Quasi-Boolean Algebras: a Generalization of Boolean Algebras

Yajie Lv, Wenjuan Chen

***Abstract*—In the present paper, we introduce the notion of quasi-Boolean algebras as a generalization of Boolean algebras. First we discuss some properties of quasi-Boolean algebras. Next we define ideals and filters of quasi-Boolean algebras and investigate the related properties. We also show that there is a one-to-one correspondence between the set of ideals and the set of ideal congruences on a quasi-Boolean algebra. Finally, we present the relationship between quasi-Boolean algebras and Boolean quasi-rings.**

***Index Terms*—Boolean algebras, quasi-lattices, quasi-Boolean algebras, ideals, Boolean quasi-rings**

I. Introduction

RECENTLY, quantum computational logics have been received more and more attentions. Many authors considered that these logics were closely related with fuzzy logics [8]. In order to study these new forms of non-classical logics, some logical algebras had been introduced and the known results showed that these algebras were generalizations of well-known algebras associated with fuzzy logics [3], [6]. For example, Ledda et al. introduced quasi-MV algebras and pointed out that quasi-MV algebras were generalization of MV-algebras [12]. Chen and Wang defined quasi-BL algebras and showed that quasi-BL algebras generalized BL-algebras [7].

In [10], quasi-Boolean algebras were introduced by Iorgulescu in order to generalize the relationship between MV-algebras and lattice ordered groups. It was proved that any quasi-Wajsberg algebra defined in [1] is a quasi-Boolean algebra. Since quasi-MV algebras are equivalent to quasi-Wajsberg algebras, it is natural to obtain that any quasi-MV algebra is a quasi-Boolean algebra. Compared the relationship between Boolean algebras and MV-algebras and considered the important role of Boolean algebras in fuzzy logics, we wish to find a more suitable way to define quasi-Boolean algebras which generalize Boolean algebras in the setting of quantum computational logics.

In 1993, Chajda introduced q-lattices and presented some elementary results of a q-lattice [4]. Subsequently, an algebra of quasiordered logic based on a q-lattice was defined in [5]. The concepts of algebra of quasiordered logic as a generalization of Boolean algebra is similar to the case of quasi-MV algebras generalizing MV-algebras. However, in the algebra of quasiordered logic, the unary operation is defined by its binary operation and it does not satisfy the involution. Hence we want to redefine the quasi-Boolean algebras based on q-lattices. In addition, ideals and filters play an important role in studying the algebraic structures [9], [11], [13], [14]. These notions are dual in a Boolean algebra [2], so in this paper, we also want to study ideals and filters in a quasi-Boolean algebra. The paper is organized as follows. In Section 2, we recall some definitions and results of q-lattices. In Section 3, we introduce the notion of quasi-Boolean algebras. We also define ideals and filters of quasi-Boolean algebras and investigate the related properties. In Section 4, we present the relationship between quasi-Boolean algebras and Boolean quasi-rings.

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Y. J. Lv is a postgraduate student of the School of Mathematical Sciences, University of Jinan, Jinan, Shandong, 250022 China (e-mail: jasslyj@mail.ujn.edu.cn).

W. J. Chen (Corresponding author) is a professor of the School of Mathematical Sciences, University of Jinan, Jinan, Shandong, 250022 China (e-mail: wjchenmath@gmail.com).

II. Preliminary

In this section, we recall some definitions and results in [4], [5].

Recall that an algebra $(\Xi;\sqcup,\sqcap)$ of type $(2,2)$ is called a *q-lattice*, if it satisfies the following conditions for any $\varpi,\varrho,\varsigma\in\Xi$,

(QL1) $\varpi\sqcup\varrho=\varrho\sqcup\varpi$ and $\varpi\sqcap\varrho=\varrho\sqcap\varpi$;

(QL2) $\varpi\sqcup(\varrho\sqcup\varsigma)=(\varpi\sqcup\varrho)\sqcup\varsigma$ and $\varpi\sqcap(\varrho\sqcap\varsigma)=(\varpi\sqcap\varrho)\sqcap\varsigma$;

(QL3) $\varpi\sqcup(\varrho\sqcap\varpi)=\varpi\sqcup\varpi$ and $\varpi\sqcap(\varrho\sqcup\varpi)=\varpi\sqcap\varpi$;

(QL4) $\varpi\sqcup\varrho=\varpi\sqcup(\varrho\sqcup\varrho)$ and $\varpi\sqcap\varrho=\varpi\sqcap(\varrho\sqcap\varrho)$;

(QL5) $\varpi\sqcup\varpi=\varpi\sqcap\varpi$.

