Some Semigroups Characterized in Terms of Cubic Bipolar Fuzzy Ideals

Thiti Gaketem, Natthinee Deetae, Pannawit Khamrot

***Abstract*—In this paper, we give the concepts of cubic bipolar fuzzy subsemigroups and provide some properties of cubic bipolar fuzzy subsemigroups. We discuss the relationship between a subsemigroup and the characteristics of cubic bipolar fuzzy subsemigroups. The results reveal beneficial application of the characterization of regular, intra-regular, and semisimple semigroups in terms of cubic bipolar fuzzy ideals is very useful for applications. Moreover, we discuss the image and pre-image of cubic bipolar fuzzy subsemigroups.**

***Index Terms*—Cubic bipolar fuzzy subsemigroup, regular, intra-regular and semisimple semigroup**

I. Introduction

IN THIS section, some basic definitions are given as the follows.

A *subsemigroup* M of a semigroup F if $M^2 \subseteq M$. A *left (right) ideal* of a semigroup F if $FM \subseteq M$ ($MF \subseteq M$). An *ideal* of a semigroup F if it is a left ideal and a right ideal of F. A semigroup F is called a *regular* if for each $r \in F$, there exists $k \in S$ such that $r = rkr$.

For any $\nu_i \in [0,1]$ where $i \in \mathcal{K}$, define

$$\bigvee_{i\in\mathcal{K}} \nu_i := \sup_{i\in\mathcal{K}}\{\nu_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{K}} \nu_i := \inf_{i\in\mathcal{K}}\{\nu_i\}.$$

We note here that for any $\nu, \xi \in [0,1]$, we have

$$\nu \vee \xi = \max\{\nu, \xi\} \quad \text{and} \quad \nu \wedge \xi = \min\{\nu, \xi\}.$$

The theory of fuzzy sets was studied by Zadeh in 1965 [1], which he gave the definition as follows: A fuzzy set ω of a non-empty set F is a function from F into the closed interval $[0,1]$, i.e, $\omega : F \to [0,1]$.

The concept was applied in many areas such as robotics, computer science, medical science, theoretical physics, control engineering, information science, measure theory, logic, set theory, topology, etc. In 1979, Kuroki [2] used knowledge of a fuzzy set in semigroup theory and various kinds of ideals in semigroups and characterized them.

In 1975 Zadeh [3] was interested in interval valued fuzzy sets as an extension of fuzzy sets. He gave the concepts of interval-valued fuzzy sets as follows:

Let $CS[0,1]$ be the set of all closed subintervals of $[0,1]$, i.e.,

$$CS[0,1] = \{\overline{\nu} = [\nu_l, \nu_u] \mid 0 \le \nu_l \le \nu_u \le 1\},$$

where ν_l is a lower interval value of $\overline{\nu}$ and ν_u is an upper interval value of $\overline{\nu}$.

We note that $[\nu, \nu] = \{\nu\}$ for all $\nu \in [0,1]$. For $\nu = 0$ or 1, we shall denote $[0,0]$ by $\overline{0}$ and $[1,1]$ by $\overline{1}$.

For $\overline{\nu} := [\nu_l, \nu_u]$ and $\overline{\xi} := [\xi_l, \xi_u]$ in $CS[0,1]$, the operations "$\preceq$", "$=$", "$\curlywedge$", "$\curlyvee$" are defined as follows:

(1) $\overline{\nu} \preceq \overline{\xi}$ if and only if $\nu_l \le \xi_l$ and $\nu_u \le \xi_u$
(2) $\overline{\nu} = \overline{\xi}$ if and only if $\nu_l = \xi_l$ and $\nu_u = \xi_u$
(3) $\overline{\nu} \curlywedge \overline{\xi} = [(\nu_l \wedge \xi_l), (\nu_u \wedge \xi_u)]$
(4) $\overline{\nu} \curlyvee \overline{\xi} = [(\nu_l \vee \xi_l), (\nu_u \vee \xi_u)]$.
If $\overline{\nu} \succeq \overline{\xi}$, we mean $\overline{\xi} \preceq \overline{\nu}$.

The following proposition is a tool used to prove the next sections.

Proposition 1.1. *[4] For $\overline{\nu}, \overline{\xi}, \overline{\rho} \in CS[0,1]$, then the following properties are true:*

(1) $\overline{\nu} \curlywedge \overline{\nu} = \overline{\nu}$ *and* $\overline{\nu} \curlyvee \overline{\nu} = \overline{\nu}$,
(2) $\overline{\nu} \curlywedge \overline{\xi} = \overline{\xi} \curlywedge \overline{\nu}$ *and* $\overline{\nu} \curlyvee \overline{\xi} = \overline{\xi} \curlyvee \overline{\nu}$,
(3) $(\overline{\nu} \curlywedge \overline{\xi}) \curlywedge \overline{\rho} = \overline{\nu} \curlywedge (\overline{\xi} \curlywedge \overline{\rho})$ *and* $(\overline{\nu} \curlyvee \overline{\xi}) \curlyvee \overline{\rho} = \overline{\nu} \curlyvee (\overline{\xi} \curlyvee \overline{\rho})$,
(4) $(\overline{\nu} \curlywedge \overline{\xi}) \curlyvee \overline{\rho} = (\overline{\nu} \curlyvee \overline{\rho}) \curlywedge (\overline{\xi} \curlyvee \overline{\rho})$ *and* $(\overline{\nu} \curlyvee \overline{\xi}) \curlywedge \overline{\rho} = (\overline{\nu} \curlywedge \overline{\rho}) \curlyvee (\overline{\xi} \curlywedge \overline{\rho})$,
(5) *If* $\overline{\nu} \preceq \overline{\xi}$, *then* $\overline{\nu} \curlywedge \overline{\rho} \preceq \overline{\xi} \curlywedge \overline{\rho}$ *and* $\overline{\nu} \curlyvee \overline{\rho} \preceq \overline{\xi} \curlyvee \overline{\rho}$.

For each interval $\overline{\nu}_i := [(\nu_l)_i, (\nu_u)_i] \in CS[0,1]$, $i \in \mathcal{K}$, where $\mathcal{K}$ is an index set, we define

$$\curlywedge_{i\in\mathcal{K}} \overline{\nu}_i := [\bigwedge_{i\in\mathcal{K}} (\nu_l)_i, \bigwedge_{i\in\mathcal{K}} (\nu_u)_i] \quad \text{and} \quad \curlyvee_{i\in\mathcal{K}} \overline{\nu}_i := [\bigvee_{i\in\mathcal{K}} (\nu_l)_i, \bigvee_{i\in\mathcal{K}} (\nu_u)_i].$$

Definition 1.2. *[3] An interval valued fuzzy set (shortly, IVF set) of a non-empty set F is a function $\overline{\mu} : F \to CS[0,1]$.*

Definition 1.3. *[5] Let M be a subset of a non-empty set F. An interval valued characteristic function (shortly, IVCF) $\overline{\chi}_M$ of F is defined to be a function $\overline{\chi}_M : F \to CS[0,1]$ by*

$$\overline{\chi}_M(r) = \begin{cases} \overline{1} & \text{if} \quad r \in M \\ \overline{0} & \text{if} \quad r \notin M \end{cases}$$

for all $r \in F$.

For two IVF subsets $\overline{\mu}$ and $\overline{\lambda}$ of a non-empty set F, define

(1) $\overline{\mu} \sqsubseteq \overline{\lambda} \Leftrightarrow \overline{\mu}(r) \preceq \overline{\lambda}(r)$ for all $r \in F$,
(2) $\overline{\mu} = \overline{\lambda} \Leftrightarrow \overline{\mu} \sqsubseteq \overline{\lambda}$ and $\overline{\lambda} \sqsubseteq \overline{\mu}$,
(3) $(\overline{\mu} \sqcap \overline{\lambda})(r) = \overline{\mu}(r) \curlywedge \overline{\lambda}(r)$ for all $r \in F$.

For $r \in F$, define $A_r := \{(k, o) \in F \times F \mid r = ko\}$.

For two IVF sets $\overline{\mu}$ and $\overline{\lambda}$ of a semigroup F, define the product $\overline{\mu} \bigcirc \overline{\lambda}$ is defined as follows for all $r \in F$,

$$(\overline{\mu} \bigcirc \overline{\lambda})(r) = \begin{cases} \curlyvee_{(k,o)\in A_r} \{\overline{\mu}(k) \curlywedge \overline{\lambda}(o)\} & \text{if} \quad A_r \neq \emptyset \\ \overline{0} & \text{if} \quad A_r = \emptyset. \end{cases}$$

Definition 1.4. *[6] An IVF subset $\overline{\mu}$ of a semigroup F is said to be an* IVF subsemigroup *of F if $\overline{\mu}(r_1 r_2) \succeq \overline{\mu}(r_1) \curlywedge \overline{\mu}(r_2)$ for all $r_1, r_2 \in F$.*

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Definition 1.5. *[6] An IVF subset $\overline{\mu}$ of a semigroup F is said to be an* IVF left (right) ideal *of F if* $\overline{\mu}(r_1r_2) \succeq \overline{\mu}(r_2)$ $(\overline{\mu}(r_1r_2) \succeq \overline{\mu}(r_1))$ *for all $r_1, r_2 \in F$.*
An IVF subset $\overline{\mu}$ of a semigroup F is called an IVF ideal *of F if it is both an IVF left ideal and an IVF right ideal of S.*

In 1994, Zhang [7] introduced the notion of bipolar fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. It is the extension of fuzzy sets and used for decision analysis, modeling, and algebraic structures. In 2000, Lee [8] used the term bipolar valued fuzzy sets and applied it to algebraic structures.

Definition 1.6. *A* bipolar fuzzy set *(shortly, BF set) ω on F is an object having the form*

$$\omega := \{(r, \omega^p(r), \omega^n(r)) \mid r \in F\},$$

where $\omega^p : F \to [0, 1]$ and $\omega^n : F \to [-1, 0]$.

Remark 1.7. *For the sake of simplicity we shall use the symbol $\omega = (F; \omega^p, \omega^n)$ for the BF set $\omega = \{(r, \omega^p(r), \omega^n(r)) \mid r \in F\}$.*

The products $\omega^p * \psi^p$ and $\omega^n * \psi^n$ were defined as follows: For $r \in F$

$$(\omega^p * \psi^p)(r) = \begin{cases} \bigvee\limits_{(k,o)\in A_r} \{\omega^p(k) \wedge \psi^p(o)\} & \text{if} \quad A_r \neq \emptyset \\ 0 & \text{if} \quad A_r = \emptyset \end{cases}$$

and

$$(\omega^n * \psi^n)(r) = \begin{cases} \bigwedge\limits_{(k,o)\in A_r} \{\omega^n(k) \vee \psi^n(o)\} & \text{if} \quad A_r \neq \emptyset \\ 0 & \text{if} \quad A_r = \emptyset, \end{cases}$$

where $A_r := \{(k, o) \in F \times F \mid r = ko\}$.

Definition 1.8. *Let M be a non-empty set of a semigroup F. A* positive characteristic function and a negative characteristic function *are respectively defined as*

$$\chi^p_M : F \to [0, 1], u \mapsto \chi^p_M(r) := \begin{cases} 1 & r \in M \\ 0 & r \notin M \end{cases}$$

and

$$\chi^n_M : F \to [-1, 0], u \mapsto \chi^n_M(r) := \begin{cases} -1 & r \in M \\ 0 & r \notin M. \end{cases}$$

Definition 1.9. *[9] A BF set $\omega = (F; \omega^p, \omega^n)$ on a semigroup F is called a* BF subsemigroup *on F if $\omega^p(r_1r_2) \geq \omega^p(r_1) \wedge \omega^p(r_2)$ and $\omega^n(r_1r_2) \leq \omega^n(r_1) \vee \omega^n(r_2)$ for all $r_1, r_2 \in F$.*

Definition 1.10. *[9] A BF set $\omega = (F; \omega^p, \omega^n)$ on a semigroup F is called a* BF left (right) ideal *on F if $\omega^p(r_1r_2) \geq \omega^p(r_2)$ $(\omega^p(r_1r_2) \geq \omega^p(r_1))$ and $\omega^n(r_1r_2) \leq \omega^n(r_2)$ $(\omega^n(r_1r_2) \leq \omega^n(r_1))$ for all $r_1, r_2 \in F$.*

Moreover, in 2017, Kavikumar et al. [10] used acknowledge of a BF set to finite switchboard state machines.

