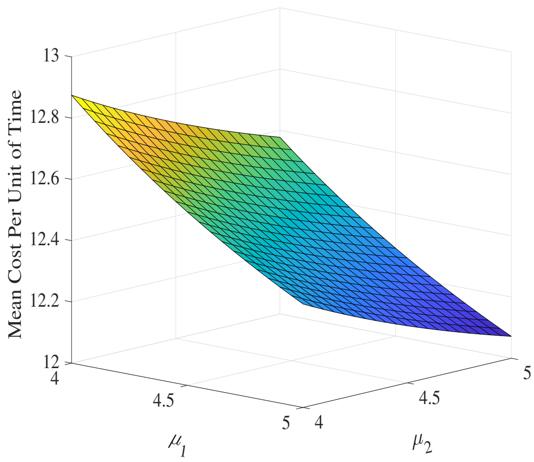

Fig. 9. The trend of mean cost per unit of time Z versus μ_1 and μ_2 ($\alpha = 0.5$, $\lambda = 2$, $C_1 = 10$ and $C_2 = 5$).

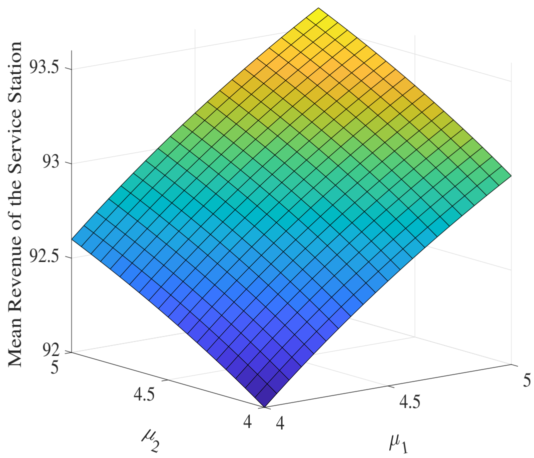

Fig. 10. The trend of mean revenue of service providers versus μ_1 and μ_2 ($\alpha = 0.5$, $\lambda = 2$, $K = 50$, $e_1 = 10$ and $e_2 = 12$).

C. *Individual expected income function*

Suppose that a customer receives F units of revenue after the service is completed and spends G units per unit of time to stay in the system. Let U represent the expected net revenue of the customer, then $U = F - GS$, and S is the mean sojourn time.

$$\begin{aligned} U =& F - G\{[\frac{\lambda\left(-\alpha^2\mu_1\lambda - \alpha\lambda\mu_2 + {\mu_2}^2 + \alpha\mu_1\left(\lambda + \mu_2\right)\right)}{\mu_2\left(-\alpha\mu_1\lambda + \left(\mu_1 - \lambda\right)\mu_2\right)} \\ &+ 1][1 - \alpha{\mu_1}^{-1} + \alpha{\mu_1}^{-1} + {\mu_2}^{-1}]\}. \end{aligned} \tag{17}$$

Let $F = 30$, $G = 10$. It can be seen from Figure 11 that individual expected income increases with the increase of service rate, and the first service rate μ_1 has a greater impact on individual expected income U, indicating that the larger μ_1 is, the higher the revenue will be.

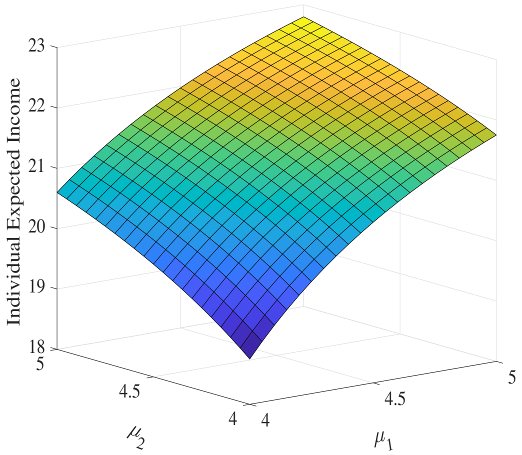

Fig. 11. The trend of individual expected income versus μ_1 and μ_2 ($\alpha = 0.5$, $\lambda = 2$, $F = 30$ and $G = 10$).

VIII. Conclusion

In this paper, we study a single service station queueing system with priority when customers need to receive secondary service and variable service rates after feedback. We obtain the steady-state probability distribution of the system state by using the matrix geometric solution method of the two-dimensional Markov process. Moreover, we get the performance indexes of the system from the probability distribution of the system state. In addition, the trend of each performance index is simulated in specific cases to analyze the model more intuitively. Finally, in connection with the actual situation, three optimization functions were established from different perspectives, and after analysis, it was concluded that μ_1 has more influence on each function. Therefore, we should continuously improve the service rate of the service station for the first time customer receiving service as much as possible to make the cost minimized, the benefit maximized, and the social benefit reached the optimal value. According to the research in this paper, the model with variable service rates and feedback can be extended to multi-sever systems in the future.

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