On any q-lattice $(\Xi;\sqcup,\sqcap)$, one can define $\varpi\preceq\varrho$ by $\varpi\sqcap\varrho=\varpi\sqcap\varpi$, or $\varpi\sqcup\varrho=\varrho\sqcup\varrho$. Then the relation $\preceq$ is *quasi-ordering*. A q-lattice $(\Xi;\sqcup,\sqcap)$ is called *distributive*, if it satisfies (D1) $\varpi\sqcup(\varrho\sqcap\varsigma)=(\varpi\sqcup\varrho)\sqcap(\varpi\sqcup\varsigma)$ and (D2) $\varpi\sqcap(\varrho\sqcup\varsigma)=(\varpi\sqcap\varrho)\sqcup(\varpi\sqcap\varsigma)$. Similarly to lattices, we can show that a q-lattice satisfies (D1) if and only if it satisfies (D2). A *bounded* q-lattice $(\Xi;\sqcup,\sqcap)$ means that there exist elements 0 and 1 in Ξ such that $\varpi\sqcap 0=0$ and $\varpi\sqcup 1=1$ for any $\varpi\in\Xi$. Let $(\Xi;\sqcup,\sqcap)$ be a bounded q-lattice and $\varpi\in\Xi$. An element $\varrho\in\Xi$ is called a *complement* of ϖ, if $\varpi\sqcap\varrho=0$ and $\varpi\sqcup\varrho=1$. For any $\varpi\in\Xi$, if it has a complement, then $(\Xi;\sqcup,\sqcap)$ is called a *complemented* q-lattice.

Let $(\Xi;\sqcup,\sqcap)$ be a complemented distributive q-lattice. Define a unary operation * on $(\Xi;\sqcup,\sqcap)$ by $\varpi\mapsto\varpi^\star$ with $\varpi^\star=\varrho\sqcup\varrho$, where ϱ is a complement of ϖ.

Lemma 1: [5] Let $(\Xi;\sqcup,\sqcap)$ be a complemented distributive q-lattice and $\varpi\in\Xi$. Then $\varpi^\star$ is a complement of ϖ.

Proposition 1: [5] Let $(\Xi;\sqcup,\sqcap)$ be a complemented distributive q-lattice. Then for any $\varpi,\varrho\in\Xi$, we have

(1) if $\varpi\sqcup\varrho=1$ and $\varpi\sqcap\varrho=0$, then $(\varpi\sqcap\varpi)^\star=\varrho\sqcap\varrho$;

(2) $(\varpi\sqcap\varrho)^\star=\varpi^\star\sqcup\varrho^\star$ and $(\varpi\sqcup\varrho)^\star=\varpi^\star\sqcap\varrho^\star$;

(3) if $\varpi\preceq\varrho$, then $\varrho^\star\preceq\varpi^\star$.

An algebra $(\Xi;\sqcup,\sqcap,^\star,0,1)$ is called *an algebra of quasiordered logic*, if its reduct $(\Xi;\sqcup,\sqcap)$ is a complemented distributive q-lattice, 0 and 1 are a zero and a unit of $(\Xi;\sqcup,\sqcap)$, respectively, and * is a unary operation defined as above.

III. Quasi-Boolean algebra and ideals

In this section, we give the definition of quasi-Boolean algebras and discuss some basic properties of quasi-Boolean algebras. We also investigate the properties of ideals and filters in a quasi-Boolean algebra.

Definition 1: An algebra $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ of type $(2, 2, 1, 0, 0)$ is called a *generalized quasi-Boolean algebra*, if the following conditions are satisfied:

(1) $(\Omega; \sqcup, \sqcap)$ is a distributive q-lattice;

(2) $\varpi \sqcap 0 = 0$ and $\varpi \sqcup 1 = 1$;

(3) $\varpi \sqcap \varpi^\star = 0$ and $\varpi \sqcup \varpi^\star = 1$;

(4) $(\varpi \sqcap \varpi)^\star = \varpi^\star \sqcup \varpi^\star$.

