Oscillatory and Asymptotic Properties of Fractional Delay Dynamic Equations on Time Scales Involving Conformable Fractional Derivative

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Abstract—**In this work, we research oscillation for a class of fractional dynamic equations on time scales involving delay term. By use of a generalized Riccati function, inequality technique, and especially a certain technique dealing with the delay term, some new sufficient conditions for oscillation and asymptotic behaviour are proposed. The established results unify continuous and discrete analysis as two special cases of arbitrary time scales, and are further extensions of the corresponding oscillatory and asymptotic results for delay dynamic equations involving derivatives of integer order. We also present some examples for the established results.**

Index Terms—**oscillation; asymptotic behavior; dynamic equations; time scales**

I. Introduction

For a long time, research on analytical or semi-analytical solutions of various differential equations has been a hot topic [1-3]. Besides, it is well known that research on qualitative properties of solutions of differential and difference equations is also very important in the case their solutions are unknown, such as the stability, existence and so on [4-6]. Oscillation belongs to the range of qualitative properties analysis. In the last few decades, research for oscillation of various equations including differential equations, difference equations has been a hot topic in the literature, and much effort has been done to establish new oscillatory criteria for these equations especially fractional differential equations so far [7-9]. In [10], Hilger initiated the theory of time scale trying to treat continuous and discrete analysis in a consistent way. Based on the theory of time scale, Many authors have taken research in oscillation of various dynamic equations on time scales (see [11-25] for example). In the research for oscillation of dynamic equations on time scales, we notice that little attention has been paid to the research of oscillation of fractional order dynamic equations on time scales so far in the literature. In [26], Feng and Meng researched oscillation for a class of fractional order dynamic equations on time scales as follows

$$(a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma})^{(\alpha)} + p(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma} + q(t)f(x(t)) = 0, \tag{1.1}$$

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where $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of order α [27]. Based on the properties of conformable fractional calculus, some oscillatory and asymptotic criteria for this equation were established. However, to our best knowledge, there has been few results on oscillatory and asymptotic behaviour for fractional dynamic equations on time scales involving delay term so far.

Motivated by the above analysis, and based on (1.1), in this paper, we further consider oscillatory and asymptotic behavior of dynamic equation involving delay term, and are concerned with the following fractional delay dynamic equation with damping term on time scales:

$$(a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\nu})^{(\alpha)} + p(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\nu} + q(t)f(x(\kappa(t))) = 0,\ t \in \mathbf{T}_0, \tag{1.2}$$

where $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of order α, $\mathbf{T}$ is an arbitrary time scale, $\mathbf{T}_0 = [t_0, \infty) \bigcap \mathbf{T}$, $a,\ r,\ p,\ q \in C_{rd}(\mathbf{T}_0, \mathbf{R}_+)$, $f \in C(\mathbf{R}, \mathbf{R})$ satisfying $xf(x) > 0$, $\frac{f(x)}{x^{\nu}} \geq L > 0$ for $x \neq 0$, $\kappa \in C_{rd}(\mathbf{R}, \mathbf{R})$ is the delay function satisfying $\kappa(t) \leq t$, $\kappa^{\Delta}(t) \geq 0$ and $\lim_{t\to\infty} \kappa(t) = \infty$, $\nu \geq 1$ is a quotient of two odd positive integers.

A solution of Eq. (1.2) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Eq. (1.2) is said to be oscillatory in case all its solutions are oscillatory.

The next of this paper will be organized as follows. In Section 2, we present some basis for the theory of the time scale and the conformable fractional calculus. In Section 3, by use of the properties of conformable fractional calculus, a generalized Riccati function and inequality technique, we establish some new oscillatory and asymptotic criteria for Eq. (1.2). In Section 4, we present some examples for the established results. Some conclusions are given at the end of this paper.

Throughout this paper, $\mathbf{R}$ denotes the set of real numbers and $\mathbf{R}_+ = (0, \infty)$, while $\mathbf{Z}$ denotes the set of integers. $\widetilde{p}(t) = t^{\alpha-1}p(t)$, $\theta_1(t,a) = \int_a^t \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\nu}}}{a^{\frac{1}{\nu}}(s)}\Delta^{\alpha}s$, $\theta_2(t,a) = \int_a^t \frac{\theta_1(s,a)}{r(s)}\Delta^{\alpha}s$, $t_i \in \mathbf{T}$, $i = 0, 1, ..., 6$. For an interval $[a,b]$, $[a,b]_{\mathbf{T}} := [a,b] \bigcap \mathbf{T}$, and we always assume $\kappa \circ \sigma = \sigma \circ \kappa$.

II. Basis for the theory of time scale and conformable fractional calculus

A time scale is an arbitrary nonempty closed subset of the real numbers. $\mathbf{T}$ denotes an arbitrary time scale. On $\mathbf{T}$ we define the forward and backward jump operators $\sigma \in (\mathbf{T}, \mathbf{T})$ and $\rho \in (\mathbf{T}, \mathbf{T})$ such that $\sigma(t) = \inf\{s \in \mathbf{T}, s > t\}$, $\rho(t) = \sup\{s \in \mathbf{T}, s < t\}$. A point $t \in \mathbf{T}$ is said to be left-dense if $\rho(t) = t$ and $t \neq \inf \mathbf{T}$, right-dense if $\sigma(t) = t$ and $t \neq \sup \mathbf{T}$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The set $\mathbf{T}^\kappa$ is defined to be $\mathbf{T}$ if $\mathbf{T}$ does not have a left-scattered maximum, otherwise it is $\mathbf{T}$ without the left-scattered maximum. A function $f \in (\mathbf{T}, \mathbf{R})$ is called rd-continuous if it is continuous at right-dense points and if the left-sided limits exist at left-dense points, while f is called regressive if $1 + \mu(t)f(t) \neq 0$, where $\mu(t) = \sigma(t) - t$. C_{rd} denotes the set of rd-continuous functions, while $\mathcal{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathcal{R}^+ = \{f | f \in \mathcal{R},\ 1 + \mu(t)f(t) > 0,\ \forall t \in \mathbf{T}\}$.

Definition 2.1: For some $t \in \mathbf{T}^\kappa$, and a function $f \in (\mathbf{T}, \mathbf{R})$, the *delta derivative* of f at t is denoted by $f^\Delta(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}$ of t satisfying

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in \mathcal{U}$.

Note that if $\mathbf{T} = \mathbf{R}$, then $f^\Delta(t)$ becomes the usual derivative $f'(t)$, while $f^\Delta(t) = f(t+1) - f(t)$ if $\mathbf{T} = \mathbf{Z}$, which represents the forward difference.

Definition 2.2: For $p \in \mathcal{R}$, the *exponential function* is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \text{ for } s, t \in \mathbf{T}.$$

Due to [28, Theorem 5.2], if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for $\forall s,\ t \in \mathbf{T}$.

For more details about the calculus of time scales, we refer to [29].

The following are some important definitions and theorems [27] for the conformable fractional calculus on time scales (see also in [26]).

Definition 2.3 [27, Definition 1]. For $t \in \mathbf{T}^\kappa$, $\alpha \in (0, 1]$, and a function $f \in (\mathbf{T}, \mathbf{R})$, the fractional derivative of α order for f at t is denoted by $f^{(\alpha)}(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}$ of t satisfying

$$|[f(\sigma(t)) - f(s)]t^{1-\alpha} - f^{(\alpha)}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in \mathcal{U}$.

Definition 2.4 [27, Definition 28]. If $F^{(\alpha)}(t) = f(t),\ t \in \mathbf{T}^\kappa$, then F is called an α-order *antiderivative* of f, and the Cauchy α-fractional integral of f is defined by

$$\int_a^b f(t)\Delta^\alpha t = \int_a^b f(t)t^{\alpha-1}\Delta t = F(b) - F(a),$$

where $a,\ b \in \mathbf{T}$.

Theorem 2.5 [27, Theorem 4]. For $t \in \mathbf{T}^\kappa$, $\alpha \in (0, 1]$, and a function $f \in (\mathbf{T}, \mathbf{R})$, the following conclusions hold:

(i). If f is conformal fractional differentiable of order α at $t > 0$, then f is continuous at t.

(ii). If f is continuous at t and t is right-scattered, then f is conformable fractional differentiable of order α at t with $f^{(\alpha)}(t) = \dfrac{f(\sigma(t)) - f(t)}{\sigma(t) - t}t^{1-\alpha} = \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}t^{1-\alpha}$.

(iii). If t is right-dense, then f is conformable fractional differentiable of order α at t if, and only if, the limit $\lim\limits_{s\to t} \dfrac{f(s) - f(t)}{s - t}t^{1-\alpha}$ exists as a finite number. In this case, $f^{(\alpha)}(t) = \lim\limits_{s\to t} \dfrac{f(s) - f(t)}{s - t}t^{1-\alpha}$.

