Regularity in Semigraphs

JYOTI SHETTY, SUDHAKARA G, K ARATHI BHAT*

Abstract—**In this article, we have introduced a variety of regular semigraphs using the concept of the degree of a vertex in a semigraph. The concept of the degree of a vertex has variations due to the variety of vertices (end vertex, mid vertex, and mid-end vertex) in semigraphs. Two more regular semigraphs have been defined with the help of the binomial incidence matrix of a semigraph which determines the semigraph uniquely up to isomorphism. The interconnection between the variety of regular semigraphs has been discussed. In the latter part, the minimum number of edges in the semigraph which is regular of all kinds have been given.**

Index Terms—**Semigraph, binomial incidence matrix, regularity, column sum, row sum.**

I. Introduction

In graph theory, a regular graph is a graph where every vertex has the same degree. A regular graph with vertices of degree k is called a k-regular graph or regular graph of degree k. Regular graphs of degree at most 2 are easy to classify: A 0-regular graph consists of disconnected vertices, a 1-regular graph consists of disconnected edges, and a 2-regular graph consists of a disjoint union of cycles. A 3-regular graph is known as a cubic graph.

Lots of studies have been taken place on regular graphs. Not only these graphs are fascinating geometrically [3] but also they are applicable in network theory [8].

Several graphs like Random regular graph, Cage graph, Strongly regular graph, Moore graph, Petersen graph, etc use the concept of regular graphs. The colouring of regular graphs is one of the well-explored fields of research [6]. The existence of the Hamiltonian cycle in random regular graphs has been studied [20]. Properties of random regular graphs have been extensively studied [4]. The existence of matching in any graph has been studied by Tutte, Petersen has used it to find some interesting results on 3-regular graphs [12]. The eigenvalues of the adjacency matrix have been used to study matching in regular graphs [2], [21]. Matchings in random regular graphs have also been explored in [14] and on bipartite regular graphs [12]. The spectral property of regular graphs is studied by [7], [21]. Overall the above-mentioned studies took place due to the single idea of regularity in graphs. This motivates us to study regularity in semigraph, we believe which is going to bring out a lot of study area in semigraph to existence.

Manuscript received 26 May 2022; revised 30 September 2022

Jyoti Shetty is an Assistant Professor in the Department of Mathematics, Manipal Institute of Technology Bengaluru, Manipal Academy of Higher Education, Manipal, Karnataka, India (email: shetty.jyoti@manipal.edu).

Sudhakara G is Professor and Head in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India-576104 (email: sudhakara.g@manipal.edu).

K Arathi Bhat is an Assistant Professor - Selection Grade in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India-576104 (email: arathi.bhat@manipal.edu).

*Corresponding author : K Arathi Bhat.

Extending the idea of regular graphs to its generalized structure called semigraphs is of interest due to the existence of a variety of degrees for a vertex in a semigraph [9]. Also, two more regular semigraphs have been defined using the concept of an incidence matrix of a semigraph which determines the semigraph uniquely up to isomorphism called the binomial incidence matrix of the semigraph.

Authors in [15] make use of a property of binomial coefficient while defining the binomial incidence matrix, which not only represents a semigraph uniquely but also has the following property. The (i, j) entry of the matrix gives information about the position of every vertex v_i incident on e_j from either end vertex on the edge e_j and also the size of the edge e_j. And addition operation in semigraph also been studied [17] and more publications related to semigraph by same authors are [16], [18], [19]

Readers are referred to [10] for all the elementary notations and definitions not described but used in this article.

II. Preliminaries

In this section, we give basics of semigraph [9] which are required for the later developments in the article.

Definition 2.1: A semigraph G is a pair (V, E) where V is a non empty set whose elements are called vertices of G, and E is a set of k-tuples of distinct vertices, called edges of G, for various $k \geq 2$, satisfying the following conditions.

1) Any two edges of G can have at most one vertex in common.
2) Two edges $(a_1, a_2, ..., a_p)$ and $(b_1, b_2, ..., b_q)$ are said to be equal if and only if
   - $p = q$ and
   - either $a_i = b_i$ for $1 \leq i \leq p$ or $a_i = b_{p-i+1}$ for $1 \leq i \leq p$.

Note 2.1: Let E be the set of vertices on the edge e then the size of E is called the size of the edge e and it is usually denoted by $|E|$.

Let $G = (V, E)$ be a semigraph and let $e = (u_1, u_2, \ldots, u_k)$ be an edge of G. Then u_1 and u_k are called the end vertices and u_i, $2 \leq i \leq k-1$, are called the mid vertices of e. Two vertices of G are adjacent if there is an edge containing both of them. An edge is said to be incident on every vertex on the edge. Two edges of G are adjacent if they have a vertex in common. Two vertices are consecutively adjacent if they are consecutive on the edge containing them.

Like a graph, a semigraph G also has a geometric representation on the plane. Vertices of G are represented either by dots or by small circles according to whether they are end vertices or mid vertices of the edge containing them and edges of G by curves passing through all the vertices on them. When a mid vertex v of an edge e_1 is an end vertex of another edge, say e_2, then a small tangent is drawn to the circle representing vertex v where e_2 meets v. A semigraph G, and its representation is given in Example 2.1.