A generalized quasi-Boolean algebra $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ with the condition (5) $\varpi^{\star\star} = \varpi$ is called a *quasi-Boolean algebra*.

Following from the definition, a generalized quasi-Boolean algebra is a complemented distributive q-lattice with the unary operation satisfying $(\varpi \sqcap \varpi)^\star = \varpi^\star \sqcup \varpi^\star$, while a quasi-Boolean algebra is a generalized quasi-Boolean algebra satisfying the unary operation with involution.

Remark 1: It is easy to see that an algebra of quasiordered logic $(\Xi; \sqcup, \sqcap, ^\star, 0, 1)$ is a generalized quasi-Boolean algebra, since $\varpi^\star \sqcup \varpi^\star = (\varpi \sqcap \varpi)^\star$ for $\varpi \in \Xi$. Moreover, an algebra of quasiordered logic $(\Xi; \sqcup, \sqcap, ^\star, 0, 1)$ with $\varpi^{\star\star} = \varpi$ is a quasi-Boolean algebra.

Remark 2: In any quasi-Boolean algebra $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$, its reduct $(\Omega; \sqcup, ^\star, 0)$ or $(\Omega; \sqcap, ^\star, 1)$ is a quasi-MV algebra.

Proposition 2: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra. Then for any $\varpi, \varrho, \kappa, \iota \in \Omega$, we have

(1) $\varpi \preceq \varpi \sqcup \varrho$ and $\varpi \sqcap \varrho \preceq \varpi$;

(2) if $\varpi \preceq \varrho$ and $\kappa \preceq \iota$, then $\varpi \sqcap \kappa \preceq \varrho \sqcap \iota$ and $\varpi \sqcup \kappa \preceq \varrho \sqcup \iota$;

(3) $(\varpi \sqcup \varrho)^\star = \varpi^\star \sqcap \varrho^\star$ and $(\varpi \sqcap \varrho)^\star = \varpi^\star \sqcup \varrho^\star$;

(4) if $\varpi \preceq \varrho$, then $\varrho^\star \preceq \varpi^\star$.

Definition 2: An algebra $(\Xi; \sqcup, \sqcap, ^\star, 0, 1)$ of type $(2, 2, 1, 0, 0)$ is called a *quasi-De Morgan algebra*, if the following conditions are satisfied:

(1) $(\Xi; \sqcup, \sqcap)$ is a distributive q-lattice;

(2) $(\varpi \sqcap \varrho)^\star = \varpi^\star \sqcup \varrho^\star$ and $(\varpi \sqcup \varrho)^\star = \varpi^\star \sqcap \varrho^\star$;

(3) $\varpi^{\star\star} = \varpi$.

Hence any quasi-Boolean algebra is a quasi-De Morgan algebra.

Below we see the relationship between quasi-Boolean algebras and Boolean algebras. Obviously, any Boolean algebra is a quasi-Boolean algebra. However, a quasi-Boolean algebra is not a Boolean algebra in general. It is easy to show the following result.

Proposition 3: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra. Then the following conditions are equivalent:

(1) $(\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ is a Boolean algebra;

(2) $(\Omega; \sqcup, \sqcap)$ is a lattice;

(3) the induced quasiorder $\preceq$ is a partial order.

Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra. Denote $\mathcal{R}(\Omega) = \{\varpi \in \Omega | \varpi \sqcap \varpi = \varpi\}$. Obviously, $0, 1 \in \mathcal{R}(\Omega)$ and then $\mathcal{R}(\Omega)$ is a non-empty subset of Ω, so $(\mathcal{R}(\Omega); \sqcup, \sqcap, ^\star, 0, 1)$ is a Boolean algebra.

Definition 3: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra. A subset Σ of Ω is called an *ideal* of $\mathbf{\Omega}$, if the following conditions are satisfied:

(1) $0 \in \Sigma$;

(2) if $\varpi, \varrho \in \Sigma$, then $\varpi \sqcup \varrho \in \Sigma$;

(3) if $\varpi \in \Sigma$ and $\varrho \preceq \varpi$, then $\varrho \in \Sigma$.