(iv). If f is fractional differentiable of order α at t, then $f(\sigma(t)) = f(t) + \mu(t)t^{1-\alpha}f^{(\alpha)}(t)$.

Corollary 2.6. According to the definition of the conformable fractional differentiable of order α, it holds that $f^{(\alpha)}(t) = t^{1-\alpha}f^\Delta(t)$, where $f^\Delta(t)$ is the usual Δ derivative in the case $\alpha = 1$. Furthermore, if $f^{(\alpha)}(t) > 0$ (< 0) for $t > 0$, then f is increasing (decreasing) for $t > 0$.

Theorem 2.7: Let $\widetilde{p}(t) = t^{\alpha-1}p(t)$, $\alpha \in (0, 1]$. If $\widetilde{p} \in \mathcal{R}$, and fix $t_0 \in \mathbf{T}$, then the *exponential function* $e_{\widetilde{p}}(t, t_0)$ is the unique solution of the following initial value problem

$$\begin{cases} y^{(\alpha)}(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases}$$

Proof. By [28, Theorem 5.1], if $p \in \mathcal{R}$, and fix $t_0 \in \mathbf{T}$, then the *exponential function* $e_p(t, t_0)$ is the unique solution of the following initial value problem

$$\begin{cases} y^\Delta(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases}$$

So according to Corollary 2.6, one has

$$(e_{\widetilde{p}}(t, t_0))^{(\alpha)} = t^{1-\alpha}(e_{\widetilde{p}}(t, t_0))^\Delta = t^{1-\alpha}\widetilde{p}(t)e_{\widetilde{p}}(t, t_0) = p(t)e_{\widetilde{p}}(t, t_0).$$

So the proof is complete.

Theorem 2.8 [27, Theorem 15]. Assume $f,\ g \in (\mathbf{T}, \mathbf{R})$ are conformable fractional differentiable of order α. Then

(i). $(f + g)^{(\alpha)}(t) = f^{(\alpha)}(t) + g^{(\alpha)}(t)$.

(ii). $(fg)^{(\alpha)}(t) = f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t) = f^{(\alpha)}(t)g(\sigma(t)) + f(t)g^{(\alpha)}(t)$.

(iii). $(\dfrac{1}{f})^{(\alpha)}(t) = -\dfrac{f^{(\alpha)}(t)}{f(t)f(\sigma(t))}$.

(iv). $(\dfrac{f}{g})^{(\alpha)}(t) = \dfrac{f^{(\alpha)}(t)g(t) - f(t)g^{(\alpha)}(t)}{g(t)g(\sigma(t))}$.

Theorem 2.9. Let $\alpha \in (0, 1]$, $f,\ g$ be two rd-continuous functions. Then

$$\int_a^b f^{(\alpha)}(t)g(t)\Delta^\alpha t = [f(t)g(t)]_a^b - \int_a^b f(\sigma(t))g^{(\alpha)}(t)\Delta^\alpha t.$$

The proof of Theorem 2.9 can be reached by fulfilling α-fractional integral for the first equality in Theorem 2.8 (ii).

III. Main Results

The following lemmas are necessary for proving our main results.

Lemma 3.1. Suppose $-\frac{\widetilde{p}}{a} \in \mathcal{R}_+$, and assume that

$$\int_{t_0}^\infty \frac{[e_{-\frac{\tilde{p}}{a}}(s,t_0)]^{\frac{1}{\nu}}}{a^{\frac{1}{\nu}}(s)}\Delta^\alpha s = \infty, \tag{3.1}$$

$$\int_{t_0}^\infty \frac{1}{r(s)}\Delta^\alpha s = \infty, \tag{3.2}$$

and x is eventually a positive solution of Eq. (1.2). Then there exists a sufficiently large T_1^* such that

$$\Big(\frac{a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^\nu}{e_{-\frac{\tilde{p}}{a}}(t,t_0)}\Big)^{(\alpha)} < 0,\ [r(t)x^{(\alpha)}(t)]^{(\alpha)} > 0 \text{ on}$$

$[T_1^*,\infty)_{\mathbf{T}}$.

Lemma 3.2. Under the conditions of Lemma 3.1, furthermore, assume that

$$\lim_{t\to\infty}\sup \int_{t_0}^t \Big[\frac{1}{r(\xi)}\int_\xi^\infty \Big(\frac{e_{-\frac{\tilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty \frac{q(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\Big)^{\frac{1}{\nu}}$$
$$\Delta^\alpha\tau\Big]\Delta^\alpha\xi = \infty. \tag{3.3}$$

Then either there exists a sufficiently large T_2^* such that $x^{(\alpha)}(t) > 0$ on $[T_2^*,\infty)_{\mathbf{T}}$ or $\lim_{t\to\infty} x(t) = 0$.

Lemma 3.3. Assume $-\frac{\widetilde{p}}{a} \in \mathcal{R}_+$, and x is eventually a positive solution of Eq. (1.2) such that

$$[r(t)x^{(\alpha)}(t)]^{(\alpha)} > 0,\ x^{(\alpha)}(t) > 0 \text{ on } [T_3^*,\infty)_{\mathbf{T}},$$

where $T_3^* \geq t_0$ is sufficiently large. Then for $t \in [T_3^*,\infty)_{\mathbf{T}}$ it holds that

$$x^{(\alpha)}(t) \geq \frac{\theta_1(t,T_3^*)}{r(t)}\Big\{\frac{a^{\frac{1}{\nu}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(t,t_0)]^{\frac{1}{\nu}}}\Big\},$$

$$x(t) \geq \theta_2(t,T_3^*)\Big\{\frac{a^{\frac{1}{\nu}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(t,t_0)]^{\frac{1}{\nu}}}\Big\}.$$

The proof of Lemmas 3.1-3.3 are similar to [26, Lemmas 2.1-2.2] with some difference on the delay term, which are omitted here.

Lemma 3.4 [30, Theorem 41]. Assume that X and Y are nonnegative real numbers. Then

$$\lambda XY^{\lambda-1} - X^\lambda \leq (\lambda-1)Y^\lambda \text{ for all } \lambda > 1.$$

Theorem 3.5. Suppose $-\frac{\widetilde{p}}{a} \in \mathcal{R}_+$, and assume that (3.1), (3.2), (3.3) hold, and for all sufficiently large $T \in \mathbf{T}$,

$$\lim_{t\to\infty}\sup \int_T^t \Big\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)} - \varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}$$
$$+\frac{\varpi_1(s)\kappa^\Delta(s)\theta_1(\kappa(s),T)[a(\sigma(s))\varpi_2(\sigma(s))]^{1+\frac{1}{\nu}}}{r(\kappa(s))}$$
$$-\Big[\frac{r(\kappa(s))\varpi_1^{(\alpha)}(s) + (\nu+1)\varpi_1(s)\kappa^\Delta(s)\theta_1(\kappa(s),T)}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa^\Delta(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)}\Big]^{\nu+1}$$
$$[a(\sigma(s))\varpi_2(\sigma(s))]^{\frac{\nu+1}{\nu}}\Big\}\Delta^\alpha s\Big\} = \infty, \tag{3.4}$$

where ϖ_1, ϖ_2 are two given nonnegative functions on $\mathbf{T}$ with $\varpi_1(t) > 0$. Then every solution of Eq. (1.2) is oscillatory or tends to zero.

Proof. Assume (1.2) has a nonoscillatory solution x on $\mathbf{T}_0$. Without loss of generality, we may assume $x(t) > 0$, $x(\kappa(t)) > 0$ on $[t_1,\infty)_{\mathbf{T}}$, where t_1 is sufficiently large. By Lemmas 3.1 and 3.2, there exists sufficiently large t_2 such that $[r(t)x^{(\alpha)}(t)]^{(\alpha)} > 0$ on $[t_2,\infty)_{\mathbf{T}}$, and either $x^{(\alpha)}(t) > 0$ on $[t_2,\infty)_{\mathbf{T}}$ or $\lim_{t\to\infty} x(t) = 0$. Now we assume $x^{(\alpha)}(t) > 0$ on $[t_2,\infty)_{\mathbf{T}}$. Since $\lim_{t\to\infty}\kappa(t) = \infty$, there exists $t_3 > t_2$ such that $\kappa(t) > t_2$ on $[t_3,\infty)_{\mathbf{T}}$. So $x^{(\alpha)}(\kappa(t)) > 0$ on $[t_3,\infty)_{\mathbf{T}}$.