Example 2.1: Let $G = (V, E)$ be a semigraph with $V(G) = \{u_1, u_2, \ldots, u_7\}$ and $E(G) = \{e_1 = (u_1, u_2, u_3);\ e_2 = (u_1, u_4, u_5, u_7);\ e_3 = (u_2, u_5);\ e_4 = (u_3, u_6, u_7)\}$. It can be represented as shown in Figure 1.

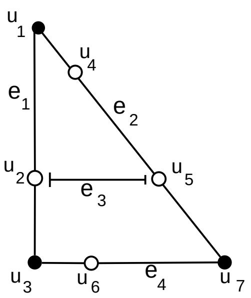

Fig. 1. Semigraph G with vertex set $\{u_1, u_2, \ldots, u_7\}$ and edges $\{e_1 = (u_1, u_2, u_3);\ e_2 = (u_1, u_4, u_5, u_7);\ e_3 = (u_2, u_5);\ e_4 = (u_3, u_6, u_7)\}$

If a vertex v is a mid vertex of more than one edge of G, say $e_1, e_2, \ldots, e_t$ then v is represented as a small regular polygon with $2t$ corners $c_1, c_2, \ldots, c_{2t}$ with the convention the curve representing the edge e_i meets the polygon precisely at c_i and c_{t+i}, $i + t$ reduced modulo 2t, $i \in \{1, 2, \ldots, t\}$.
Figure 2 shows the representation of a vertex v in a semigraph, when it is a mid vertex of three edges.

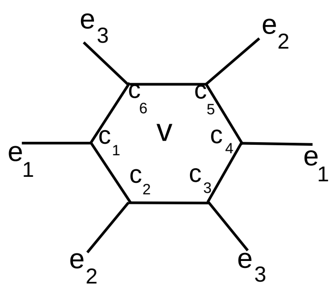

Fig. 2. vertex v is a mid vertex of three edges e_1, e_2 and e_3 (it is represented by a regular hexagon)

A complete semigraph is a semigraph in which every two vertices are adjacent. The following theorem characterizes a complete semigraph [16].

Theorem 2.1: Let G be a semigraph on n vertices and m edges $e_1, e_2, \ldots, e_m$ of sizes $n_1, n_2, \ldots, n_m$, respectively. Then G is complete if and only if $\binom{n}{2} = \sum_{i=1}^{m} \binom{n_i}{2}$.
Since the edge of a semigraph can have two or more vertices on it, the concept of degree of a vertex has the following variations.

Definition 2.2: Let $G = (V, E)$ be a semigraph and v be a vertex of G.

1) Degree of v, denoted by $\deg v$, is the number of edges having v as an end vertex.
2) Edge degree of v, denoted by $\deg_e v$, is the number of edges containing v.
3) Adjacent degree of v, denoted by $\deg_a v$, is the number of vertices adjacent to v.
4) Consecutive adjacent degree of v, denoted by $\deg_{ca} v$, is the number of vertices which are consecutively adjacent to v.

Example 2.2: Consider a semigraph with 8 vertices and 4 edges as shown in Figure 3. The vertices of the semigraph are labelled with the various degree concepts defined.

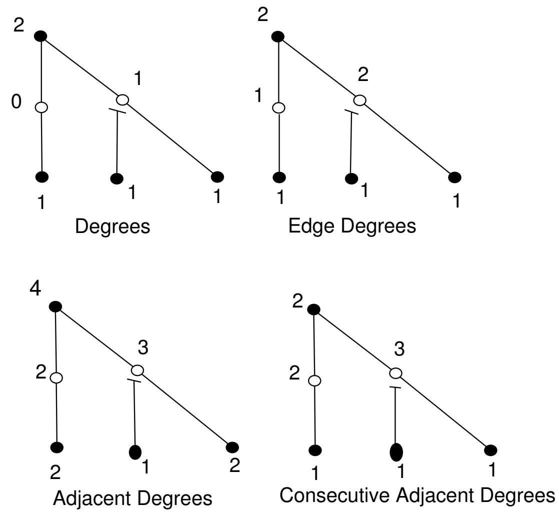

Fig. 3. A semigraph where vertices are labelled with their degrees, edge degrees, adjacent degrees and consecutive adjacent degrees

For any vertex v in a semigraph G, it can be noted that,

$$\deg v \leq \deg_e v \leq \deg_{ca} v \leq \deg_a v$$

Definition 2.3: In a semigraph, an edge of size at least three is known as a semiedge.

Definition 2.4: A semigraph is a k-uniform semigraph if all of its edges have size k.

Definition 2.5: Two semigraphs G_1 and G_2 on the same set of vertices are adjacency disjoint if any edge in G_2 has at most one vertex in common with any edge of G_1.

A semigraph $G' = (V', E')$ is a subsemigraph of a semigraph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. The union $G_1 \cup G_2$ of two adjacency disjoint semigraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a semigraph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

Definition 2.6: A complete semigraph is a semigraph in which every two vertices are adjacent. In addition, if every vertex is an end vertex of some edge in a complete semigraph then it is called a strongly complete semigraph. A semigraph which consists of a single edge of size $n \geq 3$ is complete but not strongly complete and is denoted by E_n^c. The strongly complete semigraph on n vertices with one edge of size $(n-1)$ and all other edges of size two is denoted by T_{n-1}^1.