A subset Φ of Ω is called a *filter* of $\mathbf{\Omega}$, if the following conditions are satisfied:

(1) $1 \in \Phi$;

(2) if $\varpi, \varrho \in \Phi$, then $\varpi \sqcap \varrho \in \Phi$;

(3) if $\varpi \in \Phi$ and $\varpi \preceq \varrho$, then $\varrho \in \Phi$.

Definition 4: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra. A subset Σ of Ω is called a *weak ideal* of $\mathbf{\Omega}$, if the following conditions are satisfied:

(1) $0 \in \Sigma$;

(2) if $\varpi, \varrho \in \Sigma$, then $\varpi \sqcup \varrho \in \Sigma$;

(3) if $\varpi \in \Sigma$ and $\varrho \preceq \varpi$, then $\varrho \sqcap \varrho \in \Sigma$.

A subset Φ of Ω is called a *weak filter* of $\mathbf{\Omega}$, if the following conditions are satisfied:

(1) $1 \in \Phi$;

(2) if $\varpi, \varrho \in \Phi$, then $\varpi \sqcap \varrho \in \Phi$;

(3) if $\varpi \in \Phi$ and $\varpi \preceq \varrho$, then $\varrho \sqcup \varrho \in \Phi$.

Proposition 4: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra. Then any ideal (filter) is a weak ideal (weak filter).

Proposition 5: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra. Then the set of ideals (filters) of a quasi-Boolean algebra is closed under arbitrary intersection.

Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra and Γ be a non-empty subset of Ω. *The ideal generated* by Γ is the least ideal containing Γ and is denoted by $(\Gamma]$. Dually, *the filter generated* by Γ is the least filter containing Γ and is denoted by $[\Gamma)$.

Lemma 2: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra and Γ be a non-empty subset of Ω. Then

(1) $(\Gamma] = \{\varpi \in \Omega | \varpi \preceq \varpi_1 \sqcup \varpi_2 \sqcup \cdots \sqcup \varpi_n, \text{ for some } \varpi_1, \varpi_2, \ldots, \varpi_n \in \Gamma\}$;

(2) $[\Gamma) = \{\varpi \in \Omega | \varpi_1 \sqcap \varpi_2 \sqcap \cdots \sqcap \varpi_n \preceq \varpi, \text{ for some } \varpi_1, \varpi_2, \ldots, \varpi_n \in \Gamma\}$.

Proposition 6: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra. Then

(1) For any $\Sigma \subseteq \Omega$, Σ is a (weak) ideal if and only if $\Sigma^\star$ is a (weak) filter;

(2) For any $\Phi \subseteq \Omega$, Φ is a (weak) filter if and only if $\Phi^\star$ is a (weak) ideal.

Proof: Let Σ be an ideal of $\mathbf{\Omega}$. Then $0 \in \Sigma$ and then $1 = 0^\star \in \Sigma^\star$. If $\varpi^\star, \varrho^\star \in \Sigma^\star$, then $\varpi, \varrho \in \Sigma$ and we have $\varpi \sqcup \varrho \in \Sigma$, it turns out that $\varpi^\star \sqcap \varrho^\star = (\varpi \sqcup \varrho)^\star \in \Sigma^\star$ by Proposition 2. Let $\varpi^\star \in \Sigma^\star$ and $\varrho \in \Omega$ with $\varpi^\star \preceq \varrho$. Then $\varrho^\star \preceq \varpi^{\star\star} = \varpi$. Since Σ is an ideal of $\mathbf{\Omega}$, we have $\varrho^\star \in \Sigma$, so $\varrho = \varrho^{\star\star} \in \Sigma^\star$. Hence $\Sigma^\star$ is a filter of $\mathbf{\Omega}$. Conversely, if $\Sigma^\star$ is a filter of $\mathbf{\Omega}$, then $1 \in \Sigma^\star$ and then $0 = 1^\star \in \Sigma^{\star\star} = \Sigma$. If $\varpi, \varrho \in \Sigma$, then $\varpi^\star, \varrho^\star \in \Sigma^\star$ and $\varpi \sqcup \varrho = (\varpi^\star \sqcap \varrho^\star)^\star$. Since $\Sigma^\star$ is a filter of $\mathbf{\Omega}$, we have $\varpi^\star \sqcap \varrho^\star \in \Sigma^\star$, so $\varpi \sqcup \varrho = (\varpi^\star \sqcap \varrho^\star)^\star \in \Sigma^{\star\star} = \Sigma$. Let $\varpi \in \Sigma$ and $\varrho \in \Omega$ with $\varrho \preceq \varpi$. Then $\varpi^\star \preceq \varrho^\star$, it follows that $\varpi^\star \in \Sigma^\star$ and then $\varrho^\star \in \Sigma^\star$, so $\varrho = \varrho^{\star\star} \in \Sigma^{\star\star} = \Sigma$. Hence Σ is an ideal of $\mathbf{\Omega}$. The rest can be proved similarly or dually. ■

Since ideals and filters are dual in a quasi-Boolean algebra, we only discuss the properties of ideals in the following.

Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra and ϑ be a binary relation on $\mathbf{\Omega}$. Then ϑ is called an *ideal* *congruence*, if ϑ is a congruence on $\mathbf{\Omega}$ and $\langle \varpi\sqcap\varpi, \varrho\sqcap\varrho\rangle \in \vartheta$ implies $\langle \varpi, \varrho\rangle \in \vartheta$ for any $\varpi, \varrho \in \Omega$.

Lemma 3: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra and ϑ be an ideal congruence on $\mathbf{\Omega}$. Then the set $0/\vartheta = \{\varpi \in \Omega | \langle \varpi, 0\rangle \in \vartheta\}$ is an ideal of $\mathbf{\Omega}$.

Proof: Since $\langle 0, 0\rangle \in \vartheta$, we have $0 \in 0/\vartheta$. If $\varpi, \varrho \in 0/\vartheta$, then $\langle \varpi, 0\rangle \in \vartheta$ and $\langle \varrho, 0\rangle \in \vartheta$, it turns out that $\langle \varpi \sqcup \varrho, 0\rangle = \langle \varpi\sqcup\varrho, 0\sqcup 0\rangle \in \vartheta$, so $\varpi\sqcup\varrho \in 0/\vartheta$. Let $\varpi \in 0/\vartheta$ and $\varrho \in \Omega$ with $\varrho \preceq \varpi$. Then $\langle \varpi, 0\rangle \in \vartheta$ and then $\langle \varpi\sqcap\varrho, 0\sqcap\varrho\rangle \in \vartheta$, it follows that $\langle \varrho \sqcap \varrho, 0 \sqcap 0\rangle \in \vartheta$. Note that ϑ is an ideal congruence on $\mathbf{\Omega}$, we have $\langle \varrho, 0\rangle \in \vartheta$, so $\varrho \in 0/\vartheta$. Hence the set $0/\vartheta$ is an ideal of $\mathbf{\Omega}$. ■

Lemma 4: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra and Σ be an ideal of $\mathbf{\Omega}$. Then the binary relation ϑ defined by $\langle \varpi, \varrho\rangle \in \vartheta$ if and only if $(\varpi\sqcap\varrho^\star)\sqcup(\varpi^\star\sqcap\varrho) \in \Sigma$ is an ideal congruence on $\mathbf{\Omega}$.