In 2018, Wei et al. [11] studied the concept of interval valued bipolar fuzzy set with a generalization of BF set. It was a study of values of positive and negative functions.

Definition 1.11. *An* interval valued bipolar fuzzy set *(shortly, IVBF set) $\mathfrak{C} = (F; \overline{\mu}^p, \overline{\mu}^n)$ of a non-empty set F if* $\overline{\mu}^p : F \to CS[0, 1]$ *and* $\overline{\mu}^n : F \to CS[-1, 0]$.

In 2012, Jun et al. [12] introduced a new notion, known as a cubic set, and investigated several properties of cubic sets. They introduced cubic subsemigroups and cubic left (right) ideals of semigroups as follows.

Definition 1.12. *[12] A* cubic set *$\mathcal{C}$ of a non-empty set F is a structure of the form*

$$\mathcal{C} = \{\langle r, \overline{\mu}(r), \omega(r)\rangle \mid r \in F\}$$

and denoted by $\mathcal{C} = \langle \overline{\mu}, \omega\rangle$ where $\overline{\mu}$ is an IVF set and ω is a fuzzy set. In this case, we will use

$$\mathcal{C}(r) = \langle \overline{\mu}(r), \omega(r)\rangle = \langle [\mu_l(r), \mu_u(r)], \omega(r)\rangle$$

for all $r \in F$.

Definition 1.13. *[12] A cubic set $\mathcal{C} = \langle \overline{\mu}, \omega\rangle$ of a semigroup F is called*

(1) *a* cubic subsemigroup *of F if $\overline{\mu}(r_1r_2) \succeq \overline{\mu}(r_1) \curlywedge \overline{\mu}(r_2)$ and $\omega(r_1r_2) \leq \omega(r_1) \vee (r_2)$ for all $r_1, r_2 \in F$,*
(2) *a* cubic left*(right)ideal of F if $\overline{\mu}(r_1r_2) \succeq \overline{\mu}(r_2)$ $(\overline{\mu}(r_1r_2) \succeq \overline{\mu}(r_1))$ and $\omega(r_1r_2) \leq \omega(r_2)$($\omega(r_1r_2) \leq \omega(r_1)$) for all $r_1, r_2 \in F$.*
A cubic ideal *of F if it is both a cubic left ideal and a cubic right ideal of F.*

Riaz and Tehrim [13] discussed the concept of cubic bipolar fuzzy sets and applied it to decision-making and problem solving such that [14], [15], [16].

In this paper, we consider the concepts of cubic bipolar fuzzy subsemigroups and ideals. We provide properties of cubic bipolar fuzzy subsemigroups and ideals. The regular, intra-regular, and semisimple semigroups are characterized in terms of cubic bipolar fuzzy ideals.

II. Cubic Bipolar Fuzzy Subsemigroup and Ideals in Semigroups

In this section, we give the concepts of cubic bipolar fuzzy subsemigroups and ideals in semigroups. Also, we study the important properties for reference in the next part.

Definition 2.1. *A* cubic bipolar fuzzy set *(shortly, CBF set) $\ddot{\mathfrak{C}}$ of a set F if*

$$\ddot{\mathfrak{C}} = \{\langle r, (\overline{\mu}^p(r), \overline{\mu}^n(r)), (\omega^n(r), \omega^p(r))\rangle \mid r \in F\}$$

and denoted by $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega\rangle$ where $\overline{\mu} = (F; \overline{\mu}^p, \overline{\mu}^n)$ is an IVBF set and $\omega = (F; \omega^n, \omega^p)$ is a BF set.

Definition 2.2. *A CBF set $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega\rangle$ of a semigroup F is called a* cubic bipolar fuzzy subsemigroup *(shortly, CBF subsemigroup) of F if* $\overline{\mu}^p(r_1r_2) \succeq \overline{\mu}^p(r_1) \curlywedge \overline{\mu}^p(r_2)$, $\overline{\mu}^n(r_1r_2) \preceq \overline{\mu}^n(r_1) \curlyvee \overline{\mu}^n(r_2)$ *and* $\omega^p(r_1r_2) \geq \omega^p(r_1) \wedge \omega^p(r_2)$, $\omega^n(r_1r_2) \leq \omega^n(r_1) \vee \omega^n(r_2)$ *for all $r_1, r_2 \in F$.*

The following example satisfies definition 2.2.

Example 2.3. *Let F be a semigroup defined by the following table:*

$\cdot$	a	b	c
a	a	b	c
b	b	b	c
c	c	c	b

A CBF set $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega\rangle$ in F as follows: $\overline{\mu}^p(a) = [0.6, 0.7]$, $\overline{\mu}^p(b) = [0.4, 0.5]$, $\overline{\mu}^p(c) = [0.1, 0.2]$, $\overline{\mu}^n(a) =$

$[-0.9, -0.8]$, $\overline{\mu}^n(b) = [-0.7, -0.6]$, $\overline{\mu}^p(c) = [-0.3, -0.2]$ *and* $\omega^p(a) = 0.7$, $\omega^p(b) = 0.4$, $\omega^p(c) = 0.2$, $\omega^n(a) = -0.7$, $\omega^n(b) = -0.3$, $\omega^n(c) = -0.2$ *Thus* $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ *is a CBF subsemigroup of* F.

Definition 2.4. *A CBF set* $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ *of a semigroup* F *is called a* cubic bipolar fuzzy left ideal *(shortly, CBF left ideal) of* F *if*
$\overline{\mu}^n(r_1r_2) \succeq \overline{\mu}^p(r_2)$, $\overline{\mu}^n(r_1r_2) \preceq \overline{\mu}^n(r_2)$ *and*
$\omega^p(r_1r_2) \geq \omega^p(r_2)$, $\omega^n(r_1r_2) \leq \omega^n(r_2)$ *for all* $r_1, r_2 \in F$.

Definition 2.5. *A CBF set* $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ *of a semigroup* F *is called a* cubic bipolar fuzzy right ideal *(shortly, CBF rihgt ideal) of* F *if*
$\overline{\mu}^p(r_1r_2) \succeq \overline{\mu}^p(r_1)$, $\overline{\mu}^n(r_1r_2) \preceq \overline{\mu}^n(r_1)$ *and*
$\omega^p(r_1r_2) \geq \omega^p(r_1)$, $\omega^n(r_1r_2) \leq \omega^n(r_1)$ *for all* $r_1, r_2 \in F$.

Definition 2.6. *A cubic bipolar set* $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ *of a semigroup* F *is called a* cubic cubic biopolar fuzzy ideal *(shortly, CBF ideal) of* F *if it is a CBF left ideal and a CBF right ideal of* F.

The following example is a CBF ideal of a semigroup.

Example 2.7. *Let* $F = \{a, b, c\}$ *be a semigroup with the following Cayley table:*

$\cdot$	a	b	c
a	a	b	c
b	b	b	b
c	c	b	b

A CBF set $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ *in* F *as follows:* $\overline{\mu}^p(a) = [0.1, 0.3]$, $\overline{\mu}^p(b) = [0.2, 0.4]$, $\overline{\mu}^p(c) = [0.1, 0.4]$, $\overline{\mu}^n(a) = [-0.1, -0.3]$, $\overline{\mu}^n(b) = [-0.2, -0.4]$, $\overline{\mu}^n(c) = [-0.1, -0.4]$ *and* $\omega^p(a) = 0.1$, $\omega^p(b) = 0.4$, $\omega^p(c) = 0.5$, $\omega^n(a) = -0.1$, $\omega^n(b) = -0.3$, $\omega^n(c) = -0.4$. *Thus* $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ *is a CBF ideal of* F.

Next, we study the intersection of CBF set as defined.
Let $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ are CBF sets of F. Define $\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 = \langle \overline{\mu} \sqcap \overline{\lambda}, \omega \cap \psi \rangle$ where
$(\overline{\mu}^p \sqcap \overline{\lambda}^p)(r) = \overline{\mu}^p(r) \curlywedge \overline{\lambda}^p(r)$, $(\overline{\mu}^n \sqcap \overline{\lambda}^n)(r) = \overline{\mu}^n(r) \curlyvee \overline{\lambda}^n(r)$ and
$(\omega^p \cap \psi^p)(r) = \omega^p(r) \wedge \psi^p(r)$, $(\omega^n \cap \psi^n)(r) = \omega^n(r) \vee \psi^n(r)$ for all $r \in F$.

The following lemma shows the positive of an intersection and the negative of a union of two CBF subsemigroups and ideals.

Lemma 2.8. *Let* F *be a semigroup. Then the following properties hold:*

1. *The positive of an intersection of two CBF subsemigroups of a semigroup* F *is a positive of CBF subsemigroup of* F.
2. *The positive of an intersection of two CBF left (right) ideals of a semigroup* F *is a positive of CBF left (right) ideal of* F.

Proof:

1. Assume that $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ are positive of a CBF subsemigroups of F and let $r_1, r_2 \in F$. Then
$(\overline{\mu}^p \sqcap \overline{\lambda}^p)(r_1r_2) = (\overline{\mu}^p(r_1r_2) \curlywedge \overline{\lambda}^p(r_1r_2)$
$\succeq (\overline{\mu}^p(r_1) \curlywedge \overline{\mu}^p(r_2)) \curlywedge (\overline{\lambda}^p(r_1) \curlywedge \overline{\mu}^p(r_2))$
$= (\overline{\mu}^p(r_1) \curlywedge \overline{\lambda}^p(r_1)) \curlywedge (\overline{\lambda}^p(r_2) \curlywedge \overline{\lambda}^p(r_2))$
$= (\overline{\mu}^p \sqcap \overline{\lambda}^p)(r_1) \curlywedge (\overline{\mu}^p \sqcap \overline{\lambda}^p)(r_2)$.
And
$(\omega^p \cap \psi^p)(r_1r_2) = (\omega^p(r_1r_2) \wedge \psi^p(r_1r_2))$
$\geq (\omega^p(r_1) \wedge \omega^p(r_2)) \wedge (\psi^p(r_1) \wedge \psi^p(r_2))$
$= (\omega^p(r_1) \wedge \psi^p(r_1)) \wedge (\omega^p(r_2) \wedge \psi^p(r_2))$
$= (\omega^p \cap \psi^p)(r_1) \wedge (\omega^p \cap \psi^p)(r_2)$.
Thus, $(\overline{\mu}^p \sqcap \overline{\lambda}^p)(r_1r_2) \succeq (\overline{\mu}^p \sqcap \overline{\lambda}^p)(r_1) \curlywedge (\overline{\mu}^p \sqcap \overline{\lambda}^p)(r_2)$, and $(\omega^p \cap \psi^p)(r_1r_2) \geq (\omega^p \cap \psi^p)(r_1) \wedge (\omega^p \sqcap \psi^p)(r_2)$.
Hence the positive of an intersection is a CBF subsemigroup of F.
The second can be proved in a similar way as used in the first.

■

Lemma 2.9. *Let* F *be a semigroup. Then the following properties hold.*

1. *The negative of an intersection of two CBF subsemigroups of a semigroup* F *is a negative of CBF subsemigroup of* F.
2. *The negative of an intersection of two CBF left (right) ideals of a semigroup* F *is a negative of CBF left (right) ideals of* F.