Proposition 2.2: [9] For a (n, m) semigraph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{E_1, E_2, \ldots, E_m\}$.

$$\sum_{i=1}^{n} \deg v_i = 2m \qquad \text{(II.1)}$$

$$\sum_{i=1}^{n} \deg_e v_i = \sum_{i=1}^{m} |E_i| \qquad \text{(II.2)}$$

$$\sum_{i=1}^{n} (\deg_a v_i + \deg_e v_i) = \sum_{i=1}^{m} |E_i|^2 \qquad \text{(II.3)}$$

The incidence matrix with binomial coefficients, called the binomial incidence matrix is defined formally as below.

Definition 2.7: [15] Let $G = (V, E)$ be a semigraph with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let size of the edge e_j be $n_j + 1$, $1 \le j \le m$. The binomial incidence matrix of G, denoted by $\mathscr{B}(G)$, is a $n \times m$ matrix, whose rows are indexed by the vertex set and columns are indexed by the edge set of G. The column corresponding to e_j in the binomial incidence matrix consists of entries $0, {}^{n_j}C_0, \ldots, {}^{n_j}C_{n_j}$, where nonzero entries correspond to the vertices on the edge. The entries ${}^{n_j}C_0$ and ${}^{n_j}C_{n_j}$ correspond to the end vertices of the edge e_j. The (i, j) entry of $\mathscr{B}(G)$ is given by,
b_{ij}, if vertex u_i and edge e_j are incident and u_i is the r^{th} vertex from the end vertex of e_j with entry ${}^{n_j}C_0, 0 \le r \le n_j$ and
0, if vertex u_i and edge e_j are not incident on each other.

The binomial incidence matrix $\mathscr{B}(G)$ of semigraph G given in Example 2.1 is given by

$$\mathscr{B}(\mathbf{G}) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} {}^2C_0 & {}^3C_0 & 0 & 0 \\ {}^2C_1 & 0 & {}^1C_0 & 0 \\ {}^2C_2 & 0 & 0 & {}^2C_0 \\ 0 & {}^3C_1 & 0 & 0 \\ 0 & {}^3C_2 & {}^1C_1 & 0 \\ 0 & 0 & 0 & {}^2C_1 \\ 0 & {}^3C_3 & 0 & {}^2C_2 \end{pmatrix} \end{array}$$

We refer the interested readers to [9], [15]–[17].

III. Regular Semigraphs

Vertices in Semigraph have four variety of degrees. So, each kind of degree can be used to define a different type of a regular semigraph.

Definition 3.1: Let $G = (V, E)$ be a semigraph. The semigraph G is said to be

1) D_k-regular if every vertex has degree k
2) ED_k-regular if every vertex has edge degree k
3) AD_k-regular if every vertex has adjacent degree k
4) CAD_k-regular if every vertex has consecutive adjacent degree k.

Example 3.1: The four variety of regular semigraphs defined on the basis of degree concepts in semigraph are shown in Figure 4.

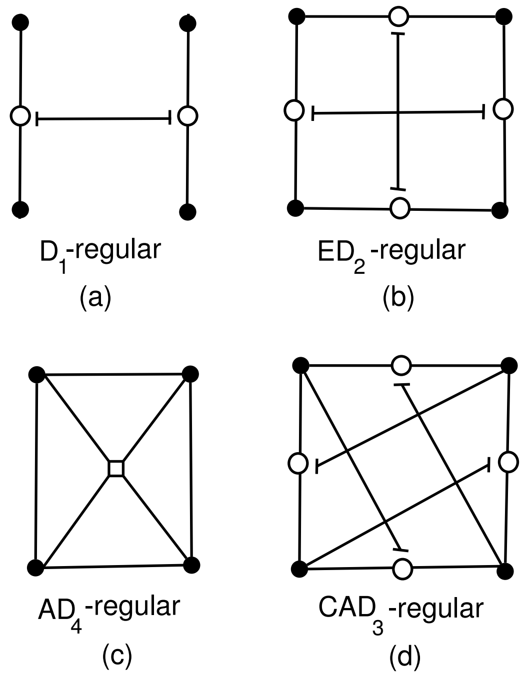

Fig. 4. Four regular semigraphs with $\deg v = 1$, $\deg_e v = 2$, $\deg_a v = 4$ and $\deg_{ca} v = 3$, respectively

Note 3.1: The four regular semigraphs defined above are depending upon a variety of degree of a vertex in semigraphs and it is interesting to note the following.

1) A semigraph which is D-regular need not be ED, AD and CAD-regular as shown in Figure 4(a).
2) A semigraph which is ED-regular need not be D, AD and CAD-regular as shown in Figure 4(b).
3) A semigraph which is AD-regular need not be D, ED and CAD-regular as shown in Figure 4(c).
4) A semigraph which is CAD-regular need not be D, ED and AD-regular as shown in Figure 4(d).