Proof: For any $\varpi \in \Omega$, since $(\varpi \sqcap \varpi^\star) \sqcup (\varpi^\star \sqcap \varpi) = 0 \in \Sigma$, we have $\langle \varpi, \varpi\rangle \in \vartheta$. If $\langle \varpi, \varrho\rangle \in \vartheta$, then $(\varpi \sqcap \varrho^\star) \sqcup (\varpi^\star \sqcap \varrho) \in \Sigma$, so $\langle \varrho, \varpi\rangle = (\varrho \sqcap \varpi^\star) \sqcup (\varrho^\star \sqcap \varpi) = (\varpi \sqcap \varrho^\star) \sqcup (\varpi^\star \sqcap \varrho) \in \Sigma$. Let $\langle \varpi, \varrho\rangle \in \vartheta$ and $\langle \varrho, \varsigma\rangle \in \vartheta$. Then $(\varpi \sqcap \varrho^\star) \sqcup (\varpi^\star \sqcap \varrho) \in \Sigma$ and $(\varrho \sqcap \varsigma^\star) \sqcup (\varrho^\star \sqcap \varsigma) \in \Sigma$. Since Σ is an ideal of $\mathbf{\Omega}$, we have $((\varpi \sqcap \varrho^\star) \sqcup (\varpi^\star \sqcap \varrho)) \sqcup ((\varrho\sqcap\varsigma^\star)\sqcup(\varrho^\star\sqcap\varsigma)) \in \Sigma$. We calculate $(\varpi\sqcap\varsigma^\star)\sqcup(\varpi^\star\sqcap\varsigma) = (\varpi\sqcap\varsigma^\star\sqcap(\varrho\sqcup\varrho^\star))\sqcup(\varpi^\star\sqcap\varsigma\sqcap(\varrho\sqcup\varrho^\star)) = (\varpi\sqcap\varsigma^\star\sqcap\varrho)\sqcup(\varpi\sqcap\varsigma^\star\sqcap\varrho^\star)\sqcup(\varpi^\star\sqcap\varsigma\sqcap\varrho)\sqcup(\varpi^\star\sqcap\varsigma\sqcap\varrho^\star) \preceq (\varsigma^\star\sqcap\varrho)\sqcup(\varpi\sqcap\varrho^\star)\sqcup(\varpi^\star\sqcap\varrho)\sqcup(\varsigma\sqcap\varrho^\star) = ((\varpi\sqcap\varrho^\star)\sqcup(\varpi^\star\sqcap\varrho))\sqcup((\varrho\sqcap\varsigma^\star)\sqcup(\varrho^\star\sqcap\varsigma))$ by Proposition 2, so $(\varpi \sqcap \varsigma^\star) \sqcup (\varpi^\star \sqcap \varsigma) \in \Sigma$ and then $\langle \varpi, \varsigma\rangle \in \vartheta$. Hence the binary relation ϑ is an equivalent relation on $\mathbf{\Omega}$. It is easy to see that if $\langle \varpi, \varrho\rangle \in \vartheta$, then $\langle \varpi^\star, \varrho^\star\rangle \in \vartheta$. For any $\langle \varpi, \varrho\rangle \in \vartheta$ and $\langle \kappa, \iota\rangle \in \vartheta$, then $(\varpi \sqcap \varrho^\star) \sqcup (\varpi^\star \sqcap \varrho) \in \Sigma$ and $(\kappa \sqcap \iota^\star) \sqcup (\kappa^\star \sqcap \iota) \in \Sigma$. Since $((\varpi\sqcup\kappa)\sqcap(\varrho\sqcup\iota)^\star)\sqcup((\varpi\sqcup\kappa)^\star\sqcap(\varrho\sqcup\iota)) = ((\varpi\sqcup\kappa)\sqcap(\varrho^\star\sqcap\iota^\star))\sqcup((\varpi^\star\sqcap\kappa^\star)\sqcap(\varrho\sqcup\iota)) = (\varpi\sqcap\varrho^\star\sqcap\iota^\star)\sqcup(\kappa\sqcap\varrho^\star\sqcap\iota^\star)\sqcup(\varpi^\star\sqcap\kappa^\star\sqcap\varrho)\sqcup(\varpi^\star\sqcap\kappa^\star\sqcap\iota) \preceq ((\varpi\sqcap\varrho^\star)\sqcup(\varpi^\star\sqcap\varrho))\sqcup((\kappa\sqcap\iota^\star)\sqcup(\kappa^\star\sqcap\iota))$, it follows that $((\varpi\sqcup\kappa)\sqcap(\varrho\sqcup\iota)^\star)\sqcup((\varpi\sqcup\kappa)^\star\sqcap(\varrho\sqcup\iota)) \in \Sigma$, so $\langle \varpi \sqcup \kappa, \varrho \sqcup \iota\rangle \in \vartheta$. Similarly, we can prove $\langle \varpi \sqcap \kappa, \varrho \sqcap \iota\rangle \in \vartheta$. Hence the equivalent relation ϑ is a congruence on $\mathbf{\Omega}$. Finally, if $\langle \varpi\sqcap\varpi, \varrho\sqcap\varrho\rangle \in \vartheta$, then $((\varpi\sqcap\varpi)\sqcap(\varrho\sqcap\varrho)^\star)\sqcup((\varpi\sqcap\varpi)^\star\sqcap(\varrho\sqcap\varrho)) \in \Sigma$, it follows that $(\varpi \sqcap \varrho^\star) \sqcup (\varpi^\star \sqcap \varrho) \in \Sigma$, so $\langle \varpi, \varrho\rangle \in \vartheta$. Hence the relation ϑ is an ideal congruence on $\mathbf{\Omega}$. ■

Theorem 1: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra. Then there exists a one-to-one correspondence between the set of ideals and the set of ideal congruences on $\mathbf{\Omega}$.