Define a generalized Riccati function:

$$\omega(t) = \varpi_1(t)a(t)\Big[\frac{([r(t)x^{(\alpha)}(t)]^{(\alpha)})^\nu}{x^\nu(\kappa(t))e_{-\frac{\tilde{p}}{a}}(t,t_0)} + \varpi_2(t)\Big].$$

Then for $t \in [t_3,\infty)_{\mathbf{T}}$, by Theorem 2.8 (iii) and Theorem 2.7 one can deduce that

$$\omega^{(\alpha)}(t) = \frac{\varpi_1(t)}{x^\nu(\kappa(t))}\Big\{\frac{a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^\nu}{e_{-\frac{\tilde{p}}{a}}(t,t_0)}\Big\}^{(\alpha)}$$
$$+\Big[\frac{\varpi_1(t)}{x^\nu(\kappa(t))}\Big]^{(\alpha)}\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^\nu}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}$$
$$+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)} + \varpi_1^{(\alpha)}(t)a(\sigma(t))\varpi_2(\sigma(t))$$
$$= \frac{\varpi_1(t)}{x^\nu(\kappa(t))}\frac{1}{e_{-\frac{\tilde{p}}{a}}(t,t_0)e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}$$
$$\{e_{-\frac{\tilde{p}}{a}}(t,t_0)(a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^\nu)^{(\alpha)}$$
$$-(e_{-\frac{\tilde{p}}{a}}(t,t_0))^{(\alpha)}a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^\nu\}$$
$$+\Big[\frac{x^\nu(\kappa(t))\varpi_1^{(\alpha)}(t) - (x^\nu(\kappa(t)))^{(\alpha)}\varpi_1(t)}{x^\nu(\kappa(t))x^\nu(\kappa(\sigma(t)))}\Big]$$
$$\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^\nu}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}$$
$$+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)} + \varpi_1^{(\alpha)}(t)a(\sigma(t))\varpi_2(\sigma(t))$$
$$= \frac{\varpi_1(t)}{x^\nu(\kappa(t))}$$
$$\Big[\frac{(a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^\nu)^{(\alpha)} + p(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^\nu}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}\Big]$$
$$+\frac{\varpi_1^{(\alpha)}(t)}{\varpi_1(\sigma(t))}\omega(\sigma(t))$$
$$-\Big[\frac{\varpi_1(t)(x^\nu(\kappa(t)))^{(\alpha)}}{x^\nu(\kappa(t))}\Big]\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^\nu}{x^\nu(\kappa(\sigma(t)))e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}$$

$$+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)}$$

$$= -\frac{\varpi_1(t)}{x^\nu(\kappa(t))}[\frac{q(t)f(x(\kappa(t)))}{e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)}] + \frac{\varpi_1^{(\alpha)}(t)}{\varpi_1(\sigma(t))}\omega(\sigma(t))$$
$$-[\frac{\varpi_1(t)(x^\nu(\kappa(t)))^{(\alpha)}}{x^\nu(\kappa(t))}]\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^\nu}{x^\nu(\kappa(\sigma(t)))e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)}$$
$$+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)}$$

$$\leq -L\frac{q(t)\varpi_1(t)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)} + \frac{\varpi_1^{(\alpha)}(t)}{\varpi_1(\sigma(t))}\omega(\sigma(t))$$
$$-[\frac{\varpi_1(t)(x^\nu(\kappa(t)))^{(\alpha)}}{x^\nu(\kappa(t))}]\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^\nu}{x^\nu(\kappa(\sigma(t)))e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)}$$
$$+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)}.$$

Furthermore, according to [29, Theorem 1.87 and 1.93], and the assumption $\kappa\circ\sigma=\sigma\circ\kappa$, one has

$$(x^\nu(\kappa(t)))^\Delta \geq \nu x^{\nu-1}(\kappa(t))(x(\kappa(t)))^\Delta$$
$$= \nu x^{\nu-1}(\kappa(t))x^\Delta(\kappa(t))\kappa^\Delta(t).$$

So by Corollary 2.6 it holds that

$$(x^\nu(\kappa(t)))^{(\alpha)} = t^{1-\alpha}(x^\nu(\kappa(t)))^\Delta$$
$$\geq t^{1-\alpha}\nu x^{\nu-1}(\kappa(t))x^\Delta(\kappa(t))\kappa^\Delta(t)$$
$$= \nu x^{\nu-1}(\kappa(t))x^{(\alpha)}(\kappa(t))\kappa^\Delta(t),$$

which implies

$$\omega^{(\alpha)}(t) \leq -L\frac{q(t)\varpi_1(t)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)} + \frac{\varpi_1^{(\alpha)}(t)}{\varpi_1(\sigma(t))}\omega(\sigma(t))$$
$$+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)}$$
$$-\varpi_1(t)[\frac{\nu x^{\nu-1}(\kappa(t))x^{(\alpha)}(\kappa(t))\kappa^\Delta(t)}{x^\nu(\kappa(t))}]$$
$$\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^\nu}{x^\nu(\kappa(\sigma(t)))e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)}. \quad (3.5)$$

According to Lemma 3.3 and $x^{(\alpha)}(t)>0$, one has

$$\omega^{(\alpha)}(t) \leq -L\frac{q(t)\varpi_1(t)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)} + \frac{\varpi_1^{(\alpha)}(t)}{\varpi_1(\sigma(t))}\omega(\sigma(t))$$
$$+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)} - [\frac{\nu\varpi_1(t)\kappa^\Delta(t)}{x(\kappa(\sigma(t)))}]$$
$$\{\frac{\theta_1(\kappa(t),t_2)}{r(\kappa(t))}[\frac{a^{\frac{1}{\nu}}(\kappa(t))[r(\kappa(t))x^{(\alpha)}(\kappa(t))]^{(\alpha)}}{[e_{\widetilde{-\frac{p}{a}}}(\kappa(t),t_0)]^{\frac{1}{\nu}}}]\}$$
$$\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^\nu}{x^\nu(\kappa(\sigma(t)))e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)}.$$

By Lemma 3.1, $\frac{a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^\nu}{e_{\widetilde{-\frac{p}{a}}}(t,t_0)}$ is decreasing on $[t_2,\infty)_{\mathbf{T}}$. So $\frac{(a(\kappa(t)))^{\frac{1}{\nu}}[r(\kappa(t))x^{(\alpha)}(\kappa(t))]^{(\alpha)}}{[e_{\widetilde{-\frac{p}{a}}}(\kappa(t),t_0)]^{\frac{1}{\nu}}} > \frac{(a(\sigma(t)))^{\frac{1}{\nu}}[r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)}}{[e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)]^{\frac{1}{\nu}}}$ for $t\in[t_3,\infty)_{\mathbf{T}}$. Furthermore, one can obtain that

$$\omega^{(\alpha)}(t) \leq -L\frac{q(t)\varpi_1(t)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)} + \frac{\varpi_1^{(\alpha)}(t)}{\varpi_1(\sigma(t))}\omega(\sigma(t))$$
$$+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)} - \nu\frac{\varpi_1(t)\kappa^\Delta(t)\theta_1(\kappa(t),t_2)}{r(\kappa(t))}$$
$$[\frac{\omega(\sigma(t))}{\varpi_1(\sigma(t))} - a(\sigma(t))\varpi_2(\sigma(t))]^{1+\frac{1}{\nu}}. \quad (3.6)$$

Using the following inequality (see [31, Eq. (3.12)]):

$$(u-v)^{1+\frac{1}{\nu}} \geq u^{1+\frac{1}{\nu}} + \frac{1}{\nu}v^{1+\frac{1}{\nu}} - (1+\frac{1}{\nu})v^{\frac{1}{\nu}}u,$$

one can deduce that

$$[\frac{\omega(\sigma(t))}{\varpi_1(\sigma(t))} - a(\sigma(t))\varpi_2(\sigma(t))]^{1+\frac{1}{\nu}}$$
$$\geq \frac{\omega^{1+\frac{1}{\nu}}(\sigma(t))}{\varpi_1^{1+\frac{1}{\nu}}(\sigma(t))} + \frac{1}{\nu}[a(\sigma(t))\varpi_2(\sigma(t))]^{1+\frac{1}{\nu}}$$
$$-(1+\frac{1}{\nu})\frac{[a(\sigma(t))\varpi_2(\sigma(t))]^{\frac{1}{\nu}}\omega(\sigma(t))}{\varpi_1(\sigma(t))}. \quad (3.7)$$

Then if follows from a combination of (3.6) and (3.7) that

$$\omega^{(\alpha)}(t) \leq -L\frac{q(t)\varpi_1(t)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)} + \varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)}$$
$$-\frac{\varpi_1(t)\kappa^\Delta(t)\theta_1(\kappa(t),t_2)[a(\sigma(t))\varpi_2(\sigma(t))]^{1+\frac{1}{\nu}}}{r(\kappa(t))}$$
$$+\frac{\omega(\sigma(t))}{r(\kappa(t))\varpi_1(\sigma(t))}\{r(\kappa(t))\varpi_1^{(\alpha)}(t)$$
$$+(\nu+1)\varpi_1(t)\kappa^\Delta(t)\theta_1(\kappa(t),t_2)[a(\sigma(t))\varpi_2(\sigma(t))]^{\frac{1}{\nu}}\}$$
$$-\nu\frac{\varpi_1(t)\kappa^\Delta(t)\theta_1(\kappa(t),t_2)}{r(\kappa(t))}\frac{\omega^{1+\frac{1}{\nu}}(\sigma(t))}{\varpi_1^{1+\frac{1}{\nu}}(\sigma(t))}. \quad (3.8)$$