Definition 3.2: A semigraph is said to be vertex-regular if the semigraph is D, ED, AD and CAD-regular.

Note 3.2: Semigraphs given in Figure 5(a) and Figure 6 are vertex-regular semigraphs.

At the same time using binomial incidence matrix of a semigraph, in particular, the row sum of the binomial incidence matrix and the column sum of the binomial incidence matrix, regular semigraphs have been defined as follows.

Definition 3.3: A semigraph is said to be

1) R-regular if the binomial incidence matrix of the semigraph has constant row sum.
2) C-regular if the binomial incidence matrix of the semigraph has constant column sum.

Clearly, k-uniform semigraph are the only C-regular semigraphs, with sum of each column is equal to ${}^{k-1}C_0 + {}^{k-1}C_1 + \ldots + {}^{k-1}C_{k-1} = 2^{k-1}$.

Example 3.2: The semigraph G as shown in Figure 5(a) is a R-regular semigraph which is not C-regular. The semigraph H as shown in Figure 5(b) is a C-regular semigraph but not R-regular.

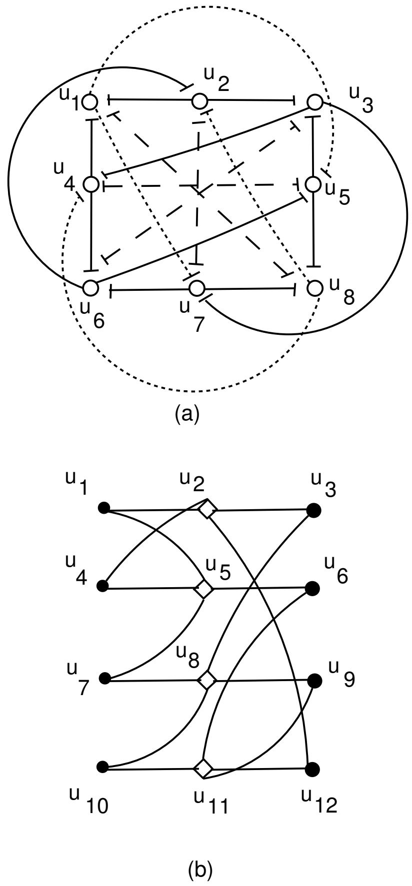

Fig. 5. R-regular Semigraph G and C-regular Semigraph H

The row sum in $\mathscr{B}(G)$ is 2^2+1 and the column sum is 2^2 or 2. Similarly, the column sum in $\mathscr{B}(H)$ is 2^2 and the row sum is 2^2 or 2.

After defining R-regular semigraph and C-regular semigraph it is straight forward to think about semigraphs which have all of its row sum equal and also all of its column sum equal.

Definition 3.4: A semigraph is said to be RC-regular if the semigraph is both R-regular and C-regular.

Example 3.3: The semigraph $G=(V,E)$ as shown in Figure 6 with $V(G)=\{u_1,u_2,\ldots,u_8\}$ and $E(G)=\{e_1=(u_1,u_2,u_3); e_2=(u_1,u_4,u_6); e_3=(u_3,u_5,u_8); e_4=(u_6,u_7,u_8);$
$e_5=(u_5,u_1,u_7); e_6=(u_4,u_3,u_7); e_7=(u_2,u_6,u_5); e_8=(u_2,u_8,u_4)\}$ is RC-regular.

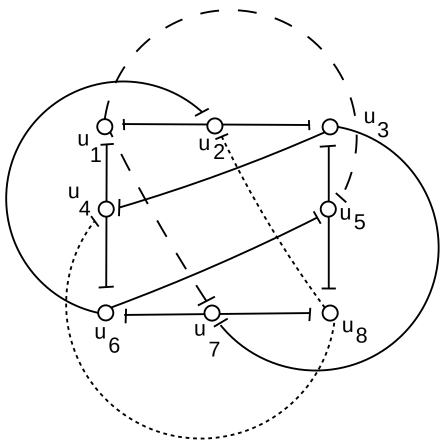

Fig. 6. RC-regular semigraph

The sum of each column in $\mathscr{B}(G)$ is equal to 2^2 and the sum of each row is also equal to 2^2.

Theorem 3.1: In a RC-regular semigraph the sum of each column is equal and it is the same as the sum of each row.

Proof: Let G be a semigraph which is RC-regular. In particular when G is C-regular, then G is a k-uniform semigraph for some positive integer k. Each column of $\mathscr{B}(G)$ has entries ${}^{k-1}C_0, {}^{k-1}C_1, \ldots, {}^{k-1}C_{k-1}$, sum of which is 2^{k-1}. As the semigraph G is R-regular, the sum of all the row entries same. Since G is k-uniform semigraph, entries in any row are from the set $\{{}^{k-1}C_0, {}^{k-1}C_1, \ldots, {}^{k-1}C_{k-1}\}$, where ${}^{k-1}C_t = {}^{k-1}C_{k-1-t}$, $0 \le t \le k-1$ and each vertex should appear in every k position exactly once in any of the edge of G to get the row sum equal, which results in row sum equal to 2^{k-1}. ■

Theorem 3.2: A uniform semigraph is R-regular if and only if it is vertex-regular.