Proof: Let Σ be an ideal of $\mathbf{\Omega}$. Then ϑ_Σ defined in Lemma 4 is an ideal congruence on $\mathbf{\Omega}$. Moreover, since $\varpi \preceq \varpi \sqcup \varpi = \varpi \sqcap \varpi \preceq \varpi$, we have $\Sigma_{\vartheta_\Sigma} = \{\varpi \in \Omega | \langle \varpi, 0\rangle \in \vartheta_\Sigma\} = \{\varpi \in \Omega | \varpi \sqcap \varpi \in \Sigma\} = \Sigma$. Conversely, let ϑ be an ideal congruence on $\mathbf{\Omega}$. Then $\Sigma_\vartheta = 0/\vartheta$ defined in Lemma 3 is an ideal of $\mathbf{\Omega}$. Moreover, $\vartheta_{\Sigma_\vartheta} = \{\langle \varpi, \varrho\rangle | (\varpi\sqcap\varrho^\star)\sqcup(\varpi^\star\sqcap\varrho) \in \Sigma_\vartheta\} = \{\langle \varpi, \varrho\rangle | \langle (\varpi \sqcap \varrho^\star) \sqcup (\varpi^\star \sqcap \varrho), 0\rangle \in \vartheta\} = \vartheta$. Indeed, for any $\langle \varpi, \varrho\rangle \in \vartheta$, we have $\langle \varpi \sqcap \varrho^\star, 0\rangle = \langle \varpi \sqcap \varrho^\star, \varrho \sqcap \varrho^\star\rangle \in \vartheta$ and $\langle \varpi^\star \sqcap \varrho, 0\rangle = \langle \varpi^\star \sqcap \varrho, \varpi^\star \sqcap \varpi\rangle \in \vartheta$, so $\langle (\varpi \sqcap \varrho^\star) \sqcup (\varpi^\star \sqcap \varrho), 0\rangle \in \vartheta$ and then $\langle \varpi, \varrho\rangle \in \vartheta_{\Sigma_\vartheta}$. For any $\langle \varpi, \varrho\rangle \in \vartheta_{\Sigma_\vartheta}$, then $\langle (\varpi \sqcap \varrho^\star) \sqcup (\varpi^\star \sqcap \varrho), 0\rangle \in \vartheta$. We calculate that $\langle (\varpi \sqcap \varrho^\star) \sqcup (\varpi^\star \sqcap \varrho), 0\rangle = \langle (\varpi \sqcup \varpi^\star) \sqcap (\varrho^\star \sqcup \varpi^\star) \sqcap (\varpi \sqcup \varrho) \sqcap (\varrho^\star \sqcup \varrho), 0\rangle = \langle (\varpi \sqcap \varrho)^\star \sqcap (\varpi \sqcup \varrho), 0\rangle \in \vartheta$, it turns out that $\langle (\varpi \sqcap \varrho) \sqcup ((\varpi \sqcap \varrho)^\star \sqcap (\varpi \sqcup \varrho)), (\varpi \sqcap \varrho) \sqcup 0\rangle = \langle \varpi \sqcup \varrho, \varpi \sqcap \varrho\rangle \in \vartheta$, so $\langle (\varpi \sqcup \varrho) \sqcap \varpi, (\varpi \sqcap \varrho) \sqcap \varpi\rangle = \langle \varpi \sqcap \varpi, \varpi \sqcap \varrho\rangle \in \vartheta$. Similarly, we have $\langle \varrho \sqcap \varrho, \varpi \sqcap \varrho\rangle \in \vartheta$. Hence we have $\langle \varpi \sqcap \varpi, \varrho \sqcap \varrho\rangle \in \vartheta$. Note that ϑ is an ideal congruence on $\mathbf{\Omega}$, we get $\langle \varpi, \varrho\rangle \in \vartheta$. ■

IV. BOOLEAN QUASI-RINGS

It is well-known that Boolean algebras can be regards as rings. Below we discuss the similar results for quasi-Boolean algebras.