Proof:

1. Assume that $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ are negative of a CBF subsemigroups of F, and let $r_1, r_2 \in F$. Then
$(\overline{\mu}^n \sqcap \overline{\lambda}^n)(r_1r_2) = \overline{\mu}^n(r_1r_2) \curlyvee \overline{\lambda}^n(r_1r_2)$
$\preceq (\overline{\mu}^n(r_1) \curlyvee \overline{\mu}^n(r_2)) \curlyvee (\overline{\lambda}^n(r_1) \curlyvee \overline{\mu}^n(r_2))$
$= (\overline{\mu}^n(r_1) \curlyvee \overline{\lambda}^n(r_1)) \curlyvee (\overline{\lambda}^n(r_2) \curlyvee \overline{\lambda}^n(r_2))$
$= (\overline{\mu}^n \sqcap \overline{\lambda}^n)(r_1) \curlyvee (\overline{\mu}^n \sqcap \overline{\lambda}^n)(r_2)$.
And
$(\omega^n \cap \psi^n)(r_1r_2) = \omega^n(r_1r_2) \vee \psi^n(r_1r_2)$
$\leq (\omega^n(r_1) \vee \omega^n(r_2)) \vee (\psi^n(r_1) \vee \overline{\mu}^n(r_2))$
$= (\omega^n((r_1) \vee \psi^n(r_1)) \curlyvee (\psi^n(r_2) \vee \psi^n(r_2))$
$= (\omega^n \cap \psi^n)(r_1)) \vee (\omega^n \cap \psi^n)(r_2))$.
Thus, $(\overline{\mu}^n \sqcap \overline{\lambda}^n)(r_1r_2) \preceq (\overline{\mu}^n \sqcap \overline{\lambda}^n)(r_1) \curlyvee (\overline{\mu}^n \sqcap \overline{\lambda}^n)(r_2)$ and $(\omega^n \cap \psi^n)(r_1r_2) \leq (\omega^n \cap \psi^n)(r_1) \vee (\omega^n \cap \psi^n)(r_2)$.
Hence the negative of an intersetion is a CBF subsemigroup of F.
The second can be proved in a similar way as used in the first.

■

The following result is an immediate consequence of Lemma 2.8 and Lemma 2.9.

Theorem 2.10. *Let* F *be a semigroup. Then the following properties hold:*

1. *The intersection of two CBF subsemigroups of a semigroup* F *is a CBF subsemigroup of* F.
2. *The intersection of two CBF left (right) ideals of a semigroup* F *is a CBF left (right) ideal of* F.

Next, we provide the definition of the characteristic cubic bipolar fuzzy function. Let M be a non-empty subset of F. The characteristic cubic bipolar fuzzy set (shortly, CCBF set) $\chi_M = \langle \overline{\mu}_{\chi_M}, \omega_{\chi_M} \rangle$ is defined as follows:

$$\overline{\mu}^p_{\chi_M}(r) = \begin{cases} \overline{1} & \text{if} \quad r \in M \\ \overline{0} & \text{if} \quad r \notin M \end{cases}, \quad \overline{\mu}^n_{\chi_M}(r) = \begin{cases} \overline{-1} & \text{if} \quad r \in M \\ \overline{0} & \text{if} \quad r \notin M \end{cases}$$

for all $r \in F$ and ω_{χ_M} are characteristics bipolar fuzzy set.

In the following lemmas, we give a relationship between a subsemigroup (left ideal, right ideal, ideal) and the CCBF function.

Lemma 2.11. *If M is a subsemigroup of a semigroup F, then the CCBF function $\chi_M = \langle \overline{\mu}_{\chi_M}, \omega_{\chi_M} \rangle$ is a CBF subsemigroup of F.*

Proof: Suppose that M is a subsemigroup of F and let $r_1, r_2 \in F$.

If $r_1, r_2 \in M$, then $r_1 r_2 \in M$. Thus, $\overline{1} = \overline{\mu}^p_{\chi_M}(r_1) = \overline{\mu}^p_{\chi_M}(r_2) = \overline{\mu}^p_{\chi_M}(r_1 r_2)$, $\overline{-1} = \overline{\mu}^n_{\chi_M}(r_1) = \overline{\mu}^n_{\chi_M}(r_2) = \overline{\mu}^n_{\chi_M}(r_1 r_2)$ and $1 = \omega^p_{\chi_M}(r_1) = \omega^p_{\chi_M}(r_2) = \omega^p_{\chi_M}(r_1 r_2)$, $-1 = \omega^n_{\chi_M}(r_1) = \omega^n_{\chi_M}(r_2) = \omega^n_{\chi_M}(r_1 r_2)$.
Hence, $\overline{\mu}^p_{\chi_M}(r_1 r_2) \succeq \overline{\mu}^p_{\chi_M}(r_1) \curlywedge \overline{\mu}^p_{\chi_M}(r_2)$, $\overline{\mu}^n_{\chi_M}(r_1 r_2) \preceq \overline{\mu}^n_{\chi_M}(r_1) \curlyvee \overline{\mu}^n_{\chi_M}(r_2)$ and $\omega^p_{\chi_M}(r_1 r_2) \geq \omega^p_{\chi_M}(r_1) \wedge \omega^p_{\chi_M}(r_2)$, $\omega^n_{\chi_M}(r_1 r_2) \leq \omega^n_{\chi_M}(r_1) \vee \omega^n_{\chi_M}(r_2)$.

If $r_1 \notin M$ or $r_2 \notin M$, then
$\overline{\mu}^p_{\chi_M}(r_1 r_2) \succeq \overline{\mu}^p_{\chi_M}(r_1) \curlywedge \overline{\mu}^p_{\chi_M}(r_2)$,
$\overline{\mu}^n_{\chi_M}(r_1 r_2) \preceq \overline{\mu}^n_{\chi_M}(r_1) \curlyvee \overline{\mu}^n_{\chi_M}(r_2)$ and
$\omega^p_{\chi_M}(r_1 r_2) \geq \omega^p_{\chi_M}(r_1) \wedge \omega^p_{\chi_M}(r_2)$,
$\omega^n_{\chi_M}(r_1 r_2) \leq \omega^n_{\chi_M}(r_1) \vee \omega^n_{\chi_M}(r_2)$.
Thus $\chi_M = \langle \overline{\mu}_{\chi_M}, \omega_{\chi_M} \rangle$ is a CBF subsemigroup of F. ■

Lemma 2.12. *If $\chi_M = \langle \overline{\mu}_{\chi_M}, \omega_{\chi_M} \rangle$ is a CBF subsemigroup of F, then M is a subsemigroup of a semigroup F.*

Proof: Suppose that $\chi_M = \langle \overline{\mu}_{\chi_M}, \omega_{\chi_M} \rangle$ is a CBF subsemigroup of F, and let $r_1, r_2 \in M$. Then $\overline{\mu}^p_{\chi_M}(r_1) = \overline{\mu}^p_{\chi_M}(r_2) = \overline{1}$, $\overline{\mu}^n_{\chi_M}(r_1) = \overline{\mu}^n_{\chi_M}(r_2) = \overline{-1}$ and $\omega^p_{\chi_M}(r_1) = \omega^p_{\chi_M}(r_2) = 1$, $\omega^n_{\chi_M}(r_1) = \omega^n_{\chi_M}(r_2) = -1$. By assumption,

$$\begin{cases} \overline{\mu}^p_{\chi_M}(r_1 r_2) \succeq \overline{\mu}^p_{\chi_M}(r_1) \curlywedge \overline{\mu}^p_{\chi_M}(r_2), \\ \overline{\mu}^n_{\chi_M}(r_1 r_2) \preceq \overline{\mu}^n_{\chi_M}(r_1) \curlyvee \overline{\mu}^n_{\chi_M}(r_2) \text{ and} \\ \omega^p_{\chi_M}(r_1 r_2) \geq \omega^p_{\chi_M}(r_1) \wedge \omega^p_{\chi_M}(r_2), \\ \omega^n_{\chi_M}(r_1 r_2) \leq \omega^n_{\chi_M}(r_1) \vee \omega^n_{\chi_M}(r_2). \end{cases} \quad (1)$$

If $r_1 r_2 \notin M$, then by (1) $\overline{0} \succeq \overline{1}$, $\overline{0} \preceq \overline{-1}$ and $0 \geq 1$, $0 \leq -1$. It is a contradiction. Hence $r_1 r_2 \in M$. Therefore, M is a subsemigroup of F. ■

The following result is an immediate consequence of Lemma 2.11 and Lemm 2.12.

Theorem 2.13. *Let M be a non-empty subset of a semigroup F. Then M is a subsemigroup of F if and only if $\chi_M = \langle \overline{\mu}_{\chi_M}, \omega_{\chi_M} \rangle$ is a CBF subsemigroup of F.*

Lemma 2.14. *If M is a left (right) of a semigroup F, then the CCBF function $\chi_M = \langle \overline{\mu}_{\chi_M}, \omega_{\chi_M} \rangle$ is a CBF left (right) of F.*

Lemma 2.15. *If $\chi_M = \langle \overline{\mu}_{\chi_M}, \omega_{\chi_M} \rangle$ is a CBF left (right) of F, then M is a left (right) of a semigroup F.*

The following result is an immediate consequence of Lemma 2.14 and Lemm 2.15.

Theorem 2.16. *Let M be a non-empty subset of a semigroup F. Then M is a left (right) ideal of F if and only if $\chi_M = \langle \overline{\mu}_{\chi_M}, \omega_{\chi_M} \rangle$ is a CBF left (right) of F.*

The following definition is of the $(\overline{s}, \overline{t})$-level and (s, t)-level subset of a CBF set.

Definition 2.17. *Let $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ be a CBF set with $(\overline{s}, \overline{t}) \in CS[-1, 0] \times CS[0, 1]$ and $(s, t) \in [-1, 0] \times [0, 1]$. Define the set $\mathcal{U}^{(\overline{t}, \overline{s})}_{\overline{\mu}} = \{r \in F \mid \overline{\mu}^p(r) \succeq \overline{t}, \overline{\mu}^n(r) \preceq \overline{s}\}$ and $\mathcal{U}^{(t,s)}_{\omega} = \{r \in F \mid \omega^p(r) \geq t, \omega^n(r) \leq s\}$ is called and $(\overline{s}, \overline{t})$-**level** and (s, t)-**level subset** of a CBF set of F.*

In the following theorems, we give a relationship between a subsemigroup (left ideal, right ideal, ideal) and the $(\overline{s}, \overline{t})$-level and (s, t)-level subset of a CBF set.

Theorem 2.18. *A CBF set $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF subsemigroup of a semigroup F if and only if the level sets $\mathcal{U}^{(\overline{t}, \overline{s})}_{\overline{\mu}}$ and $\mathcal{U}^{(t,s)}_{\omega}$ are subsemigroups of F for all*
$(\overline{s}, \overline{t}) \in CS[-1, 0] \times CS[0, 1]$ and $(s, t) \in [-1, 0] \times [0, 1]$.

Proof: Let $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ be a CBF subsemigroup of F and let $r_1, r_2 \in F, (\overline{s}, \overline{t}) \in CS[-1, 0] \times CS[0, 1]$ and $(s, t) \in [-1, 0] \times [0, 1]$.