Proof: In a binomial incidence matrix of a semigraph with n vertices, the following are true;

1) The number of $1's$ in the row corresponds to a vertex u_i is $\deg u_i$, $1 \le i \le n$,
2) The number of non zero entries in a row corresponding to the vertex u_i is the edge degree of u_i i.e $\deg_e u_i$, $1 \le i \le n$.
3) The number of non zero entries along the columns for which row of a vertex u_i has a non zero entry is $\deg_a u_i$, $1 \le i \le n$.
4) The number of $1's$ in the row that corresponds to a vertex u_i plus twice the number of non zero entries other than $1's$ in the same row is $\deg_{ca} u_i$, $1 \le i \le n$.

Let G be a k-uniform semigraph. The column entries in $\mathscr{B}(G)$ are ${}^{k-1}C_0, {}^{k-1}C_1, \ldots, {}^{k-1}C_{k-1}$, where ${}^{k-1}C_t = {}^{k-1}C_{k-1-t}$ and middle entry in the sequence is given by

$$middle\ entry = \begin{cases} {}^{k-1}C_{\frac{k-1}{2}} & \text{when } k \text{ is odd} \\ \\ {}^{k}C_{\frac{k-2}{2}} & \text{when } k \text{ is even.} \end{cases}$$

Suppose that G is a R-regular semigraph. Each entry till middle entry should appear at least once in each row and repeat an equal number of times, to get the sum of each row same.

So, in the binomial incidence matrix of a k-uniform semigraph G which is R-regular with n vertices, the following are true;

1) The number of $1's$ in each row of $\mathscr{B}(G)$ is the same, hence G is $D_{\deg u_i}$-regular.
2) The number of non zero entries in each row of $\mathscr{B}(G)$ is the same, hence G is $ED_{\deg_e u_i}$-regular.
3) The number of non zero entries along with the columns for which row of each vertex of $\mathscr{B}(G)$ has a non zero entry is the same, hence G is $AD_{\deg_a u_i}$-regular.
4) The number of $1's$ in each row plus twice the number of non zero entries other than $1's$ in the same row of $\mathscr{B}(G)$ is the same, hence G is $CAD_{\deg_{ca} u_i}$-regular and $\deg_{ca} u_i = \deg u_i + 2\deg_e u_i$, $1 \le i \le n$.

Conversely, if a k-uniform semigraph G is vertex-regular then it follows that G is R-regular. ■

Corollary 3.3: If a semigraph is RC-regular then it is vertex-regular.

Proof: The proof directly follows from the proof of Theorem 3.2. ■

Theorem 3.4: If a semigraph is D and ED-regular then it is CAD-regular.

Proof: Let G be a semigraph which is D-regular and ED-regular with the vertex set $\{v_1, v_2, \ldots, v_p\}$. As the semigraph G is D-regular, it follows that for each vertex of G number of edges containing them as an end vertex is fixed, say k_1. Each edge containing v_i as an end vertex contribute 1 to $\deg_{ca} v_i$.
From the definition of ED-regular, it follows that for each vertex of G number of edges containing them is fixed. Number of edges containing them as a mid vertex is fixed, say k_2. Each edge containing v_i as a mid vertex contribute 2 to $\deg_{ca} v_i$.

Hence,

$$\deg_{ca} v_i = k_1 + 2k_2, \quad \forall \quad 1 \leq i \leq p,$$

which implies that, the consecutive adjacent degree of every vertex in G is same. Hence, G is a CAD-regular semigraph. ■

Theorem 3.5: If a semigraph is D and CAD-regular then it is ED-regular.

Proof: Let G be a semigraph which is D-regular and CAD-regular with the vertex set $\{v_1, v_2, \ldots, v_p\}$.
As the semigraph G is D-regular, from the definition it follows that for each vertex of G, the number of edges containing them as an end vertex is fixed, say, k_1. Each edge containing v_i as an end vertex contribute 1 to $\deg_{ca} v_i$.
The semigraph G is also CAD_{k_0}-regular, for some positive integer k_0. Then, the number of vertices consecutively adjacent to v_i, $\forall\ 1 \leq i \leq p$, is fixed. Therefore,

$$k_1 + 2k_2 = k_0,$$

where k_2 is the number of edges containing v_i, $\forall\ 1 \leq i \leq p$ as a mid vertex. Hence,

$$k_2 = \frac{k_0 - k_1}{2}, \quad \text{is also fixed.}$$

So, number of edges containing v_i, $\forall\ 1 \leq i \leq p$ is fixed as well. Hence the result. ■

Theorem 3.6: If a semigraph is ED and CAD-regular then it is D-regular.