Setting

$$\lambda = 1+\frac{1}{\nu},\ X^\lambda = \nu\frac{\varpi_1(t)\kappa^\Delta(t)\theta_1(\kappa(t),t_2)}{r(\kappa(t))}\frac{\omega^{1+\frac{1}{\nu}}(\sigma(t))}{\varpi_1^{1+\frac{1}{\nu}}(\sigma(t))},$$

$$Y^{\lambda-1} = \frac{\nu^{\frac{1}{\nu+1}}}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(t))\varpi_1^{\frac{\nu}{\nu+1}}(t)(\kappa^\Delta(t))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(t),t_2)}$$

$$\{r(\kappa(t))\varpi_1^{(\alpha)}(t) + (\nu+1)\varpi_1(t)\kappa^\Delta(t)\theta_1(\kappa(t),t_2)$$
$$[a(\sigma(t))\varpi_2(\sigma(t))]^{\frac{1}{\nu}}\},$$

Applying Lemma 3.4 in (3.7) one has

$$\omega^{(\alpha)}(t) \leq -L\frac{q(t)\varpi_1(t)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(t),t_0)} + \varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)}$$
$$-\frac{\varpi_1(t)\kappa^\Delta(t)\theta_1(\kappa(t),t_2)[a(\sigma(t))\varpi_2(\sigma(t))]^{1+\frac{1}{\nu}}}{r(\kappa(t))}+$$

$$[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(t))\varpi_1^{\frac{\nu}{\nu+1}}(t)(\kappa^\Delta(t))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(t),t_2)}]^{\nu+1}$$

$$\{r(\kappa(t))\varpi_1^{(\alpha)}(t) + (\nu+1)\varpi_1(t)\kappa^\Delta(t)\theta_1(\kappa(t),t_2)$$
$$[a(\sigma(t))\varpi_2(\sigma(t))]^{\frac{1}{\nu}}\}^{\nu+1}. \quad (3.9)$$

Substituting t with s in (3.9), fulfilling α-fractional integral for (3.9) with respect to s from t_3 to t yields that

$$\int_{t_3}^t\{L\frac{q(s)\varpi_1(s)}{e_{\widetilde{-\frac{p}{a}}}(\sigma(s),t_0)} - \varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}$$
$$+\frac{\varpi_1(s)\kappa^\Delta(s)\theta_1(\kappa(s),t_2)[a(\sigma(s))\varpi_2(\sigma(s))]^{1+\frac{1}{\nu}}}{r(\kappa(s))}-$$

$$[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa^{\Delta}(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),t_2)}]^{\nu+1}$$

$$\{r(\kappa(s))\varpi_1^{(\alpha)}(s)+(\nu+1)\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)[a(\sigma(s))\varpi_2(\sigma(s))]^{\frac{1}{\nu}}\}^{\nu+1}\Delta^{\alpha}s$$

$$\leq \omega(t_3)-\omega(t)\leq\omega(t_3)<\infty.$$

So it also holds that

$$\int_{t_2}^{t}\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}+\frac{\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)[a(\sigma(s))\varpi_2(\sigma(s))]^{1+\frac{1}{\nu}}}{r(\kappa(s))}$$

$$-[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa^{\Delta}(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),t_2)}]^{\nu+1}$$

$$\{r(\kappa(s))\varpi_1^{(\alpha)}(s)+(\nu+1)\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)[a(\sigma(s))\varpi_2(\sigma(s))]^{\frac{1}{\nu}}\}^{\nu+1}\Delta^{\alpha}s<\infty,$$

which contradicts (3.4), and the proof is complete.

Theorem 3.6. Suppose $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$, and (3.1), (3.2), (3.3) hold. If for all sufficiently large $T\in\mathbf{T}$,

$$\lim_{t\to\infty}\sup\{\int_{T}^{t}\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}+\frac{\nu\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),T)\theta_2^{\nu-1}(\kappa(\sigma(s),T))a^2(\sigma(s))\varpi_2^2(\sigma(s))}{r(\kappa(s))}$$

$$-\frac{1}{4\nu r(\kappa(s))\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),T)\theta_2^{\nu-1}(\kappa(\sigma(s),T))}[r(\kappa(s))\varpi_1^{(\alpha)}(s)+2\nu\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),T)\theta_2^{\nu-1}(\kappa(\sigma(s),T))a(\sigma(s))\varpi_2(\sigma(s))]^2\}\Delta^{\alpha}s\}=\infty, \tag{3.10}$$

where ϖ_1, ϖ_2 are defined as in Theorem 3.5. Then every solution of Eq. (1.2) is oscillatory or tends to zero.

Proof. Assume (1.2) has a nonoscillatory solution x on $\mathbf{T}_0$. Similar to Theorem 3.5, we may assume $x(t)>0$ on $[t_1,\infty)_{\mathbf{T}}$, where t_1 is sufficiently large. By Lemmas 3.1 and 3.2, there exists sufficiently large t_2 such that $[r(t)x^{(\alpha)}(t)]^{(\alpha)}>0$ on $[t_2,\infty)_{\mathbf{T}}$, and either $x^{(\alpha)}(t)>0$ on $[t_2,\infty)_{\mathbf{T}}$ or $\lim_{t\to\infty}x(t)=0$. Now we assume $x^{(\alpha)}(t)>0$, $x^{(\alpha)}(\kappa(t))>0$ on $[t_3,\infty)_{\mathbf{T}}$, where $t_3>t_2$ is sufficiently large. Let $\omega(t)$ be defined as in Theorem 3.5. By Lemma 3.3, for $t\in[t_3,\infty)_{\mathbf{T}}$, we have the following observation:

$$\frac{x^{(\alpha)}(\kappa(t))}{x(\kappa(t))}\geq\frac{x^{(\alpha)}(\kappa(t))}{x(\kappa(\sigma(t)))}=\frac{x^{(\alpha)}(\kappa(t))}{x^{\nu}(\kappa(\sigma(t)))}x^{\nu-1}(\kappa(\sigma(t)))$$

$$\geq\frac{\theta_1(\kappa(t),t_2)}{r(\kappa(t))x^{\nu}(\kappa(\sigma(t)))}\{\frac{a^{\frac{1}{\nu}}(\kappa(t))[r(\kappa(t))x^{(\alpha)}(\kappa(t))]^{(\alpha)}}{[e_{-\frac{\widetilde{p}}{a}}(\kappa(t),t_0)]^{\frac{1}{\nu}}}\}x^{\nu-1}(\kappa(\sigma(t)))$$

$$\geq\frac{\theta_1(\kappa(t),t_2)}{r(\kappa(t))x^{\nu}(\kappa(\sigma(t)))}\{\frac{a^{\frac{1}{\nu}}(\kappa(t))[r(\kappa(t))x^{(\alpha)}(\kappa(t))]^{(\alpha)}}{[e_{-\frac{\widetilde{p}}{a}}(\kappa(t),t_0)]^{\frac{1}{\nu}}}\}\times\theta_2^{\nu-1}(\kappa(\sigma(t)),t_2)\{\frac{a^{\frac{1}{\nu}}(\kappa(\sigma(t)))[r(\kappa(\sigma(t)))x^{(\alpha)}(\kappa(\sigma(t)))]^{(\alpha)}}{[e_{-\frac{\widetilde{p}}{a}}(\kappa(\sigma(t)),t_0)]^{\frac{1}{\nu}}}\}^{\nu-1}$$

$$\geq\frac{\theta_1(\kappa(t),t_2)}{r(\kappa(t))x^{\nu}(\kappa(\sigma(t)))}\{\frac{a^{\frac{1}{\nu}}(\kappa(t))[r(\kappa(t))x^{(\alpha)}(\kappa(t))]^{(\alpha)}}{[e_{-\frac{\widetilde{p}}{a}}(\kappa(t),t_0)]^{\frac{1}{\nu}}}\}\times\theta_2^{\nu-1}(\kappa(\sigma(t)),t_2)\{\frac{a^{\frac{1}{\nu}}(\sigma(t))[r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)}}{[e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)]^{\frac{1}{\nu}}}\}^{\nu-1}$$

$$\geq\frac{\theta_1(\kappa(t),t_2)\theta_2^{\nu-1}(\kappa(\sigma(t),t_2))}{r(\kappa(t))}\{\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\nu}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)x^{\nu}(\kappa(\sigma(t)))}\}. \tag{3.11}$$