If r_1, r_2 are elements of $\mathcal{U}^{(\overline{t}, \overline{s})}_{\overline{\mu}}$ and $\mathcal{U}^{(t,s)}_{\omega}$, then $\overline{\mu}^p(r_1) \succeq \overline{t}$, $\overline{\mu}^p(r_2) \succeq \overline{t}, \overline{\mu}^n(r_1) \preceq \overline{s}$, $\overline{\mu}^n(r_1) \preceq \overline{s}$ and $\omega^p(r_1) \geq t, \omega^p(r_2) \geq t$, $\omega^n(r_1) \leq s, \omega^n(r_2) \leq s$. By assumption,
$\overline{\mu}^p(r_1 r_2) \succeq \overline{\mu}^p(r_1) \curlywedge \overline{\mu}^p(r_2), \overline{\mu}^p(r_1 r_2) \preceq \overline{\mu}^p(r_1) \curlyvee \overline{\mu}^p(r_2)$ and $\omega^p(r_1 r_2) \geq \omega^p(r_1) \wedge \omega^p(r_2)$, $\omega^n(r_1 r_2) \leq \omega^n(r_1) \vee \omega^p(r_2)$.
Thus $r_1 r_2$ is an element of $\mathcal{U}^{(\overline{t}, \overline{s})}_{\overline{\mu}}$ and $\mathcal{U}^{(t,s)}_{\omega}$.
Hence $\mathcal{U}^{(\overline{t}, \overline{s})}_{\overline{\mu}}$ and $\mathcal{U}^{(t,s)}_{\omega}$ are subsemigroups of F.

Conversely, suppose that $\mathcal{U}^{(\overline{t}, \overline{s})}_{\overline{\mu}}$ and $\mathcal{U}^{(t,s)}_{\omega}$ are subsemigroups of F. Let $r_1, r_2 \in F, (\overline{s}, \overline{t}) \in CS[-1, 0] \times CS[0, 1]$ and $(s, t) \in [-1, 0] \times [0, 1]$.
By assumption, $r_1 r_2$ is an element of $\mathcal{U}^{(\overline{t}, \overline{s})}_{\overline{\mu}}$ and $\mathcal{U}^{(t,s)}_{\omega}$. Thus, $\overline{\mu}^p(r_1 r_2) \succeq \overline{\mu}^p(r_1) \curlywedge \overline{\mu}^p(r_2), \overline{\mu}^p(r_1 r_2) \preceq \overline{\mu}^p(r_1) \curlyvee \overline{\mu}^p(r_2)$ and $\omega^p(r_1 r_2) \geq \omega^p(r_1) \wedge \omega^p(r_2)$, $\omega^n(r_1 r_2) \leq \omega^n(r_1) \vee \omega^p(r_2)$.
Hence $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF subsemigroup of F. ■

Theorem 2.19. *A CBF set $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF left (right) ideal of a semigroup F if and only if the level set $\mathcal{U}^{(\overline{t}, \overline{s})}_{\overline{\mu}}$ and $\mathcal{U}^{(t,s)}_{\omega}$ are left (right) ideals of F for all $(\overline{s}, \overline{t}) \in CS[-1, 0] \times CS[0, 1]$ and $(s, t) \in [-1, 0] \times [0, 1]$.*

Next, we study the subset and product of CBF set as defined.
Let $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ be CBF sets of a semigroup F. Define

1) $\ddot{\mathfrak{C}}_1 \sqsubseteq \ddot{\mathfrak{C}}_2$ if and only if $\overline{\mu}^p(r) \preceq \overline{\lambda}^p(r)$, $\overline{\mu}^n(r) \succeq \overline{\lambda}^n(r)$ and $\omega^p(r) \leq \psi^p(r)$, $\omega^n(r) \geq \psi^n(r)$, for all $r \in F$.
2) $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 = \langle \overline{\mu} \bigcirc \overline{\lambda}, \omega * \psi \rangle$ and define $\overline{\mu} \bigcirc \overline{\lambda}$ as follows. For $r \in F$

$$(\overline{\mu}^p \bigcirc \overline{\lambda}^p)(r) = \begin{cases} \underset{(k,o) \in A_r}{\curlyvee} \{\overline{\mu}^p(k) \curlywedge \overline{\lambda}^p(o)\} & \text{if } A_r \neq \emptyset \\ \overline{0} & \text{if } A_r = \emptyset, \end{cases}$$

$$(\overline{\mu}^n \bigcirc \overline{\lambda}^n)(r) = \begin{cases} \underset{(k,o) \in A_r}{\curlywedge} \{\overline{\mu}^n(k) \curlyvee \overline{\lambda}^n(o)\} & \text{if } A_r \neq \emptyset \\ \overline{0} & \text{if } A_r = \emptyset, \end{cases}$$

and $\omega * \psi$ is a product of a BF set.

Next, we study equivalent conditions of important properties for CBF subsemigroups of semigroups as shown in the following theorems.

Theorem 2.20. *A CBF set $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF subsemigroup of a semigroup F if and only if $\ddot{\mathfrak{C}} \circledast \ddot{\mathfrak{C}} \sqsubseteq \ddot{\mathfrak{C}}$.*

Proof: $(\Rightarrow)$ Assume that $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF subsemigroup of a semigroup F, and let $r \in F$.

If $A_r = \emptyset$, then it is easy to verify that,
$(\overline{\mu}^p \bigcirc \overline{\mu}^p)(r) \preceq \overline{\mu}^p(r)$, $(\overline{\mu}^n \bigcirc \overline{\mu}^n)(r) \succeq \overline{\mu}^n(r)$ and $(\omega^p * \omega^p)(r) \leq \omega^p(r)$, $(\omega^n * \omega^n)(r) \geq \omega^n(r)$.

If $A_r \neq \emptyset$, then

$$\begin{aligned}(\overline{\mu}^p \bigcirc \overline{\mu}^p)(r) &= \curlyvee_{(k,o)\in A_r} \{\overline{\mu}^p(k) \curlywedge \overline{\mu}^p(o)\} \\ &\preceq \curlyvee_{(k,o)\in A_r} \{\overline{\mu}^p(ko)\} = \overline{\mu}^p(r),\end{aligned}$$

$$\begin{aligned}(\overline{\mu}^n \bigcirc \overline{\mu}^n)(r) &= \curlywedge_{(k,o)\in A_r} \{\overline{\mu}^n(k) \curlyvee \overline{\mu}^n(o)\} \\ &\succeq \curlywedge_{(k,o)\in A_r} \{\overline{\mu}^n(ko)\} = \overline{\mu}^n(r)\end{aligned}$$

and

$$\begin{aligned}(\omega^p * \omega^p)(r) &= \bigvee_{(k,o)\in A_r} \{\omega^p(k) \wedge \omega^p(o)\} \\ &\leq \bigvee_{(k,o)\in A_r} \{\omega^p(ko)\} = \omega^p(r),\end{aligned}$$

$$\begin{aligned}(\omega^n * \omega^n)(r) &= \bigwedge_{(k,o)\in A_r} \{\omega^n(k) \vee \omega^n(o)\} \\ &\geq \bigwedge_{(k,o)\in A_r} \{\omega^n(ko)\} = \omega^n(r),\end{aligned}$$

Thus, $(\overline{\mu}^p \bigcirc \overline{\mu}^p)(r) \preceq \overline{\mu}^n(r)$, $(\overline{\mu}^n \bigcirc \overline{\mu}^n)(r) \succeq \overline{\mu}^n(r)$ and $(\omega^p * \omega^p)(r) \leq \omega^p(r)$, $(\omega^n * \omega^n)(r) \geq \omega^n(r)$.
Hence, $\ddot{\mathfrak{C}} \circledast \ddot{\mathfrak{C}} \sqsubset \ddot{\mathfrak{C}}$.

($\Leftarrow$) Suppose $\ddot{\mathfrak{C}} \circledast \ddot{\mathfrak{C}} \sqsubset \ddot{\mathfrak{C}}$, and let $r_1, r_2 \in F$. Then $(\overline{\mu}^p \bigcirc \overline{\mu}^p)(r_1r_2) \preceq \overline{\mu}^p(r_1r_2)$, $(\overline{\mu}^n \bigcirc \overline{\mu}^n)(r_1r_2) \succeq \overline{\mu}^n(r_1r_2)$ and $(\omega^p * \omega^p)(r_1r_2) \geq \omega^p(r_1r_2)$, $(\omega^n * \omega^n)(r_1r_2) \leq \omega^n(r_1r_2)$. Thus

$$\begin{aligned}\overline{\mu}^p(r_1r_2) &\succeq (\overline{\mu}^p \bigcirc \overline{\mu}^p)(r_1r_2) \\ &= \curlyvee_{(k,o)\in A_{r_1r_2}} \{\overline{\mu}^p(k) \curlywedge \overline{\mu}^p(o)\} \\ &\succeq \overline{\mu}^p(r_1) \curlywedge \overline{\mu}^p(r_2),\end{aligned}$$

$$\begin{aligned}\overline{\mu}^n(r_1r_2) &\preceq (\overline{\mu}^n \bigcirc \overline{\mu}^n)(r_1r_2) \\ &= \curlywedge_{(k,o)\in A_{r_1r_2}} \{\overline{\mu}^n(k) \curlyvee \overline{\mu}^n(o)\} \\ &\preceq \overline{\mu}^n(r_1) \curlyvee \overline{\mu}^n(r_2)\end{aligned}$$

and

$$\begin{aligned}\omega^p(r_1r_2) &\geq (\omega^p * \omega^p)(r_1r_2) \\ &= \bigvee_{(k,o)\in A_{r_1r_2}} \{\omega^p(k) \wedge \omega^p(o)\} \\ &\geq \omega^p(r_1) \wedge \omega^p(r_2),\end{aligned}$$

$$\begin{aligned}\omega^n(r_1r_2) &\leq (\omega^n * \omega^n)(r_1r_2) \\ &= \bigwedge_{(k,o)\in A_{r_1r_2}} \{\omega^n(k) \vee \omega^n(o)\} \\ &\leq \omega^n(r_1) \vee \omega^n(r_2).\end{aligned}$$

Hence, $\overline{\mu}^p(r_1r_2) \succeq \overline{\mu}^p(r_1) \curlywedge \overline{\mu}^p(r_2)$,
$\overline{\mu}^n(r_1r_2) \preceq \overline{\mu}^n(r_1) \curlyvee \overline{\mu}^p(r_2)$ and
$\omega^p(r_1r_2) \geq \omega^p(r_1) \wedge \omega^p(r_2)$, $\omega^n(r_1r_2) \leq \omega^n(r_1) \vee \omega^n(r_2)$.
Therefore $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF subsemigroup of F. ■

Let $\ddot{\mathfrak{F}} = \langle \overline{\mathcal{F}}, \mathcal{F} \rangle$ be a CBF set of a non-empty set F. Then $\overline{\mathcal{F}}^p(r) = \overline{1}$, $\overline{\mathcal{F}}^n(r) = \overline{-1}$ and $\mathcal{F}^p(r) = 1$, $\mathcal{F}^n(r) = -1$ for all $r \in F$.

Theorem 2.21. *A CBF set $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF left (right) ideal of a semigroup F if and only if $\ddot{\mathfrak{F}} \circledast \ddot{\mathfrak{C}} \sqsubset \ddot{\mathfrak{C}}$ ($\ddot{\mathfrak{C}} \circledast \ddot{\mathfrak{F}} \sqsubset \ddot{\mathfrak{C}}$), where $\ddot{\mathfrak{F}} = \langle \overline{\mathcal{F}}, \mathcal{F} \rangle$ is a CBF set of F.*

Proof: ($\Rightarrow$) Assume that $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF left ideal of F, and let $r \in F$.

If $A_r = \emptyset$, then it is easy to verify that,
$(\overline{\mathcal{F}}^p \bigcirc \overline{\mu}^p)(r) \preceq \overline{\mu}^p(r)$, $(\overline{\mathcal{F}}^n \bigcirc \overline{\mu}^n)(r) \succeq \overline{\mu}^n(r)$
and $(\mathcal{F}^p * \omega^p)(r) \geq \omega^p(r)$, $(\mathcal{F}^n * \omega^n)(r) \leq \omega^n(r)$.