Proof: Let G be a semigraph which is ED-regular and CAD-regular with the vertex set $\{v_1, v_2, \ldots, v_p\}$. Let k_1 be the number of edges containing v_i, $1 \leq i \leq p$ as an end vertex in G. Let k_2 be the number of edges containing v_i, $1 \leq i \leq p$ as a mid vertex in G. Each edge containing v_i as an end vertex contribute 1 to $\deg_{ca} v_i$ and an edge containing v_i as a mid vertex contribute 2 to $\deg_{ca} v_i$. Hence, $k_1 + 2k_2$ is fixed. Since, G is CAD_{k_0}-regular,

$$k_1 + 2k_2 = k_0.$$

Let k_3 be the number of edges containing v_i, $1 \leq i \leq p$ is fixed. Since, G is ED_{k_3}-regular we have,

$$k_1 + k_2 = k_3.$$

Therefore,

$$k_1 = 2k_3 - k_0.$$

Therefore, k_1 is fixed in G i.e., the number of edges containing v_i, $1 \leq i \leq p$ as an end vertex in G is fixed. Hence the result. ■

Note 3.3: If a semigraph is D and CAD-regular then it need not be AD-regular.

Example 3.4: In Figure 7, we give a semigraph of 14 vertices and 7 edges which is D and CAD-regular but not AD-regular.

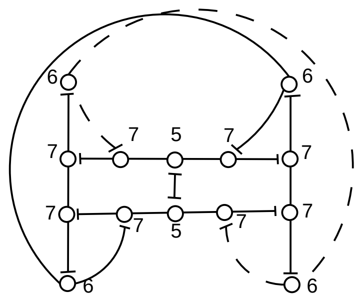

Fig. 7. A semigraph which is D_1 and CAD_3-regular where vertices are labelled with their adjacent degrees

Note 3.4: A semigraph which is D and ED-regular need not be AD-regular, which follows from Theorem 3.4, Theorem 3.5 and Example 3.4.

Note 3.5: A semigraph which is D and AD-regular need not be ED-regular and CAD-regular.

Example 3.5: The semigraph with 8 vertices and 8 edges as shown in Figure 8 is D and AD-regular but not ED-regular and CAD-regular.

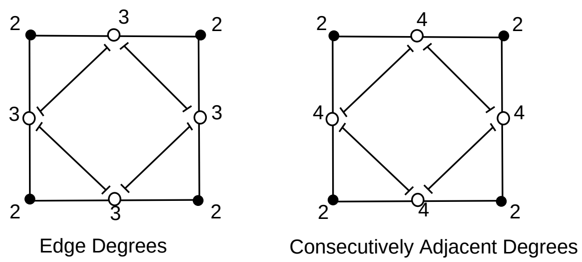

Fig. 8. A semigraph which is D_2 and AD_4-regular where vertices are labelled with their edge degrees and consecutive adjacent degrees

Note 3.6: A semigraph which is ED and AD-regular need not be D-regular and CAD-regular.

Example 3.6: The semigraph with 8 vertices and 8 edges as shown in Figure 9 is ED and AD-regular but not D-regular and CAD-regular.

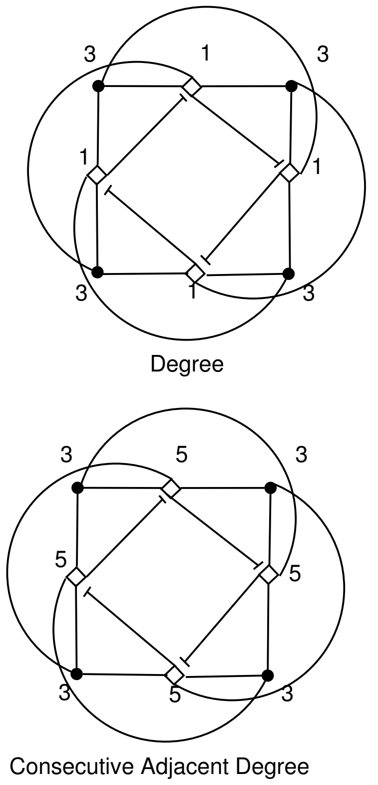

Fig. 9. A semigraph which is ED_3 and AD_6-regular where vertices are labelled with their degrees and consecutive adjacent degrees

Observation 3.7: Any ED_1-regular semigraph consist of disjoint union of E_n^c, for $n \geq 2$. Similarly, any AD_1-regular semigraph and a CAD_1-regular semigraph consist of disjoint union of K_2s.

Theorem 3.8: Let G be a (n, m) semigraph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{E_1, E_2, \ldots, E_m\}$. Let N_{me} be the number of mid end vertices in G.

When

1) G is D_k-regular,
   - $kn = 2m$
   - $N_{me} = \sum_{i=1}^{m} |E_i| - kn$
2) G is ED_k-regular,
   - $kn = \sum_{i=1}^{m} |E_i|$
   - $N_{me} = kn - 2m$
3) G is AD_k-regular,
   - $kn = \sum_{i=1}^{m} (|E_i|^2 - |E_i|)$
   - $N_{me} = \sum_{i=1}^{m} |E_i|^2 - kn - 2m$
4) G is CAD_k-regular,
   - $kn = 2\sum_{i=1}^{m} |E_i| - 2m$
   - $N_{me} = \frac{kn}{2} - m.$