Using (3.11) in (3.5) we get that

$$\omega^{(\alpha)}(t)\leq -L\frac{q(t)\varpi_1(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}+\frac{\varpi_1^{(\alpha)}(t)}{\varpi_1(\sigma(t))}\omega(\sigma(t))+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)}-\varpi_1(t)[\frac{\nu x^{\nu-1}(\kappa(t))x^{(\alpha)}(\kappa(t))\kappa^{\Delta}(t)}{x^{\nu}(\kappa(t))}]$$

$$\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\nu}}{x^{\nu}(\kappa(\sigma(t)))e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}$$

$$\leq -L\frac{q(t)\varpi_1(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}+\frac{\varpi_1^{(\alpha)}(t)}{\varpi_1(\sigma(t))}\omega(\sigma(t))+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)}$$

$$-\nu\varpi_1(t)\kappa^{\Delta}(t)\frac{\theta_1(\kappa(t),t_2)\theta_2^{\nu-1}(\kappa(\sigma(t),t_2))}{r(\kappa(t))}\{\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\nu}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)x^{\nu}(\kappa(\sigma(t)))}\}^2$$

$$=-L\frac{q(t)\varpi_1(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}+\frac{\varpi_1^{(\alpha)}(t)}{\varpi_1(\sigma(t))}\omega(\sigma(t))+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)}$$

$$-\nu\varpi_1(t)\kappa^{\Delta}(t)\frac{\theta_1(\kappa(t),t_2)\theta_2^{\nu-1}(\kappa(\sigma(t),t_2))}{r(\kappa(t))}[\frac{\omega(\sigma(t))}{\varpi_1(\sigma(t))}-a(\sigma(t))\varpi_2(\sigma(t))]^2$$

$$=-L\frac{q(t)\varpi_1(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)}-\frac{\nu\varpi_1(t)\kappa^{\Delta}(t)\theta_1(\kappa(t),t_2)\theta_2^{\nu-1}(\kappa(\sigma(t),t_2))a^2(\sigma(t))\varpi_2^2(\sigma(t))}{r(\kappa(t))}$$

$$+\frac{\omega(\sigma(t))}{r(\kappa(t))\varpi_1(\sigma(t))}[r(\kappa(t))\varpi_1^{(\alpha)}(t)+2\nu\varpi_1(t)\kappa^{\Delta}(t)\theta_1(\kappa(t),t_2)\theta_2^{\nu-1}(\kappa(\sigma(t),t_2))a(\sigma(t))\varpi_2(\sigma(t))]$$

$$-\frac{\nu\varpi_1(t)\kappa^{\Delta}(t)\theta_1(\kappa(t),t_2)\theta_2^{\nu-1}(\kappa(\sigma(t),t_2))}{r(\kappa(t))\varpi_1^2(\sigma(t))}\omega^2(\sigma(t))$$

$$\leq -L\frac{q(t)\varpi_1(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}+\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)}-\frac{\nu\varpi_1(t)\kappa^{\Delta}(t)\theta_1(\kappa(t),t_2)\theta_2^{\nu-1}(\kappa(\sigma(t),t_2))a^2(\sigma(t))\varpi_2^2(\sigma(t))}{r(\kappa(t))}$$

$$+\frac{1}{4\nu r(\kappa(t))\varpi_1(t)\kappa^{\Delta}(t)\theta_1(\kappa(t),t_2)\theta_2^{\nu-1}(\kappa(\sigma(t),t_2))} [r(\kappa(t))\varpi_1^{(\alpha)}(t)+2\nu\varpi_1(t)\kappa^{\Delta}(t)\theta_1(\kappa(t),t_2) \theta_2^{\nu-1}(\kappa(\sigma(t),t_2))a(\sigma(t))\varpi_2(\sigma(t))]^2. \quad (3.12)$$

Substituting t with s in (3.12), fulfilling α-fractional integral for (3.12) with respect to s from t_3 to t yields

$$\int_{t_3}^{t}\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}+ \frac{\nu\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)\theta_2^{\nu-1}(\kappa(\sigma(s),t_2))a^2(\sigma(s))\varpi_2^2(\sigma(s))}{r(\kappa(s))}$$

$$-\frac{1}{4\nu r(\kappa(s))\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)\theta_2^{\nu-1}(\kappa(\sigma(s),t_2))} [r(\kappa(s))\varpi_1^{(\alpha)}(s)+2\nu\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2) \theta_2^{\nu-1}(\kappa(\sigma(s),t_2))a(\sigma(s))\varpi_2(\sigma(s))]^2\}\Delta^{\alpha}s$$

$$\leq \omega(t_3)-\omega(t)\leq\omega(t_3)<\infty.$$

Then

$$\int_{t_2}^{t}\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}+ \frac{\nu\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)\theta_2^{\nu-1}(\kappa(\sigma(s),t_2))a^2(\sigma(s))\varpi_2^2(\sigma(s))}{r(\kappa(s))}$$

$$-\frac{}{4\nu r(\kappa(s))\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)\theta_2^{\nu-1}(\kappa(\sigma(s),t_2))} [r(\kappa(s))\varpi_1^{(\alpha)}(s)+2\nu\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2) \theta_2^{\nu-1}(\kappa(\sigma(s),t_2))a(\sigma(s))\varpi_2(\sigma(s))]^2\}\Delta^{\alpha}s<\infty,$$

which contradicts (3.10). So the proof is complete.

Theorem 3.7. Suppose $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$, and assume that (3.1), (3.2), (3.3) hold, and define $\mathbf{D}=\{(t,s)|t\geq s\geq t_0,\ t,s\in\mathbf{T}\}$. If there exists a function $H\in C_{rd}(\mathbf{D},\mathbf{R})$ such that

$$H(t,t)=0,\ for\ t\geq t_0,\quad H(t,s)>0,\ for\ t>s\geq t_0, \quad (3.13)$$

and H has a nonpositive continuous $\alpha-$ partial fractional derivative $H_s^{(\alpha)}(t,s)$ with respect to the second variable, and for all sufficiently large $T\in\mathbf{T}$,

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\{\int_{t_0}^{t}H(t,s)\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} -\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)} +\frac{\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),T)[a(\sigma(s))\varpi_2(\sigma(s))]^{1+\frac{1}{\nu}}}{r(\kappa(s))}-$$

$$[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa^{\Delta}(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)}]^{\nu+1}$$

$$[r(\kappa(s))\varpi_1^{(\alpha)}(s)+(\nu+1)\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),T) [a(\sigma(s))\varpi_2(\sigma(s))]^{\frac{1}{\nu}}]^{\nu+1}\}\Delta^{\alpha}s\}=\infty. \quad (3.14)$$

Then every solution of Eq. (1.2) is oscillatory or tends to zero.

Proof. Assume (1.2) has a nonoscillatory solution x on $\mathbf{T}_0$. Similar to Theorem 3.5, we may assume $x(t)>0$ on $[t_1,\infty)_{\mathbf{T}}$, where t_1 is sufficiently large. By Lemmas 3.1 and 3.2, there exists sufficiently large t_2 such that $[r(t)x^{(\alpha)}(t)]^{(\alpha)}>0$ on $[t_2,\infty)_{\mathbf{T}}$, and either $x^{(\alpha)}(t)>0$ on $[t_2,\infty)_{\mathbf{T}}$ or $\lim_{t\to\infty}x(t)=0$. Now we assume $x^{(\alpha)}(t)>0,\ x^{(\alpha)}(\kappa(t))>0$ on $[t_3,\infty)_{\mathbf{T}}$, where $t_3>t_2$ is sufficiently large. Let $\omega(t)$ be defined as in Theorem 3.5. Then by (3.9), for $t\in[t_3,\infty)_{\mathbf{T}}$, we have

$$L\frac{q(t)\varpi_1(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}-\varpi_1(t)[a(t)\varpi_2(t)]^{(\alpha)} +\frac{\varpi_1(t)\kappa^{\Delta}(t)\theta_1(\kappa(t),t_2)[a(\sigma(t))\varpi_2(\sigma(t))]^{1+\frac{1}{\nu}}}{r(\kappa(t))}$$

$$-[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(t))\varpi_1^{\frac{\nu}{\nu+1}}(t)(\kappa^{\Delta}(t))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(t),t_2)}]^{\nu+1}$$

$$[r(\kappa(t))\varpi_1^{(\alpha)}(t)+(\nu+1)\varpi_1(t)\kappa^{\Delta}(t)\theta_1(\kappa(t),t_2) [a(\sigma(t))\varpi_2(\sigma(t))]^{\frac{1}{\nu}}]^{\nu+1} \leq-\omega^{(\alpha)}(t). \quad (3.15)$$