If $A_r \neq \emptyset$, then

$$\begin{aligned}(\overline{\mathcal{F}}^p \bigcirc \overline{\mu}^p)(r) &= \curlyvee_{(k,o)\in A_r} \{\overline{\mathcal{F}}^p(k) \curlywedge \overline{\mu}^p(o)\} \\ &= \curlyvee_{(k,l)\in A_r} \{\overline{1} \curlywedge \overline{\mu}^p(o)\} \\ &= \curlyvee_{(k,o)\in A_r} \{\overline{\mu}^p(o)\} \\ &\preceq \curlyvee_{(k,o)\in A_r} \{\overline{\mu}^p(ko)\} = \overline{\mu}^p(r),\end{aligned}$$

$$\begin{aligned}(\overline{\mathcal{F}}^n \bigcirc \overline{\mu}^n)(r) &= \curlywedge_{(k,o)\in A_r} \{\overline{\mathcal{F}}^n(k) \curlyvee \overline{\mu}^n(o)\} \\ &= \curlywedge_{(k,o)\in A_r} \{\overline{-1} \curlyvee \overline{\mu}^n(o)\} \\ &= \curlywedge_{(k,o)\in A_r} \{\overline{\mu}^n(l)\} \\ &\succeq \curlyvee_{(k,o)\in A_r} \{\overline{\mu}^n(ko)\} = \overline{\mu}^n(r)\end{aligned}$$

and

$$\begin{aligned}(\mathcal{F}^p * \omega^p)(r) &= \bigvee_{(k,o)\in A_r} \{\mathcal{F}^p(k) \wedge \omega^p(o)\} \\ &= \bigvee_{(k,o)\in A_r} \{1 \wedge \omega^p(l)\} \\ &= \bigvee_{(k,o)\in A_r} \{\omega^p(l)\} \\ &\geq \bigvee_{(k,o)\in A_r} \{\omega^p(kl)\} = \omega^p(r),\end{aligned}$$

$$\begin{aligned}(\mathcal{F}^n * \omega^n)(r) &= \bigwedge_{(k,o)\in A_r} \{\mathcal{F}^n(k) \vee \omega^n(o)\} \\ &= \bigwedge_{(k,o)\in A_r} \{-1 \vee \omega^n(o)\} \\ &= \bigwedge_{(k,o)\in A_r} \{\omega^n(o)\} \\ &\leq \bigwedge_{(k,o)\in A_r} \{\omega^n(ko)\} = \omega^n(r).\end{aligned}$$

Thus, $(\overline{\mathcal{F}}^p \bigcirc \overline{\mu}^p)(r) \preceq \overline{\mu}^p(r)$, $(\overline{\mathcal{F}}^n \bigcirc \overline{\mu}^n)(r) \succeq \overline{\mu}^n(r)$ and $(\mathcal{F}^p \circ \omega^p)(r) \leq \omega^p(r)$, $(\mathcal{F}^n \circ \omega^n)(r) \geq \omega^n(r)$.
Hence, $\ddot{\mathfrak{F}} \circledast \ddot{\mathfrak{C}} \sqsubset \ddot{\mathfrak{C}}$.

($\Leftarrow$) Suppose $\ddot{\mathfrak{F}} \circledast \ddot{\mathfrak{C}} \sqsubset \ddot{\mathfrak{C}}$, and let $r_1r_2 \in F$. Then $(\overline{\mathcal{F}}^p \bigcirc \overline{\mu}^p)(r_1r_2) \preceq \overline{\mu}^p(r_1r_2)$, $(\overline{\mathcal{F}}^n \bigcirc \overline{\mu}^n)(r_1r_2) \succeq \overline{\mu}^n(r_1r_2)$ and $(\mathcal{F}^p * \omega^p)(r_1r_2) \geq \omega^p(r_1r_2)$, $(\mathcal{F}^n * \omega^n)(r_1r_2) \leq \psi^n(r_1r_2)$. Thus

$$\begin{aligned}\overline{\mu}^p(r_1r_2) &\succeq (\overline{\mathcal{F}}^p \square \overline{\mu}^p)(r_1r_2) \\ &= \curlyvee_{(k,o)\in A_{r_1r_2}} \{\overline{\mathcal{F}}^p(k) \curlywedge \overline{\mu}^p(o)\} \\ &= \curlyvee_{(k,o)\in A_{r_1r_2}} \{\overline{1} \curlywedge \overline{\mu}^p(o)\} \\ &= \curlyvee_{(k,o)\in A_{r_1r_2}} \{\overline{\mu}^p(o)\} \succeq \overline{\mu}^p(r_2),\end{aligned}$$

$$\begin{aligned}\overline{\mu}^n(r_1r_2) &\preceq (\overline{\mathcal{F}}^n \square \overline{\mu}^n)(r_1r_2) \\ &= \curlywedge_{(k,o)\in A_{r_1r_2}} \{\overline{\mathcal{F}}^n(k) \curlyvee \overline{\mu}^n(o)\} \\ &= \curlywedge_{(k,o)\in A_{r_1r_2}} \{\overline{-1} \curlyvee \overline{\mu}^n(o)\} \\ &= \curlyvee_{(k,o)\in A_{r_1r_2}} \{\overline{\mu}^n(o)\} \preceq \overline{\mu}^p(r_2),\end{aligned}$$

and

$$\begin{aligned}\omega^p(r_1r_2) &\geq (\mathcal{F}^p * \omega^p)(r_1r_2) \\ &= \bigvee_{(k,o)\in A_{r_1r_2}} \{\mathcal{F}^p(k) \wedge \omega^p(o)\} \\ &= \bigvee_{(k,o)\in A_{r_1r_2}} \{1 \wedge \psi^p(o)\} \\ &= \bigvee_{(k,o)\in A_{r_1r_2}} \{\psi^p(o)\} \geq \omega^p(r_2),\end{aligned}$$

$$\begin{array}{rcl}\omega^n(r_1r_2) & \leq & (\mathcal{F}^n * \omega^n)(r_1r_2) \\ & = & \bigwedge_{(k,o)\in A_{r_1r_2}} \{\mathcal{F}^n(k) \vee \omega^n(o)\} \\ & = & \bigwedge_{(k,o)\in A_{r_1r_2}} \{-1 \vee \psi^n(o)\} \\ & = & \bigwedge_{(k,o)\in A_{r_1r_2}} \{\omega^n(o)\} \leq \omega^n(r_2),\end{array}$$

Hence, $\overline{\mu}^p(r_1r_2) \succeq \overline{\mu}^p(r_2)$, $\overline{\mu}^n(r_1r_2) \preceq \overline{\mu}^n(r_2)$ and $\omega^p(r_1r_2) \geq \omega^p(r_2)$, $\omega^n(r_1r_2) \leq \overline{\mu}^n(r_2)$.
Therefore, $\ddot{\mathfrak{C}} = \langle\overline{\mu}, \omega\rangle$ is a CBF left ideal of F. ■

Corollary 2.22. *A CEF set $\ddot{\mathfrak{C}} = \langle\overline{\mu}, \omega\rangle$ is a CBF ideal of a semigroup F if and only if $\ddot{\mathfrak{F}} \circledast \ddot{\mathfrak{C}} \sqsubseteq \ddot{\mathfrak{C}}$ and $\ddot{\mathfrak{C}} \circledast \ddot{\mathfrak{F}} \sqsubseteq \ddot{\mathfrak{C}}$.*

Theorem 2.23. *Suppose that S is a regular semigroup. Then $\ddot{\mathfrak{C}} = \langle\overline{\mu}, \omega\rangle$ is a CBF left (right) ideal of S if and only if $\ddot{\mathfrak{F}} \circledast \ddot{\mathfrak{C}} = \ddot{\mathfrak{C}}(\ddot{\mathfrak{C}} \circledast \ddot{\mathfrak{F}} = \ddot{\mathfrak{C}})$.*

Proof: Assume that $\ddot{\mathfrak{C}} = \langle\overline{\mu}, \omega\rangle$ is a CBF left ideal of F, and let $r \in F$. Then there exists $k \in F$ such that $r = rkr$. Thus

$$\begin{array}{rcl}(\overline{\mathcal{F}}^p \bigcirc \overline{\mu}^p)(r) & = & \curlyvee_{(i,o)\in A_r} \{\overline{\mathcal{F}}^p(i) \curlywedge \overline{\mu}^p(o)\} \\ & = & \curlyvee_{(i,o)\in A_{rkr}} \{\overline{\mathcal{F}}^p(i) \curlywedge \overline{\mu}^p(o)\} \\ & \succeq & \overline{\mathcal{F}}^p(r) \curlywedge \overline{\mu}^p(kr) \\ & = & \overline{1} \curlywedge \overline{\mu}^p(kr) = \overline{\mu}^p(kr) \succeq \overline{\mu}^p(r), \\ (\overline{\mathcal{F}}^n \bigcirc \overline{\mu}^n)(r) & = & \curlywedge_{(i,o)\in A_r} \{\overline{\mathcal{F}}^n(i) \curlyvee \overline{\mu}^n(o)\} \\ & = & \curlyvee_{(i,o)\in A_{rkr}} \{\overline{\mathcal{F}}^p(i) \curlywedge \overline{\mu}^n(o)\} \\ & \preceq & \overline{\mathcal{F}}^n(r) \curlyvee \overline{\mu}^n(kr) \\ & = & \overline{-1} \curlyvee \overline{\mu}^n(kr) = \overline{\mu}^n(kr) \preceq \overline{\mu}^n(r)\end{array}$$

and

$$\begin{array}{rcl}(\mathcal{F}^p * \omega^p)(r) & = & \bigvee_{(i,o)\in A_r} \{\mathcal{F}^p(i) \wedge \omega^p(o)\} \\ & = & \bigvee_{(i,o)\in A_{rkr}} \{\mathcal{F}^p(i) \wedge \omega^p(o)\} \\ & \geq & \mathcal{F}^p(r) \wedge \omega^p(kr) \\ & = & 1 \wedge \omega^p(kr) = \omega^p(kr) \geq \omega^p(r), \\ (\mathcal{F}^n * \omega^n)(r) & = & \bigvee_{(i,o)\in A_r} \{\mathcal{F}^n(i) \wedge \omega^n(o)\} \\ & = & \bigvee_{(i,o)\in A_{rkr}} \{\mathcal{F}^n(i) \wedge \omega^n(o)\} \\ & \leq & \mathcal{F}^n(r) \wedge \omega^n(kr) \\ & = & -1 \wedge \omega^n(kr) = \omega^n(kr) \leq \omega^n(r),\end{array}$$

Hence, $\ddot{\mathfrak{C}} \sqsubseteq \ddot{\mathfrak{F}} \circledast \ddot{\mathfrak{C}}$. By Theorem 2.21, $\ddot{\mathfrak{F}} \circledast \ddot{\mathfrak{C}} \sqsubseteq \ddot{\mathfrak{C}}$.
Thus, $\ddot{\mathfrak{F}} \circledast \ddot{\mathfrak{C}} = \ddot{\mathfrak{C}}$.

For the conversion, it follows from Theorem 2.21. ■

The next corollary follows from Theorem 2.23.