Proof: From (II.2) and (II.3) we have,

$$\sum_{i=1}^{n} \deg_a v_i = \sum_{i=1}^{m} (|E_i|^2 - |E_i|). \tag{III.4}$$

From (II.1) and (II.2) we have,

$$\sum_{i=1}^{n} \deg_{ca} v_i = 2\sum_{i=1}^{n} \deg_e v_i - \sum_{i=1}^{n} \deg v_i \tag{III.5}$$

$$= 2\left(\sum_{i=1}^{m} |E_i| - m\right) \tag{III.6}$$

and

$$N_{me} = \sum_{i=1}^{n} \deg_e v_i - \sum_{i=1}^{n} \deg v_i \tag{III.7}$$

$$= \sum_{i=1}^{m} |E_i| - 2m. \tag{III.8}$$

In a D_k-regular semigraph,

$$\deg v_i = k, \quad \text{for all i}, \quad 1 \leq i \leq m.$$

From (II.1)

$$\sum_{i=1}^{n} \deg v_i = \sum_{i=1}^{n} k$$

$$2m = kn$$

From (III.7) we have,

$$N_{me} = \sum_{i=1}^{m} |E_i| - kn$$

Similarly rest of the things can be proved. ■

Observation 3.9: All graphs are semigraphs, but a semigraph is not a graph when it has a semiedge. Consider semigraphs with at least one semiedge. Then, the minimum number of edges required for a semigraph to be D-regular is given in Figure 10(a) and that of CAD-regular semigraph is given in Figure 10(b). A semigraph with a minimum number of edges such that it is ED-regular as well as AD-regular is shown in Figure 10(c).

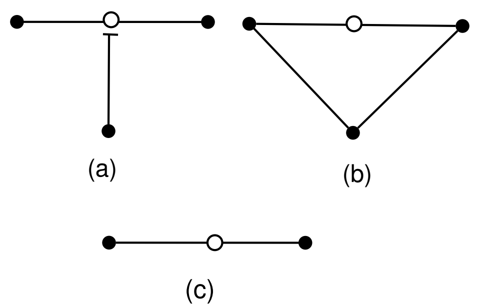

Fig. 10. D-regular semigraph, CAD-regular semigraph and ED-regular, AD-regular semigraph with a minimum number of edges possible

A. Number of complete semigraphs

In this section we give expression for number of complete semigraphs on n vertices having m edges. First we observe the following.

Observation 3.10: Complete semigraph on n vertices is an AD_{n-1}-regular semigraph.

We can observe that there are several complete semigraphs on n vertices having m edges, for different values of m.

Example 3.7: The two complete semigraphs on 8 vertices and 19 edges are as shown in Figure 11.

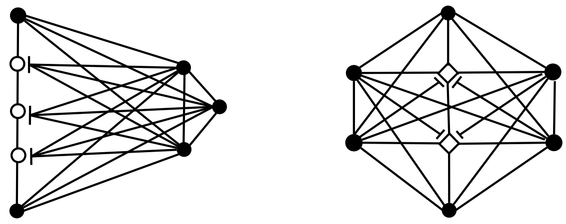

Fig. 11. Complete semigraphs on 8 vertices and 19 edges

The first semigraph has one edge of size 5 and eighteen edges of size 2, whereas the second semigraph has one edge of size 4, two edges of size 3 and sixteen edges of size 2. Counting the number of different complete semigraphs on a given number of vertices and edges, is a difficult problem.

Theorem 3.11: The number of non isomorphic complete semigraphs on n vertices having m edges is equal to

$$\sum_{m=1}^{\binom{n}{2}} \sum_{j=1}^{N} \sigma_j(n,m),$$

where $\sigma_j(n,m)$ is denote the number of non isomorphic complete semigraphs on n vertices having m edges of sizes $n_{j_1}, n_{j_2}, \ldots, n_{j_m}, 1 \leq j \leq N$.

Proof: From Theorem 2.1 we know that unless $\binom{n}{2} = \sum_{i=1}^{m} \binom{n_{j_i}}{2}$ the semigraph on n vertices and m edges of size $n_1, n_2, \ldots, n_m$ cannot be complete. Hence, the number of complete semigraphs on n vertices having m edges of given size $n_1, n_2, \ldots, m_m$ cannot exceed the number of different ways in which $\binom{n}{2}$ can be written as sum of m smaller triangular numbers namely, $\binom{n_1}{2}, \binom{n_2}{2}, \ldots, \binom{n_m}{2.}$ Let $\mu_j(n,m)$ denote the ways in which $\binom{n}{2}$ can be written as sum of $\binom{n_{j_1}}{2}, \binom{n_{j_2}}{2}, \ldots, \binom{n_{j_m}}{2}$. That is $\binom{n}{2} = \sum_{i=1}^{m} \binom{n_{j_i}}{2}$ for $1 \leq j \leq N$. Hence, the result follows. ■

Note that, $\sigma_j(n,m)$ may be equal to zero for a given set of positive integers $n_{j_1}, n_{j_2}, \ldots, n_{j_m}$ even when $\binom{n}{2} = \sum_{i=1}^{m} \binom{n_{j_i}}{2}$. A triangular number is a number which can be written as $\dfrac{k(k+1)}{2}$ for some natural number k. There are many results in the literature [1], [11], [13] about writing a given natural numbers as the sum of r triangular numbers, for different values of r. So, the problem of finding $\sigma_j(n,m)$ is open and so is the problem of finding the number of non isomorphic complete semigraphs on n vertices.