Substituting t with s in (3.15), multiplying both sides by $H(t,s)$ and then fulfilling α-fractional integral with respect to s from t_3 to t, together with the use of Theorem 2.9 yields that

$$\int_{t_3}^{t}H(t,s)\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)} +\frac{\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)[a(\sigma(s))\varpi_2(\sigma(s))]^{1+\frac{1}{\nu}}}{r(\kappa(s))}$$

$$-[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa^{\Delta}(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),t_2)}]^{\nu+1}$$

$$[r(\kappa(s))\varpi_1^{(\alpha)}(s)+(\nu+1)\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2) [a(\sigma(s))\varpi_2(\sigma(s))]^{\frac{1}{\nu}}]^{\nu+1}\}\Delta^{\alpha}s$$

$$\begin{aligned}&\leq-\int_{t_3}^{t}H(t,s)\omega^{(\alpha)}(s)\Delta^{\alpha}s\\&=H(t,t_3)\omega(t_3)+\int_{t_3}^{t}H_s^{(\alpha)}(t,s)\omega(\sigma(s))\Delta^{\alpha}s\\&\leq H(t,t_3)\omega(t_3)\\&\leq H(t,t_0)\omega(t_3).\end{aligned}$$

So one has

$$\int_{t_0}^{t}H(t,s)\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)} +\frac{\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)[a(\sigma(s))\varpi_2(\sigma(s))]^{1+\frac{1}{\nu}}}{r(\kappa(s))}$$

$$-[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa^{\Delta}(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),t_2)}]^{\nu+1}$$

$$[r(\kappa(s))\varpi_1^{(\alpha)}(s)+(\nu+1)\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2) [a(\sigma(s))\varpi_2(\sigma(s))]^{\frac{1}{\nu}}]^{\nu+1}\}\Delta^{\alpha}s$$

$$=\int_{t_0}^{t_3}H(t,s)\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}$$

$$+\frac{\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)[a(\sigma(s))\varpi_2(\sigma(s))]^{1+\frac{1}{\nu}}}{r(\kappa(s))}$$

$$-[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa^{\Delta}(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),t_2)}]^{\nu+1}$$

$$[r(\kappa(s))\varpi_1^{(\alpha)}(s)+(\nu+1)\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)$$
$$[a(\sigma(s))\varpi_2(\sigma(s))]^{\frac{1}{\nu}}]^{\nu+1}\}\Delta^{\alpha}s$$

$$+\int_{t_3}^{t}H(t,s)\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}$$
$$+\frac{\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)[a(\sigma(s))\varpi_2(\sigma(s))]^{1+\frac{1}{\nu}}}{r(\kappa(s))}$$

$$-[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa^{\Delta}(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),t_2)}]^{\nu+1}$$

$$[r(\kappa(s))\varpi_1^{(\alpha)}(s)+(\nu+1)\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)$$
$$[a(\sigma(s))\varpi_2(\sigma(s))]^{\frac{1}{\nu}}]^{\nu+1}\}\Delta^{\alpha}s$$

$$\leq H(t,t_0)\omega(t_3)$$
$$+H(t,t_0)\int_{t_0}^{t_3}|L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}$$
$$+\frac{\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)[a(\sigma(s))\varpi_2(\sigma(s))]^{1+\frac{1}{\nu}}}{r(\kappa(s))}$$

$$-[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa^{\Delta}(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),t_2)}]^{\nu+1}$$

$$[r(\kappa(s))\varpi_1^{(\alpha)}(s)+(\nu+1)\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)$$
$$[a(\sigma(s))\varpi_2(\sigma(s))]^{\frac{1}{\nu}}]^{\nu+1}|\Delta^{\alpha}s.$$

Then

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\{\int_{t_0}^{t}H(t,s)\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}$$
$$-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}$$
$$+\frac{\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)[a(\sigma(s))\varpi_2(\sigma(s))]^{1+\frac{1}{\nu}}}{r(\kappa(s))}$$

$$-[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa^{\Delta}(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),t_2)}]^{\nu+1}$$

$$[r(\kappa(s))\varpi_1^{(\alpha)}(s)+(\nu+1)\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)$$
$$[a(\sigma(s))\varpi_2(\sigma(s))]^{\frac{1}{\nu}}]^{\nu+1}\}\Delta^{\alpha}s\}$$

$$\leq\omega(t_3)+\int_{t_0}^{t_3}|L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}$$
$$+\frac{\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)[a(\sigma(s))\varpi_2(\sigma(s))]^{1+\frac{1}{\nu}}}{r(\kappa(s))}$$

$$-[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa^{\Delta}(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),t_2)}]^{\nu+1}$$

$$[r(\kappa(s))\varpi_1^{(\alpha)}(s)+(\nu+1)\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),t_2)$$
$$[a(\sigma(s))\varpi_2(\sigma(s))]^{\frac{1}{\nu}}]^{\nu+1}|\Delta^{\alpha}s<\infty,$$

which contradicts (3.14). The proof is complete.

Based on (3.12) and the deduction process in Theorem 3.7, one can easily prove the following theorem.

Theorem 3.8. Suppose $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$, and assume that (3.1), (3.2), (3.3) hold. Let H be defined as in Theorem 3.7, and for all sufficiently large $T\in\mathbf{T}$,

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\{\int_{t_0}^{t}H(t,s)\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}$$
$$-\varpi_1(s)[a(s)\varpi_2(s)]^{(\alpha)}$$
$$+\frac{\nu\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),T)\theta_2^{\nu-1}(\kappa(\sigma(s),T))a^2(\sigma(s))\varpi_2^2(\sigma(s))}{r(\kappa(s))}$$
$$-\frac{1}{4\nu r(\kappa(s))\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),T)\theta_2^{\nu-1}(\kappa(\sigma(s),T))}$$
$$[r(\kappa(s))\varpi_1^{(\alpha)}(s)+2\nu\varpi_1(s)\kappa^{\Delta}(s)\theta_1(\kappa(s),T)$$
$$\theta_2^{\nu-1}(\kappa(\sigma(s),T))a(\sigma(s))\varpi_2(\sigma(s))]^2\}\Delta^{\alpha}s\}$$
$$=\infty. \quad (3.16)$$

Then every solution of Eq. (1.2) is oscillatory or tends to zero.

Remark 3.9. If $\kappa(t)=t$, then the established results above reduce to the main results in [26, Theorems 2.4-2.7].

In Theorems 3.5-3.8, if we take $\mathbf{T}$ for some special cases such as $\mathbf{T}=\mathbf{R}$, $\mathbf{T}=\mathbf{Z}$ and so on, then one can obtain some corollaries. For example, based on theorem 3.5, one has the following corollaries.

Corollary 3.10. Let $\mathbf{T}=\mathbf{R}$. Assume that

$$\int_{t_0}^{\infty}\frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\nu}}}{a^{\frac{1}{\nu}}(s)}s^{\alpha-1}ds=\infty, \quad (3.17)$$

$$\int_{t_0}^{\infty}\frac{1}{r(s)}s^{\alpha-1}ds=\infty, \quad (3.18)$$

$$\int_{t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\int_{\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}ds)^{\frac{1}{\nu}}d\tau]d\xi$$
$$=\infty, \quad (3.19)$$

and for all sufficiently large $T\in\mathbf{R}$,

$$\lim_{t\to\infty}\sup\{\int_{T}^{t}\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}-\varpi_1(s)[a(s)\varpi_2(s)]'s^{1-\alpha}$$
$$+\frac{\varpi_1(s)\kappa'(s)\theta_1(\kappa(s),T)[a(s)\varpi_2(s)]^{1+\frac{1}{\nu}}}{r(\kappa(s))}$$

$$-[\frac{1}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa'(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)}]^{\nu+1}$$

$$[r(\kappa(s))\varpi_1'(s)s^{1-\alpha}+(\nu+1)\varpi_1(s)\kappa'(s)\theta_1(\kappa(s),T)$$
$$[a(s)\varpi_2(s)]^{\frac{1}{\nu}}]^{\nu+1}\}ds\}=\infty. \quad (3.20)$$

Then every solution of Eq. (1.2) is oscillatory or tends to zero.