Corollary 2.24. *Suppose that F is a regular semigroup. Then $\ddot{\mathfrak{C}} = \langle\overline{\mu}, \omega\rangle$ is a CBF ideal of a semigroup F if and only if $\ddot{\mathfrak{F}} \circledast \ddot{\mathfrak{C}} = \ddot{\mathfrak{C}}$ and $\ddot{\mathfrak{C}} \circledast \ddot{\mathfrak{F}} = \ddot{\mathfrak{C}}$.*

Lemma 2.25. *If $\ddot{\mathfrak{C}}_1 = \langle\overline{\mu}, \omega\rangle$ is a CBF right ideal and $\ddot{\mathfrak{C}}_2 = \langle\overline{\lambda}, \psi\rangle$ is a CBF left ideal of a semigroup F, then $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \sqsubseteq \ddot{\mathfrak{C}}_1 \sqcap \ddot{\mathfrak{C}}_2$.*

Proof: Assume that $\ddot{\mathfrak{C}}_1 = \langle\overline{\mu}, \omega\rangle$ and $\ddot{\mathfrak{C}}_2 = \langle\overline{\lambda}, \psi\rangle$ is a CBF right ideal and a CBF left ideal of F, respectively. Let $r \in F$. Then, by Theorem 2.21, $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \sqsubseteq \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{F}} \sqsubseteq \ddot{\mathfrak{C}}_1$ and $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \sqsubseteq \ddot{\mathfrak{F}} \circledast \ddot{\mathfrak{C}}_2 \sqsubseteq \ddot{\mathfrak{C}}_2$ Hence, $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \sqsubseteq \ddot{\mathfrak{C}}_1 \sqcap \ddot{\mathfrak{C}}_2$. ■

III. Characterizing Regular, Intra-Regular and Semisimple Semigroups by Using Cubic Bipolar Fuzzy Ideals

In this section, we will use knowledge of characteristics of cubic set and bipolar fuzzy set to characterize regular, intra-regular and semisimple semigroups by using CBF left and right ideals in semigroups.

Theorem 3.1. *Let M and N be a non-empty subsets of a semigroup F. Then*

(1) $\chi_M \circledast \chi_N = \chi_{MN}$ *i.e.* $\langle\overline{\mu}_{\chi_M} \bigcirc \overline{\mu}_{\chi_N}, \omega_{\chi_M} * \omega_{\chi_N}\rangle = \langle\overline{\mu}_{\chi_{MN}}, \omega_{\chi_{MN}}\rangle$
(2) $\chi_M \overline{\sqcap} \chi_N = \chi_{M\overline{\sqcap}N}$ *i.e.* $\langle\overline{\mu}_{\chi_M} \sqcap \overline{\mu}_{\chi_N}, \omega_{\chi_M} \cap \omega_{\chi_N}\rangle = \langle\overline{\mu}_{\chi_{M\sqcap N}}, \omega_{\chi_{M\cap N}}\rangle$

Lemma 3.2. *[17] A semigroup F is regular if and only if $RL = R \cap L$ for every right ideal R and left ideal L of F.*

The following theorem shows an equivalent conditional statement for a regular semigroup.

Theorem 3.3. *A semigroup F is regular if and only if $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 = \ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2$ for every CBF right ideal $\ddot{\mathfrak{C}}_1$ and CBF left ideal $\ddot{\mathfrak{C}}_2$ of F.*

Proof: Assume that $\ddot{\mathfrak{C}}_1 = \langle\overline{\mu}, \omega\rangle$ and $\ddot{\mathfrak{C}}_2 = \langle\overline{\lambda}, \psi\rangle$ are CBF right ideal and a CBF left ideal of F, respectively. Then by Lemma 2.25, $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \sqsubseteq \ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2$. Let $r \in F$. Then there exists $t \in F$ such that $r = rtr$. Thus

$$\begin{array}{rcl}(\overline{\mu}^p \bigcirc \overline{\lambda}^p)(r) & = & \curlyvee_{(k,o)\in A_r} \{\overline{\mu}^p(k) \curlywedge \overline{\lambda}^p(o)\} \\ & = & \curlyvee_{(k,o)\in A_{rtr}} \{\overline{\mu}^p(k) \curlywedge \overline{\lambda}^p(o)\} \\ & \succeq & \overline{\mu}^p(rt) \curlywedge \overline{\lambda}^p(r) \\ & \succeq & \overline{\mu}^p(r) \curlywedge \overline{\lambda}^p(r) = (\overline{\mu}^p \sqcap \overline{\lambda}^p)(r), \\ (\overline{\mu}^n \bigcirc \overline{\lambda}^n)(r) & = & \curlywedge_{(k,o)\in A_r} \{\overline{\mu}^n(k) \curlyvee \overline{\lambda}^n(o)\} \\ & = & \curlyvee_{(k,o)\in A_{rtr}} \{\overline{\mu}^n(k) \curlyvee \overline{\lambda}^n(o)\} \\ & \preceq & \overline{\mu}^n(rt) \curlyvee \overline{\lambda}^n(r) \\ & \preceq & \overline{\mu}^n(r) \curlyvee \overline{\lambda}^n(r) = (\overline{\mu}^n \sqcap \overline{\lambda}^n)(r)\end{array}$$

and

$$\begin{array}{rcl}(\omega^p * \psi^p)(r) & = & \bigvee_{(k,o)\in A_r} \{\omega^p(k) \wedge \psi^p(o)\} \\ & = & \bigvee_{(k,o)\in A_{rtr}} \{\omega^p(k) \wedge \psi^p(o)\} \\ & \geq & \omega^p(rt) \wedge \psi^p(r) \\ & \geq & \omega^p(r) \wedge \psi^p(r) = (\omega^p \cap \psi^p)(r), \\ (\omega^n * \psi^n)(r) & = & \bigwedge_{(k,o)\in A_r} \{\omega^n(k) \vee \psi^n(o)\} \\ & = & \bigwedge_{(k,o)\in A_{rtr}} \{\omega^n(k) \vee \psi^n(o)\} \\ & \leq & \omega^n(rt) \vee \psi^n(r) \\ & \leq & \omega^n(r) \vee \psi^n(r) = (\omega^n \cap \psi^n)(r).\end{array}$$

Hence, $\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \sqsubseteq \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2$. Therefore, $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 = \ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2$.

($\Leftarrow$) Let R and L be a right ideal and a left ideal of F, respectively. Then, by Theorem 2.14, χ_R and χ_L is a CCBF IVF right ideal and a CCBF left ideal of F, respectively. By supposition and Theorem 3.1, we have

$$\begin{array}{rcl}\overline{\mu}^p_{\chi_{RL}}(r) & = & (\overline{\mu}^p_{\chi_R} \bigcirc \overline{\mu}^p_{\chi_L})(r) = (\overline{\mu}^p_{\chi_R} \sqcap \overline{\mu}^p_{\chi_L})(r) \\ & = & \overline{\mu}^p_{\chi_{R\sqcap L}}(r) = \overline{1},\end{array}$$

$$\begin{aligned}\overline{\mu}^n_{\chi_{RL}}(r) &= (\overline{\mu}^n_{\chi_R} \bigcirc \overline{\mu}^n_{\chi_L})(r) = (\overline{\mu}^n_{\chi_R} \sqcap \overline{\mu}^n_{\chi_L})(r)\\ &= \overline{\mu}^n_{\chi_{R\cap L}}(r) = \overline{-1}\end{aligned}$$

and

$$\begin{aligned}\omega^p_{\chi_{RL}}(r) &= (\omega^p_{\chi_R} * \omega^p_{\chi_L})(r) = (\omega^p_{\chi_R} \cap \omega^p_{\chi_L})(r)\\ &= \omega^p_{\chi_{R\cap L}}(r) = 1,\\ \omega^n_{\chi_{RL}}(r) &= (\omega^n_{\chi_R} * \omega^n_{\chi_L})(r) = (\omega^n_{\chi_R} \cap \omega^n_{\chi_L})(r)\\ &= \omega^n_{\chi_{R\cap L}}(r) = -1.\end{aligned}$$

Thus $r \in RL$, and so $RL = R \cap L$. It follows that by Lemma 3.2, F is regular. ■

The following definition and lemma will be used to prove Theorem 3.6.

Definition 3.4. *[17] A semigroup F is called an* intra-regular *if, for each $r \in S$, there exist $k, t \in S$ such that $r = kr^2t$.*

Lemma 3.5. *[17] A semigroup F is intra-regular if and only if $L \cap R \subseteq LR$ for every left ideal L and every right ideal R of F.*

Theorem 3.6. *A semigroup F is intra-regular if and only if $\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \overline{\sqsubset} \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2$, for every CBF left ideal $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and every CBF right ideal $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ of F.*

Proof: ($\Rightarrow$) Assume that $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ are a CBF left ideal and CBF right ideal of F, respectively. Let $r \in F$. Then there exist $e, t \in F$ such that $r = er^2t$. Thus

$$\begin{aligned}(\overline{\mu}^p \bigcirc \overline{\lambda}^p)(r) &= \curlyvee_{(k,o)\in A_r} \{\overline{\mu}^p(k) \curlywedge \overline{\lambda}^p(o)\}\\ &= \curlyvee_{(k,o)\in A_{errt}} \{\overline{\mu}^p(k) \curlywedge \overline{\lambda}^p(o)\}\\ &\succeq \overline{\mu}^p(er) \curlywedge \overline{\lambda}^p(rt)\\ &\succeq \overline{\mu}^p(r) \curlywedge \overline{\lambda}^p(r) = (\overline{\mu}^p \sqcap \overline{\lambda}^p)(r),\\ (\overline{\mu}^n \bigcirc \overline{\lambda}^n)(r) &= \curlywedge_{(k,o)\in A_r} \{\overline{\mu}^n(k) \curlyvee \overline{\lambda}^n(o)\}\\ &= \curlyvee_{(k,o)\in A_{errt}} \{\overline{\mu}^n(k) \curlyvee \overline{\lambda}^n(o)\}\\ &\preceq \overline{\mu}^n(er) \curlyvee \overline{\lambda}^n(rt)\\ &\preceq \overline{\mu}^n(r) \curlyvee \overline{\lambda}^n(r) = (\overline{\mu}^n \sqcap \overline{\lambda}^n)(r)\end{aligned}$$

and

$$\begin{aligned}(\omega^p * \psi^p)(r) &= \bigvee_{(k,o)\in A_r} \{\omega^p(k) \wedge \psi^p(o)\}\\ &= \bigvee_{(k,o)\in A_{errt}} \{\omega^p(k) \wedge \psi^p(o)\}\\ &\geq \omega^p(er) \wedge \psi^p(rt)\\ &\geq \omega^p(r) \wedge \psi^p(r) = (\omega^p \cap \psi^p)(r),\\ (\omega^n * \psi^n)(r) &= \bigwedge_{(k,o)\in A_r} \{\omega^n(k) \vee \psi^n(o)\}\\ &= \bigwedge_{(k,o)\in A_{errt}} \{\omega^n(k) \vee \psi^n(o)\}\\ &\leq \omega^n(er) \vee \psi^n(rt)\\ &\leq \omega^n(r) \vee \psi^n(r) = (\omega^n \cap \psi^n)(r).\end{aligned}$$

Hence, $\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \overline{\sqsubset} \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2$.

($\Leftarrow$) Let R and L be a right ideal and a left ideal of F respectively. Then by Theorem 2.14, χ_R and χ_L is a CCBF right ideal and a CCBF left ideal of F, respectively. By supposition and Theroem 3.1, we have

$$\begin{aligned}\overline{\mu}^p_{\chi_{RL}}(r) &= (\overline{\mu}^p_{\chi_R} \bigcirc \overline{\mu}^p_{\chi_L})(r) \succeq (\overline{\mu}^p_{\chi_R} \sqcap \overline{\mu}^p_{\chi_L})(r)\\ &= \overline{\mu}^p_{\chi_{R\cap L}}(r) = \overline{1},\\ \overline{\mu}^n_{\chi_{RL}}(r) &= (\overline{\mu}^n_{\chi_R} \bigcirc \overline{\mu}^n_{\chi_L})(r) \preceq (\overline{\mu}^n_{\chi_R} \sqcap \overline{\mu}^n_{\chi_L})(r)\\ &= \overline{\mu}^n_{\chi_{R\cap L}}(r) = \overline{-1}\end{aligned}$$

and

$$\begin{aligned}\omega^p_{\chi_{RL}}(r) &= (\omega^p_{\chi_R} * \omega^p_{\chi_L})(r) \geq (\omega^p_{\chi_R} \cap \omega^p_{\chi_L})(r)\\ &= \omega^p_{\chi_{R\cap L}}(r) = 1,\\ \omega^n_{\chi_{RL}}(r) &= (\omega^n_{\chi_R} * \omega^n_{\chi_L})(r) \leq (\omega^n_{\chi_R} \cap \omega^n_{\chi_L})(r)\\ &= \omega^n_{\chi_{R\cap L}}(r) = -1.\end{aligned}$$

Thus $r \in LR$, and so $L \cap R \subseteq LR$. It follows that by Lemma 3.5, F is intra-regular. ■

The following definition and lemma will be used to prove Theorem 3.10.