IV. Conclusions

In this article we have studied variety of regular semigraph depending upon the degree concepts of semigraphs and binomial incidence matrix of a semigraphs. The results in a semigraph like when a particular vertex regularity implies some other vertex regularity is an interesting part as those are not trivial results. Also, we believe that application of regular graphs in various field make space for regular semigraphs to find some application. Checking the existence of the Hamiltonian cycle in these variety of regular semigraphs could be a goal for future work.

References

[1] C. Adiga "On the Representations of an Integer as a Sum of Two or Four Triangular Numbers", Nihonkai Math J, vol. 3 , no.2, pp125-131, 1992.

[2] A. E. Brouwer and W. H. Haemers, "Eigenvalues and Perfect Matchings ", Linear Algebra and its Applications, vol. 395, pp155-162, 2005.

[3] A, M. Raigorodskii, "Thirty Essays on Geometric Graph Theory", Springer, 2013.

[4] Bollobas bela, "Random Graphs", Cambridge: Cambridge Univ. Pr., 2004.

[5] D. B. West, "Introduction to Graph Theory", New York, NY: Pearson, 2018.

[6] D. Galvin, D. "Counting Colorings of a Regular Graph", Graphs and Combinatorics, vol. 31, no.3, pp629-638, 2015.

[7] D. M. Cvetkovic, "Spectra of Graphs: Theory and Applications", New York: Wiley, 1998.

[8] E. Bullmore and O. Sporns, "Complex Brain Networks: Graph Theoretical Analysis of Structural and Functional Systems", Nature Reviews Neuroscience, vol. 10, no.2, pp186-198, 2009.

[9] E. Sampathkumar, C. M. Deshpande, B. Y. Bam, L. Pushpalatha and V. Swaminathan, "Semigraphs and Their Applications", India: Academy of Discrete Mathematics and Applications India, 2019.

[10] F. Harary, Graph Theory, Cambridge, MA: Perseus Books, 2001.

[11] H.M. Farkas, "Sums of Squares and Triangular Numbers ", Online J Anal Combinat, vol. 1, 2006.

[12] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, New York: Wiley, 2002.

[13] J.A. Ewell, "On sums of triangular numbers and sums of squares", Am Math Month, vol. 99, no.8, pp752-757, 1992.

[14] R. Yuster, "Maximum Matching in Regular and Almost Regular Graphs", Algorithmica, vol. 66, no.1, pp87-92, 2013.

[15] Shetty, J. and Sudhakara, G.,. "Binomial Incidence Matrix of a Semigraph". Discrete Mathematics, Algorithms and Applications, vol. 13, no. 03, 2021.

[16] Shetty, J., Sudhakara, G. and Madhusudanan, V., "On the Existence of Semigraphs and Complete Semigraphs with Given Parameters". Ain Shams Engineering Journal, vol. 12, no.4, pp4119-4124, 2021.

[17] Shetty, J., K Arathi Bhat and Sudhakara, G., "Addition Operation in Semigraphs", Applied Mathematics E-Notes, vol. 22, pp184-194, 2022.

[18] Shetty, J., Sudhakara, G., and Madhusudanan, V. "Encryption System Involving Matrix Associated with Semigraphs", IAENG International Journal of Applied Mathematics, vol. 52, no.2, pp458-465, 2022.

[19] Shetty, J. and Sudhakara, G., 2022. "Domination Number of a Bipartite Wemigraph when it is a Cycle", TWMS Journal of Applied and Engineering Mathematics, 12(1), pp167-175, 2022.

[20] T. I. Fenner and A. M. Frieze. "Hamiltonian Cycles in Random Regular Graphs", Journal of Combinatorial Theory, Series B, vol. 37, no.2, pp103-112, 1984.

[21] T. Koledin and Z. Stanic, "Regular Graphs Whose Second Largest Eigenvalue is At most 1", Novi Sad J. Math, vol. 43, no.1, pp145-153, 2013.

Jyoti Shetty received her B.Sc. from Karnataka University, Dharwad, India, in 2009 and M.Sc. degree in Mathematics from Mangalore University, Mangalore, India, in 2013. She then received her Ph.D. from Manipal Academy of Higher Education, Manipal, in 2021. At present she is working as an Assistant Professor in the Department of Mathematics at Manipal Institute of Technology Bengaluru, Manipal Academy of Higher Education, Manipal.

Sudhakara G. received his B.Sc. and M.Sc. degrees in Mathematics from Mangalore University, Mangalore, India, in 1984 and 1986, respectively. He is a Professor and Head of the Department of Mathematics, Manipal Institute of Technology, Manipal, India, where he is working as a faculty in the department since 1987. His research interests include Graph Theory, Combinatorics, and Number theory.

K Arathi Bhat received her B. Sc. Degree from Mangalore University, Mangalore, India, in 2000. and M. Sc. degree from Manipal Academy of Higher Education, Manipal, in 2011. She then received her Ph.D. from Manipal Academy of Higher Education, Manipal, in 2018. At present she is working as an Assistant Professor - Selection Grade in the Department of Mathematics at Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal. Her research interests include Algebraic graph theory, Network theory and Combinatorics.