Corollary 3.11. Let $\mathbf{T}=\mathbf{Z}$. Assume that $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$, and

$$\sum_{s=t_0}^{\infty}\frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\nu}}}{a^{\frac{1}{\nu}}(s)}s^{\alpha-1}=\infty, \tag{3.21}$$

$$\sum_{s=t_0}^{\infty}\frac{1}{r(s)}s^{\alpha-1}=\infty, \tag{3.22}$$

$$\sum_{\xi=t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)})^{\frac{1}{\nu}}] \\ =\infty, \tag{3.23}$$

and for all sufficiently large $T\in\mathbf{Z}$,

$$\lim_{t\to\infty}\sup\{\sum_{s=T}^{t-1}\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)} \\ -\varpi_1(s)[a(s+1)\varpi_2(s+1)-a(s)\varpi_2(s)]s^{1-\alpha} \\ +\frac{\varpi_1(s)(\kappa(s+1)-\kappa(s))\theta_1(\kappa(s),T)[a(s+1)\varpi_2(s+1)]^{1+\frac{1}{\nu}}}{r(\kappa(s))} \\ -[(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa(s+1)-\kappa(s))^{\frac{\nu}{\nu+1}} \\ \theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)]^{-\nu-1} \\ [r(\kappa(s))(\varpi_1(s+1)-\varpi_1(s))s^{1-\alpha} \\ +(\nu+1)\varpi_1(s)(\kappa(s+1)-\kappa(s))\theta_1(\kappa(s),T) \\ [a(s+1)\varpi_2(s+1)]^{\frac{1}{\nu}}]^{\nu+1}\}\}=\infty. \tag{3.24}$$

Then every solution of Eq. (1.2) is oscillatory or tends to zero.

Remark 3.12. We note that the established main theorems are generalizations of many existing results in the literature. If we set $\nu=1$, $\kappa(t)=t$ and the forced function $f(x)=x$ in Eq. (1.2), then Eq. (1.2) reduces to [11, Eq. (1.1)], and the results in Theorems 3.4 and 3.7 reduce to [11, Theorems 2.4, 2.6] with $L=1$.

Remark 3.13. The deduction process of Theorems 3.4-3.7 can be further applied to prove oscillation of other dynamic equations on time scales with higher fractional derivative term, such as

$$\{[(a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma})^{(\alpha)}]^{(\alpha)}\}^{(\alpha)} \\ +p(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}+q(t)f(x(t))=0.$$

Corresponding oscillatory criteria can be established by following a similar proving process with Theorems 3.4-3.7.

IV. Applications

In this section, we will present some applications for the established results above.

Example 1. Consider the following fractional delay differential equation with $\mathbf{T}=\mathbf{R}$:

$$\{t^{\frac{\nu}{3}}[((t+1)^{-\frac{2}{3}}x^{(\frac{1}{3})}(t))^{(\frac{1}{3})}]^{\nu}\}^{(\frac{1}{3})} \\ +\frac{1}{t^{\nu+\frac{5}{3}}}[((t+1)^{-\frac{2}{3}}x^{(\frac{1}{3})}(t))^{(\frac{1}{3})}]^{\nu} \\ +\frac{1}{t^{\nu+\frac{1}{3}}}x^{\nu}(t-1)[e^{x(t-1)}+1]=0,\ t\in[2,\infty), \tag{4.1}$$

where $\nu\geq 1$ is a quotient of two odd positive integers.

Compared with Eq. (1.2) we have $\alpha=\frac{1}{3}$, $a(t)=t^{\frac{\nu}{3}}$, $p(t)=\frac{1}{t^{\nu+\frac{5}{3}}}$, $\widetilde{p}(t)=t^{\alpha-1}p(t)=\frac{1}{t^{\nu+\frac{7}{3}}}$, $q(t)=\frac{1}{t^{\nu+\frac{1}{3}}}$, $\kappa(t)=t-1$, $f(x)=x^{\nu}[e^{x}+1]$, $r(t)=(t+1)^{-\frac{2}{3}}$, $t_0=2$.

One can see $\frac{f(x)}{x^{\nu}}\geq 1=L$, $\mu(t)=\sigma(t)-t=0$, and $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$. So $e_{-\frac{\widetilde{p}}{a}}(t,t_0)=e_{-\frac{\widetilde{p}}{a}}(t,2)=\exp(-\int_2^t\frac{\widetilde{p}(s)}{a(s)}ds)$, and

$1>\exp(-\int_2^t\frac{\widetilde{p}(s)}{a(s)}ds)\geq 1-\int_2^t\frac{\widetilde{p}(s)}{a(s)}ds=1-\int_2^t\frac{1}{s^{\frac{4\nu}{3}+\frac{7}{3}}}ds\geq 1-\int_2^t\frac{1}{s^{\frac{4\nu}{3}+1}}ds=1-\frac{3}{4\nu}[2^{-\frac{4}{3}\nu}-t^{-\frac{4}{3}\nu}]>\frac{1}{4}$.

Furthermore, one has the following observations:

$$\int_{t_0}^{\infty}\frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\nu}}}{a^{\frac{1}{\nu}}(s)}s^{\alpha-1}ds>\frac{1}{4^{\frac{1}{\nu}}}\int_2^{\infty}\frac{1}{s}ds=\infty, \\ \int_{t_0}^{\infty}\frac{1}{r(s)}s^{\alpha-1}ds=\int_{t_0}^{\infty}(\frac{s+1}{s})^{\frac{2}{3}}ds=\infty,$$

and

$$\int_{t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\int_{\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}ds)^{\frac{1}{\nu}}d\tau]d\xi \\ =\int_2^{\infty}(\frac{\xi+1}{\xi})^{\frac{2}{3}} \\ [\int_{\xi}^{\infty}\frac{1}{\tau}(e_{-\frac{\widetilde{p}}{a}}(\tau,2)\int_{\tau}^{\infty}\frac{1}{s^{\nu+1}e_{-\frac{\widetilde{p}}{a}}(s,2)}ds)^{\frac{1}{\nu}}d\tau]d\xi \\ >\frac{1}{4^{\frac{1}{\nu}}}\int_2^{\infty}[\int_{\xi}^{\infty}\frac{1}{\tau}(\int_{\tau}^{\infty}\frac{1}{s^{\nu+1}}ds)^{\frac{1}{\nu}}d\tau]d\xi \\ =\frac{1}{(4\nu)^{\frac{1}{\nu}}}\int_2^{\infty}[\int_{\xi}^{\infty}\frac{1}{\tau^2}d\tau]d\xi \\ =\frac{1}{(4\nu)^{\frac{1}{\nu}}}\int_2^{\infty}\frac{1}{\xi}d\xi=\infty.$$

Then (3.18)-(3.20) hold. Moreover, for a sufficiently large T, one has

$$\theta_1(t,T)=\int_T^t\frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\nu}}}{a^{\frac{1}{\nu}}(s)}s^{\alpha-1}ds\to\infty \text{ for } t\to\infty.$$

So we can take sufficiently large $T^*>T$ such that $\theta_1(\kappa(t),T)>1$ for $t\in[T^*,\infty)$. Taking $\varpi_1(t)=t^{\nu}$, $\varpi_2(t)=0$ in (3.20), considering $L=1$, one can obtain that

$$\lim_{t\to\infty}\sup\{\int_T^t\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)} \\ -[\frac{r(\kappa(s))\varpi_1'(s)s^{1-\alpha}}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa'(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)}]^{\nu+1}\} \\ s^{\alpha-1}ds\}$$

$$=\lim_{t\to\infty}\sup\{\int_T^{T^*}\{\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)} \\ -[\frac{r(\kappa(s))\varpi_1'(s)s^{1-\alpha}}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa'(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)}]^{\nu+1}\} \\ s^{\alpha-1}ds$$

$$+\int_{T^*}^{t}\{\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}$$

$$-[\frac{r(\kappa(s))\varpi_1'(s)s^{1-\alpha}}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa'(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)}]^{\nu+1}\}$$

$$s^{\alpha-1}ds\}$$

$$>\lim_{t\to\infty}\sup\{\int_{T}^{T^*}\{\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}$$

$$-[\frac{r(\kappa(s))\varpi_1'(s)s^{1-\alpha}}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa'(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)}]^{\nu+1}\}$$

$$s^{\alpha-1}ds$$

$$+\int_{T^*}^{t}[1-(\frac{\nu}{\nu+1})^{\nu+1}]\frac{1}{s}ds\}=\infty.$$

So (3.17)-(3.20) all hold, and by Corollary 3.10 we deduce that every solution of Eq. (4.1) is oscillatory or tends to zero.

Example 2. Consider the following fractional delay difference equation with $\mathbf{T}=\mathbf{Z}$:

$$\{t^{\frac{3\nu}{4}}[\Delta^{(\frac{3}{4})}((t+1)^{-\frac{1}{4}}\Delta^{(\frac{3}{4})}x(t))]^{\nu}\}^{(\frac{3}{4})}+\frac{1}{t^{\nu+\frac{5}{4}}}[\Delta^{(\frac{3}{4})}((t+1)^{-\frac{1}{4}}\Delta^{(\frac{3}{4})}x(t))]^{\nu}+\frac{M}{t^{\nu+\frac{3}{4}}}x^{\nu}(\frac{t}{2})=0,\ t\in[2,\infty)_{\mathbf{Z}},\quad(4.2)$$

where $\Delta^{(\frac{3}{4})}$ denotes the fractional difference operator of order $\frac{3}{4}$, $M>0$ is a constant, and $\nu\geq 1$ is a quotient of two odd positive integers.