Definition 3.7. *[17] A semigroup F is called* semisimple *if every ideal of F is idempotent.*

Remark 3.8. *A semigroup F is semisimple if and only if $r \in (FrF)(FrF)$ for every $r \in F$, that is there exist $w, y, z \in F$ such that $r = wryrz$.*

Lemma 3.9. *[17] A semigroup F is semisimple if and only if $I \cap J = IJ$ for every ideals I and J of F.*

Theorem 3.10. *A semigroup F is semisimple if and only if $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 = \ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2$, for every CBF ideals $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ of F.*

Proof: ($\Rightarrow$) Assume that $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ and $\ddot{\mathfrak{C}}_2 = \langle \overline{\lambda}, \psi \rangle$ are CBF ideals of F. Then, by Theorem 2.25, $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 \overline{\sqsubset} \ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2$. Let $r \in F$. Since F is semisimple, there exist $w, x, y, z \in F$ such that $r = (xry)(wrz)$. Thus

$$\begin{aligned}(\overline{\mu}^p \bigcirc \overline{\lambda}^p)(r) &= \curlyvee_{(k,o)\in A_r} \{\overline{\mu}^p(k) \curlywedge \overline{\lambda}^p(o)\}\\ &= \curlyvee_{(k,o)\in A_{(xry)(wrz)}} \{\overline{\mu}^p(k) \curlywedge \overline{\lambda}^p(o)\}\\ &\succeq (\overline{\mu}^p(xry) \curlywedge \overline{\lambda}^p(wrz))\\ &\succeq (\overline{\mu}^p(xr) \curlywedge \overline{\lambda}^p(rz))\\ &\succeq (\overline{\mu}^p(r) \curlywedge \overline{\lambda}^p(r)) = (\overline{\mu}^p \sqcap \overline{\lambda}^p)(r),\\ (\overline{\mu}^n \bigcirc \overline{\lambda}^n)(r) &= \curlywedge_{(k,o)\in A_r} \{\overline{\mu}^n(k) \curlyvee \overline{\lambda}^n(o)\}\\ &= \curlyvee_{(k,o)\in A_{(xry)(wrz)}} \{\overline{\mu}^n(k) \curlyvee \overline{\lambda}^n(o)\}\\ &\preceq (\overline{\mu}^n(xry) \curlywedge \overline{\lambda}^n(wrz))\\ &\preceq (\overline{\mu}^n(xr) \curlyvee \overline{\lambda}^n(rz))\\ &\preceq (\overline{\mu}^n(r) \curlyvee \overline{\lambda}^n(r)) = (\overline{\mu}^n \sqcap \overline{\lambda}^n)(r)\end{aligned}$$

and

$$\begin{aligned}(\omega^p * \psi^p)(r) &= \bigvee_{(k,o)\in A_r} \{\omega^p(k) \wedge \psi^p(o)\}\\ &= \bigvee_{(k,o)\in A_{(xry)(wrz)}} \{\omega^p(k) \wedge \psi^p(o)\}\\ &\geq (\omega^p(xry) \wedge \psi^p(wrz))\\ &\geq (\omega^p(xr) \wedge \psi^p(rz))\\ &\geq (\omega^p(r) \wedge \psi^p(r)) = (\omega^p \cap \psi^p)(r),\\ (\omega^n * \psi^n)(r) &= \bigwedge_{(k,o)\in A_r} \{\omega^n(k) \vee \psi^n(o)\}\\ &= \bigwedge_{(k,o)\in A_{(xry)(wrz)}} \{\omega^n(k) \vee \psi^n(o)\}\\ &\leq (\omega^n(xry) \vee \psi^n(wrz))\\ &\leq (\omega^n(xr) \vee \psi^n(rz))\\ &\leq (\omega^n(r) \vee \psi^n(r)) = (\omega^n \cap \psi^n)(r).\end{aligned}$$

Hence, $(\overline{\mu}^p \bigcirc \overline{\lambda}^p)(r) \succeq (\overline{\mu}^p \sqcap \overline{\lambda}^p)(r)$, $(\overline{\mu}^n \bigcirc \overline{\lambda}^n)(r) \preceq (\overline{\mu}^n \sqcap \overline{\lambda}^n)$ and $(\omega^p * \psi^p)(r) \geq (\omega^p \cap \psi^p)(r)$, $(\omega^n * \psi^n)(h) \leq (\omega^n \cap \psi^n)(r)$ and so, $\ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2 \overline{\sqsubset} \ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2$. Therefore, $\ddot{\mathfrak{C}}_1 \circledast \ddot{\mathfrak{C}}_2 = \ddot{\mathfrak{C}}_1 \overline{\sqcap} \ddot{\mathfrak{C}}_2$.

($\Leftarrow$) Let I and J be ideals of F. Then, by Theorem 2.14, $\overline{\chi}_I$ and $\overline{\chi}_J$ are $(\overline{s},\overline{t})$-IVF ideals of F. By supposition and Theorem 3.1, we have

$$\begin{aligned}\overline{\mu}^p_{\chi_{IJ}}(r) &= (\overline{\mu}^p_{\chi_I} \bigcirc \overline{\mu}^p_{\chi_J})(r) = (\overline{\mu}^p_{\chi_I} \sqcap \overline{\mu}^p_{\chi_J})(r)\\ &= \overline{\mu}^p_{\chi_{I\cap J}}(r) = \overline{1},\end{aligned}$$

$$\begin{aligned}\overline{\mu}^n_{\chi_{IJ}}(r) &= (\overline{\mu}^n_{\chi_I} \bigcirc \overline{\mu}^n_{\chi_J})(r) = (\overline{\mu}^n_{\chi_I} \sqcap \overline{\mu}^n_{\chi_J})(r)\\ &= \overline{\mu}^n_{\chi_{I\cap J}}(r) = \overline{-1}\end{aligned}$$

and

$$\begin{aligned}\omega^p_{\chi_{IJ}}(r) &= (\omega^p_{\chi_I} * \omega^p_{\chi_J})(r) = (\omega^p_{\chi_I} \cap \omega^p_{\chi_J})(r)\\ &= \omega^p_{\chi_{I\cap J}}(r) = 1,\end{aligned}$$

$$\begin{aligned}\omega^n_{\chi_{IJ}}(r) &= (\omega^n_{\chi_I} * \omega^n_{\chi_J})(r) = (\omega^n_{\chi_I} \cap \omega^n_{\chi_J})(r)\\ &= \omega^n_{\chi_{I\cap J}}(r) = -1.\end{aligned}$$

Thus $r \in IJ$, and so $IJ = I \cap J$. It follows that by Lemma 3.9, F is semisimple. ■

The following lemma will be used to prove Theorem 3.12.

Lemma 3.11. *[17] A semigroup F is semisimple if and only if $I^2 = I$ for every ideal I of F.*

Theorem 3.12. *A semigroup F is semisimple if and only if $\ddot{\mathfrak{C}} \circledast \ddot{\mathfrak{C}} = \ddot{\mathfrak{C}}$, for every $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$, is a CBF ideal of F.*

Proof: ($\Rightarrow$) Let $\ddot{\mathfrak{C}}_1 = \langle \overline{\mu}, \omega \rangle$ be a CBF ideal of F. Then, by Theorem 2.20 we have that $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ is a CBF subsemigroup of F if and only if $\ddot{\mathfrak{C}} \circledast \ddot{\mathfrak{C}} \sqsubseteq \ddot{\mathfrak{C}}$. Let $r \in F$. Then there exist $w, x, y, z \in F$ such that $r = (xry)(wrz)$. Thus

$$\begin{aligned}(\overline{\mu}^p \bigcirc \overline{\mu}^p)(r) &= \curlyvee_{(k,o)\in A_r} \{\overline{\mu}^p(k) \curlywedge \overline{\mu}^p(o)\}\\ &= \curlyvee_{(k,o)\in A_{(xry)(wrz)}} \{\overline{\mu}^p(k) \curlywedge \overline{\mu}^p(o)\}\\ &\succeq (\overline{\mu}^p(xry) \curlywedge \overline{\mu}^p(wrz))\\ &\succeq (\overline{\mu}^p(xr) \curlywedge \overline{\mu}^p(rz))\\ &\succeq (\overline{\mu}^p(r) \curlywedge \overline{\mu}^p(r)) = \overline{\mu}^p(r),\end{aligned}$$

$$\begin{aligned}(\overline{\mu}^n \bigcirc \overline{\mu}^n)(r) &= \curlywedge_{(k,o)\in A_r} \{\overline{\mu}^n(k) \curlyvee \overline{\mu}^n(o)\}\\ &= \curlyvee_{(k,o)\in A_{(xry)(wrz)}} \{\overline{\mu}^n(k) \curlyvee \overline{\mu}^n(o)\}\\ &\preceq (\overline{\mu}^n(xry) \curlywedge \overline{\mu}^n(wrz))\\ &\preceq (\overline{\mu}^n(xr) \curlyvee \overline{\mu}^n(rz))\\ &\preceq (\overline{\mu}^n(r) \curlyvee \overline{\mu}^n(r)) = \overline{\mu}^n(r)\end{aligned}$$

and

$$\begin{aligned}(\omega^p * \omega^p)(r) &= \bigvee_{(k,o)\in A_r} \{\omega^p(k) \wedge \omega^p(o)\}\\ &= \bigvee_{(k,o)\in A_{(xry)(wrz)}} \{\omega^p(k) \wedge \omega^p(o)\}\\ &\geq (\omega^p(xry) \wedge \omega^p(wrz))\\ &\geq (\omega^p(xr) \wedge \omega^p(rz))\\ &\geq (\omega^p(r) \wedge \omega^p(r)) = \omega^p(r),\end{aligned}$$

$$\begin{aligned}(\omega^n * \omega^n)(r) &= \bigwedge_{(k,o)\in A_r} \{\omega^n(k) \vee \omega^n(o)\}\\ &= \bigwedge_{(k,o)\in A_{(xry)(wrz)}} \{\omega^n(k) \vee \omega^n(o)\}\\ &\leq (\omega^n(xry) \vee \omega^n(wrz))\\ &\leq (\omega^n(xr) \vee \omega^n(rz))\\ &\leq (\omega^n(r) \vee \omega^n(r)) = \omega^n(r).\end{aligned}$$

Hence, $(\overline{\mu}^p \bigcirc \overline{\mu}^p)(r) \succeq \overline{\mu}^p(r)$, $(\overline{\mu}^n \bigcirc \overline{\mu}^n)(r) \preceq \overline{\mu}^n(r)$ and $(\omega^p * \omega^p)(r) \geq \omega^p(r)$, $(\omega^n * \omega^n)(r) \leq \omega^n(r)$ and so, $\ddot{\mathfrak{C}} \sqsubseteq \ddot{\mathfrak{C}} \circledast \ddot{\mathfrak{C}}$. Therefore, $\ddot{\mathfrak{C}} \circledast \ddot{\mathfrak{C}} = \ddot{\mathfrak{C}}$.