Compared with Eq. (1.2) we have $\alpha=\frac{3}{4},\ a(t)=t^{\frac{3\nu}{4}},\ p(t)=\frac{1}{t^{\nu+\frac{5}{4}}},\ \widetilde{p}(t)=t^{\alpha-1}p(t)=\frac{1}{t^{\nu+\frac{3}{2}}},\ q(t)=\frac{1}{t^{\nu+\frac{3}{4}}},\ \kappa(t)=\frac{t}{2},\ f(x)=Mx^{\nu},\ r(t)=(t+1)^{-\frac{1}{4}},\ t_0=2.$

Then $\frac{f(t)}{x^{\nu}(t)}\geq M=L,\ \mu(t)=\sigma(t)-t=1$, and

$$1-\mu(t)\frac{\widetilde{p}(t)}{a(t)}=1-\frac{1}{t^{\frac{7\nu}{4}+\frac{3}{2}}}\geq 1-\frac{1}{t}\geq 1-\frac{1}{2}>0,$$

which implies $-\frac{\widetilde{p}}{a}\in\mathcal{R}_+$. So According to [32, Lemma 2] one can deduce that

$$\begin{aligned}&e_{-\frac{\widetilde{p}}{a}}(t,t_0)=e_{-\frac{\widetilde{p}}{a}}(t,2)\geq 1-\int_2^t\frac{\widetilde{p}(s)}{a(s)}\Delta s\\&=1-\int_2^t\frac{1}{s^{\frac{7\nu}{4}+\frac{3}{2}}}\Delta s=1-\sum_{s=2}^{t-1}\frac{1}{s^{\frac{7\nu}{4}+\frac{3}{2}}}\\&\geq 1-\int_1^{t-1}\frac{1}{s^{\frac{7\nu}{4}+\frac{3}{2}}}ds\geq 1-\int_1^{t-1}\frac{1}{s^{\frac{7\nu}{4}+1}}ds\\&=1-\frac{4}{7\nu}[1-(t-1)^{-\frac{7\nu}{4}}]>\frac{3}{7},\end{aligned}$$

and

$$e_{-\frac{\widetilde{p}}{a}}(t,t_0)\leq\exp(-\int_2^t\frac{\widetilde{p}(s)}{a(s)}\Delta s)<1.$$

Then by a computation for (3.21)-(3.23) we conclude

$$\sum_{s=t_0}^{\infty}\frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\nu}}}{a^{\frac{1}{\nu}}(s)}s^{\alpha-1}=\sum_{s=2}^{\infty}\frac{[e_{-\frac{\widetilde{p}}{a}}(s,2)]^{\frac{1}{\nu}}}{a^{\frac{1}{\nu}}(s)}s^{\alpha-1}$$

$$=\sum_{s=2}^{\infty}\frac{[e_{-\frac{\widetilde{p}}{a}}(s,2)]^{\frac{1}{\nu}}}{s}>(\frac{3}{7})^{\frac{1}{\nu}}\sum_{s=2}^{\infty}\frac{1}{s}=\infty,$$

and

$$\sum_{s=t_0}^{\infty}\frac{1}{r(s)}s^{\alpha-1}=\sum_{s=t_0}^{\infty}(\frac{s+1}{s})^{\frac{1}{4}}=\infty.$$

Moreover,

$$\begin{aligned}&\sum_{\xi=t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)})^{\frac{1}{\nu}}]\\&=\sum_{\xi=2}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,2)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s+1,2)})^{\frac{1}{\nu}}]\\&>(\frac{3}{7})^{\frac{1}{\nu}}\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}\frac{1}{\tau}(\sum_{s=\tau}^{\infty}\frac{1}{s^{\nu+1}})^{\frac{1}{\nu}}]\\&\geq(\frac{3}{7})^{\frac{1}{\nu}}\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}\frac{1}{\tau}(\int_{\tau}^{\infty}\frac{1}{s^{\nu+1}}ds)^{\frac{1}{\nu}}]\\&=(\frac{3}{7\nu})^{\frac{1}{\nu}}\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}\frac{1}{\tau^2}]\\&>(\frac{3}{7\nu})^{\frac{1}{\nu}}\sum_{\xi=2}^{\infty}\sum_{\tau=\xi}^{\infty}\frac{1}{\tau(\tau+1)}\\&=(\frac{3}{7\nu})^{\frac{1}{\nu}}\sum_{\xi=2}^{\infty}\frac{1}{\xi}=\infty.\end{aligned}$$

So (3.21)-(3.23) hold. On the other hand, for a sufficiently large $T>1$, when $t\to\infty$, one has

$$\theta_1(t,T)=\sum_{s=T}^{t-1}\frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\nu}}}{a^{\frac{1}{\nu}}(s)}s^{\alpha-1}\to\infty.$$

So there exists $T^*>T$ such that $\theta_1(\kappa(t),T)>1$ for $t\in[T^*,\infty)_{\mathbf{Z}}$.

In (3.24), if we let $\varpi_1(t)=t^{\nu},\ \varpi_2(t)=0$, then by use of the inequality $(t+1)^{\nu}-t^{\nu}\leq\nu(t+1)^{\nu-1}<\nu 2^{\nu-1}t^{\nu-1}$ for $t\geq T^*$, one can deduce that

$$\lim_{t\to\infty}\sup\{\sum_{s=T}^{t-1}\{L\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)}-$$

$$[\frac{r(\kappa(s))(\varpi_1(s+1)-\varpi_1(s))s^{1-\alpha}}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa(s+1)-\kappa(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)}$$

$$]^{\nu+1}\}s^{\alpha-1}\}$$

$$=\lim_{t\to\infty}\sup\{\sum_{s=T^*}^{t-1}\{M\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)}-$$

$$[\frac{r(\kappa(s))(\varpi_1(s+1)-\varpi_1(s))s^{1-\alpha}}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa(s+1)-\kappa(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)}$$

$$]^{\nu+1}\}s^{\alpha-1}$$

$$+\sum_{s=T}^{T^*}\{M\frac{q(s)\varpi_1(s)}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)}-$$

$$[\frac{r(\kappa(s))(\varpi_1(s+1)-\varpi_1(s))s^{1-\alpha}}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa(s+1)-\kappa(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)}$$

$$]^{\nu+1}\}s^{\alpha-1}\}$$

$$> \sum_{s=T}^{T^*}\{M\frac{q(s)\varpi_1(s)}{e_{-\frac{p}{a}}^{\sim}(s+1,t_0)}-$$

$$[\frac{r(\kappa(s))(\varpi_1(s+1)-\varpi_1(s))s^{1-\alpha}}{(\nu+1)r^{\frac{1}{\nu+1}}(\kappa(s))\varpi_1^{\frac{\nu}{\nu+1}}(s)(\kappa(s+1)-\kappa(s))^{\frac{\nu}{\nu+1}}\theta_1^{\frac{\nu}{\nu+1}}(\kappa(s),T)}$$

$$]^{\nu+1}\}s^{\alpha-1} + \sum_{s=T^*}^{t-1}[M-(\frac{\nu}{\nu+1})^{\nu+1}2^{\nu^2-1}]\frac{1}{s} \to \infty$$

for $t \to \infty$,

provided that $M > (\frac{\nu}{\nu+1})^{\nu+1}2^{\nu^2-1}$. So (3.21)-(3.24) all hold, and by Corollary 3.11 we obtain that every solution of Eq. (4.2) is oscillatory or tends to zero under the condition $M > (\frac{\nu}{\nu+1})^{\nu+1}2^{\nu^2-1}$.

V. Conclusions

We have derived some new oscillatory and asymptotic criteria for a class of fractional delay dynamic equation on time scales based on the properties of conformable fractional calculus, a generalized Riccati function and inequality technique, which are extensions of the corresponding results for dynamic equations on time scales involving integer order derivative. When the time scale $\mathbf{T}$ is taken for some different cases such as $\mathbf{T} = \mathbf{R}$, $\mathbf{T} = \mathbf{Z}$, $\mathbf{T} = q^{\mathbf{Z}}$ and so on, then corresponding oscillatory and asymptotic criteria can be obtained respectively. Applications for the established results show that they are valid.

The deduction process of Theorems 3.5-3.8 can also be applied to other types of fractional delay dynamic equation on time scales, which is expected to further research. For example, the $n-th$ fractional dynamic equation on time scales with delay term and forced term as follows

$$\{[(a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma})^{(\alpha)}]^{(\alpha)}......\}^{(\alpha)}$$
$$+ p(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma} + q(t)f(x(t)) = 0,$$

where the α-order fractional derivative appears for n times.

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