($\Leftarrow$) Let I be an ideal of F. Then, by Theorem 2.14, $\overline{\chi}_I$ is an $(\overline{s},\overline{t})$-IVF ideal of F. By supposition and Theorem 3.1, we have

$$\overline{\mu}^p_{\chi_{I^2}}(r) = (\overline{\mu}^p_{\chi_I} \bigcirc \overline{\mu}^p_{\chi_I})(r) = \overline{\mu}^p_{\chi_\Im}(r) = \overline{1},$$

$$\overline{\mu}^n_{\chi_{I^2}}(r) = (\overline{\mu}^n_{\chi_I} \bigcirc \overline{\mu}^n_{\chi_I})(r) = \overline{\mu}^n_{\chi_I}(r) = \overline{-1}$$

and

$$\omega^p_{\chi_{I^2}}(r) = (\omega^p_{\chi_I} * \omega^p_{\chi_I})(r) = \omega^p_{\chi_I}(r) = 1,$$

$$\omega^n_{\chi_{I^2}}(r) = (\omega^n_{\chi_I} * \omega^n_{\chi_I})(r) = \omega^n_{\chi_I}(r) = -1,$$

Thus $r \in I^2$, and so $I^2 = I$. It follows that, by Lemma 3.11, F is semisimple. ■

IV. The Image and Pre-Image of CBF Subsemigroups

In this section, we introduce the notion of image and pre-image of the CBF subsemigroups and discuss some of their properties.

Definition 4.1. *[17] A mapping ϕ from a semigroup F_1 to a semigroup F_2 is said to be* ***homomorphism*** *if $\phi(uv) = \phi(u)\phi(v)$ for all $u, v \in F_1$.*

Definition 4.2. *Let ϕ be a mapping a semigroup set F_1 to a semigroup F_2, and let $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ be CBF set in F_1 and F_2, respectively. Then the image $\phi(\ddot{\mathfrak{C}})$ of $\ddot{\mathfrak{C}}$ is the CBF set $\phi(\ddot{\mathfrak{C}}) = \langle \phi(\overline{\mu}), \phi(\omega) \rangle$ of F_2 defined by for $r \in F_2$,*

$$\phi(\overline{\mu})^p(r) = \begin{cases} \curlyvee_{y=\phi^{-1}(r)} \overline{\mu}^p(y), & \text{if } \phi^{-1}(r) \neq 0,\\ \overline{0}, & \text{otherwise}\end{cases}$$

$$\phi(\overline{\mu})^n(r) = \begin{cases} \curlywedge_{y=\phi^{-1}(r)} \overline{\mu}^n(y), & \text{if } \phi^{-1}(r) \neq 0,\\ \overline{0}, & \text{otherwise}\end{cases}$$

and

$$\phi(\omega)^p(r) = \begin{cases} \bigvee_{y=\phi^{-1}(r)} \omega^p(y), & \text{if } \phi^{-1}(r) \neq 0,\\ 1, & \text{otherwise}\end{cases}$$

$$\phi(\omega)^n(r) = \begin{cases} \bigwedge_{y=\phi^{-1}(r)} \omega^n(y), & \text{if } \phi^{-1}(r) \neq 0,\\ -1, & \text{otherwise}\end{cases}$$

for all $r \in F_2$. The inverse image *$\phi^{-1}(\ddot{\mathfrak{C}}) = \langle \phi^{-1}(\overline{\mu}), \phi^{-1}(\omega) \rangle$ is defined by $\phi_{\phi}^{-1}(\overline{\mu}^p)(r) = \overline{\mu}(\phi(r))$, $\phi^{-1}(\overline{\mu}^n)(r) = \overline{\mu}(\phi(r))$ and $\phi^{-1}(\omega^p)(r) = \omega^p(\phi(r))$, $\phi^{-1}(\omega^n)(x) = \omega^n(\phi(r))$ for all $r \in F_1$. Then the mapping ϕ is called a* cubic bipolar transformation *(CBT), and ϕ^{-1} is called an* inverse cubic bipolar transformation *(ICBT) induced by ϕ. A CBF set $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ in X has the* CBF property *if for any subset T of X. There exists $x_0 \in T$ such that $(\overline{\mu})^p(r_0) = \curlyvee_{r\in T} (\overline{\mu})^p(r)$, $(\overline{\mu})^n(r_0) = \curlywedge_{r\in T} (\overline{\mu})^n(r)$ and $(\omega)^p(r_0) = \bigvee_{r\in T} (\omega)^p(x)$, $(\omega)^n(r_0) = \bigwedge_{r\in T} (\omega)^n(r)$.*

Theorem 4.3. *For a homomorphism $\phi : F_1 \to F_2$ of semigroups, and let $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega \rangle$ be a CBF subsemigroup of F_1 and F_2, respectively.*

(1) *$\phi(\ddot{\mathfrak{C}}) = \langle \phi(\overline{\mu}), \phi(\omega) \rangle$ is a CBF subsemigroup of F_2.*
(2) *$\phi^{-1}(\ddot{\mathfrak{C}}) = \langle \phi^{-1}(\overline{\mu}), \phi^{-1}(\omega) \rangle$ is a CBF subsemigroup of F_1.*

Proof:

(1) Let $r_1, r_2 \in F_2$. Since ϕ is a homorphism, there exist $m_1, m_2 \in F_1$ such that $\phi(m_1) = r_1$ and $\phi(m_2) = r_2$. Thus

$$\begin{aligned}\phi(\overline{\mu})^p(r_1r_2) &= \curlyvee_{k=\phi^{-1}(r_1r_2)} (\overline{\mu})^p(k)\\ &= \curlyvee_{\phi(m_1),\phi(m_2)} (\overline{\mu})^p(m_1m_2)\\ &\succeq \curlyvee_{\phi(m_1),\phi(m_2)} \{(\overline{\mu})^p(m_1) \curlywedge (\overline{\mu})^p(m_2)\}\\ &= \curlyvee_{k=\phi^{-1}(r_1r_2)} (\overline{\mu})^p(k)\\ &= \phi((\overline{\mu})^p(r_1) \curlywedge (\overline{\mu})^p(r_2)),\end{aligned}$$

$$\begin{aligned}\phi(\overline{\mu})^n(r_1r_2) &= \curlywedge_{k=\phi^{-1}(r_1r_2)} (\overline{\mu})^n(k)\\ &= \curlywedge_{\phi(m_1),\phi(m_2)} (\overline{\mu})^n(m_1m_2)\\ &\preceq \curlywedge_{\phi(m_1),\phi(m_2)} \{(\overline{\mu})^n(m_1) \curlyvee (\overline{\mu})^n(m_2)\}\\ &= \curlywedge_{k=\phi^{-1}(r_1r_2)} (\overline{\mu})^n(k)\\ &= \phi((\overline{\mu})^n(r_1) \curlyvee (\overline{\mu})^n(r_2)).\end{aligned}$$

And

$$\begin{aligned}\phi(\omega)^p(r_1r_2) &= \bigvee_{k=\phi^{-1}(r_1r_2)} (\omega)^p(k)\\ &= \bigvee_{\phi(m_1),\phi(m_2)} (\omega)^p(m_1m_2)\\ &\leq \bigvee_{\phi(m_1),\phi(m_2)} \{(\omega)^p(m_1) \wedge (\omega)^p(m_2)\}\\ &= \bigvee_{k=\phi^{-1}(r_1r_2)} (\omega)^p(k)\\ &= \phi((\omega)^p(r_1) \wedge (\omega)^p(r_2)),\end{aligned}$$

$$\begin{aligned}\phi(\omega)^n(r_1r_2) &= \bigwedge_{k=\phi^{-1}(r_1r_2)} (\omega)^n(k)\\ &= \bigwedge_{\phi(m_1),\phi(m_2)} (\omega)^n(m_1m_2)\\ &\geq \bigwedge_{\phi(m_1),\phi(m_2)} \{(\omega)^n(m_1) \curlyvee (\omega)^n(m_2)\}\\ &= \bigwedge_{k=\phi^{-1}(r_1r_2)} (\omega)^n(k)\\ &= \phi((\omega)^n(r_1) \vee (\omega)^n(r_2)).\end{aligned}$$

Thus $\phi(\overline{\mu})^p(r_1r_2) \succeq \phi((\overline{\mu})^p(r_1) \curlywedge (\overline{\mu})^p(r_2))$, $\phi(\overline{\mu})^n(r_1r_2) \preceq \phi((\overline{\mu})^n(r_1) \curlyvee (\overline{\mu})^n(r_2))$ and $\phi(\omega)^p(r_1r_2) \leq \phi((\omega)^p(r_1) \wedge (\omega)^p(r_2))$, $\phi(\omega)^n(r_1r_2) \geq \phi((\omega)^n(r_1) \vee (\omega)^n(r_2))$.
Hence $\phi(\ddot{\mathfrak{C}}) = \langle \phi(\overline{\mu}), \phi(\omega)\rangle$ is a CBF subsemigroup of F_2.

(2) Let $m_1, m_2 \in F_1$. Then

$$\begin{aligned}\phi^{-1}(\overline{\mu})^p(m_1m_2) &= (\overline{\mu})^p(\phi(m_1m_2))\\ &= (\overline{\mu})^p(\phi(m_1)\phi(m_2))\\ &\succeq (\overline{\mu})^p(\phi(m_1) \curlywedge \phi(m_2))\\ &= \phi^{-1}(\overline{\mu})^p(m_1 \curlywedge m_2),\end{aligned}$$

$$\begin{aligned}\phi^{-1}(\overline{\mu})^n(m_1m_2) &= (\overline{\mu})^n(\phi(m_1m_2))\\ &= (\overline{\mu})^n(\phi(m_1)\phi(m_2))\\ &\preceq (\overline{\mu})^n(\phi(m_1) \curlyvee \phi(m_2))\\ &= \phi^{-1}(\overline{\mu})^n(m_1 \curlyvee m_2).\end{aligned}$$

And

$$\begin{aligned}\phi^{-1}(\omega)^p(m_1m_2) &= (\omega)^p(\phi(m_1m_2))\\ &= (\omega)^p(\phi(m_1)\phi(m_2))\\ &\geq (\omega)^p(\phi(m_1) \wedge \phi(m_2))\\ &= \phi^{-1}(\omega)^p(m_1 \wedge m_2),\end{aligned}$$

$$\begin{aligned}\phi^{-1}(\omega)^n(m_1m_2) &= (\omega)^n(\phi(m_1m_2))\\ &= (\omega)^n(\phi(m_1)\phi(m_2))\\ &\leq (\omega)^n(\phi(m_1) \vee \phi(m_2))\\ &= \phi^{-1}(\omega)^n(m_1 \vee m_2).\end{aligned}$$

Thus $\phi^{-1}(\overline{\mu})^p(m_1m_2) \succeq \phi^{-1}(\overline{\mu})^p(m_1 \curlywedge m_2)$, $\phi^{-1}(\overline{\mu})^n(m_1m_2) \preceq \phi^{-1}(\overline{\mu})^n(m_1 \curlyvee m_2)$ and $\phi^{-1}(\omega)^p(m_1m_2) \geq \phi^{-1}(\omega)^p(m_1 \wedge m_2)$, $\phi^{-1}(\omega)^n(m_1m_2) \leq \phi^{-1}(\omega)^n(m_1 \vee m_2)$.
Hence $\phi^{-1}(\ddot{\mathfrak{C}}) = \langle \phi^{-1}(\overline{\mu}), \phi^{-1}(\omega)\rangle$ is a CBF subsemigroup of F_1.

■

Theorem 4.4. *For a homomorphism $\phi : F_1 \to F_2$ of semigroups, and let $\ddot{\mathfrak{C}} = \langle \overline{\mu}, \omega\rangle$ be a CBF left (right) of F_1 and F_2, respectively.*

(1) $\phi(\ddot{\mathfrak{C}}) = \langle \phi(\overline{\mu}), \phi(\omega)\rangle$ *is a CBF left (right) of* F_2.
(2) $\phi^{-1}(\ddot{\mathfrak{C}}) = \langle \phi^{-1}(\overline{\mu}), \phi^{-1}(\omega)\rangle$ *is a CBF left (right) of* F_1.

Proof: It follows Theorem 4.3. ■

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