Definition 5: Let $\mathbf{\Lambda} = (\Lambda; \oplus, \ominus, 0)$ be an algebra of type $(2, 1, 0)$ and denote the set $\Lambda \oplus 0 = \{\varpi \oplus 0 | \varpi \in \Lambda\}$. Then $\mathbf{\Lambda}$ is called a *quasi-group* if the following conditions are satisfied for any $\varpi, \varrho \in \Lambda$:

(QG1) $(\Lambda \oplus 0; \oplus, \ominus, 0)$ is a group with $0 \oplus 0 = 0$;
(QG2) $\ominus(\ominus\varpi) = \varpi$;
(QG3) $\ominus(\varpi \oplus 0) = (\ominus\varpi) \oplus 0$;
(QG4) $\varpi \oplus \varrho = (\varpi \oplus 0) \oplus (\varrho \oplus 0)$.

If $\mathbf{\Lambda} = (\Lambda; \oplus, \ominus, 0)$ is a quasi-group and for any $\varpi, \varrho \in \Lambda$, we have $\varpi \oplus \varrho = \varrho \oplus \varpi$, then $\mathbf{\Lambda}$ is *commutative*. Following from the definition, we know that if $(\Lambda \oplus 0; \oplus, \ominus, 0)$ is a commutative group, then $\mathbf{\Lambda}$ is a commutative quasi-group.

Lemma 5: Let $\mathbf{\Lambda} = (\Lambda; \oplus, \ominus, 0)$ be a quasi-group. Then the following hold for any $\varpi, \varrho, \varsigma \in \Lambda$:

(1) if $\varpi \oplus 0 \in \Lambda \oplus 0$, then $\varpi \oplus 0 = 0 \oplus \varpi$;
(2) $(\varpi \oplus \varrho) \oplus \varsigma = \varpi \oplus (\varrho \oplus \varsigma)$;
(3) $(\ominus\varpi) \oplus \varpi = \varpi \oplus (\ominus\varpi) = 0$;
(4) if $\varpi \oplus \varrho = \varpi \oplus \varsigma$, then $0 \oplus \varrho = 0 \oplus \varsigma$,
if $\varrho \oplus \varpi = \varsigma \oplus \varpi$, then $\varrho \oplus 0 = \varsigma \oplus 0$.

Proof: (1) Since $0 \oplus 0 = 0 \in \Lambda \oplus 0$, we have $\varpi \oplus 0 = 0 \oplus (\varpi \oplus 0) = (0 \oplus 0) \oplus (\varpi \oplus 0) = 0 \oplus \varpi$ by (QG4).

(2) We have $(\varpi\oplus\varrho)\oplus\varsigma = ((\varpi\oplus0)\oplus(\varrho\oplus0)\oplus0)\oplus(\varsigma\oplus0) = ((\varpi\oplus0)\oplus(\varrho\oplus0))\oplus(\varsigma\oplus0) = (\varpi\oplus0)\oplus((\varrho\oplus0)\oplus(\varsigma\oplus0)) = \varpi \oplus (\varrho \oplus \varsigma)$ by (QG4) and (QG1).

(3) We have $(\ominus\varpi) \oplus \varpi = ((\ominus\varpi) \oplus 0) \oplus (\varpi \oplus 0) = (\ominus(\varpi \oplus 0)) \oplus (\varpi \oplus 0) = 0$ by (QG4), (QG3) and (QG1).

(4) If $\varpi\oplus\varrho = \varpi\oplus\varsigma$, then $(\ominus\varpi)\oplus\varpi\oplus\varrho = (\ominus\varpi)\oplus\varpi\oplus\varsigma$, we have $0 \oplus \varrho = 0 \oplus \varsigma$. The other can be proved similarly. ■

Definition 6: Let $\mathbf{\Psi} = (\Psi; \oplus, \odot, \ominus, 0)$ be an algebra of type $(2, 2, 1, 0)$. Then $\mathbf{\Psi}$ is called a *quasi-ring* if the following conditions are satisfied for any $\varpi, \varrho, \varsigma \in \Psi$:

(QR1) $(\Psi; \oplus, \ominus, 0)$ is a commutative quasi-group;
(QR2) $(\Psi; \odot)$ is a semigroup;
(QR3) $\varpi \odot \varrho = (\varpi \odot \varrho) \oplus 0$;
(QR4) $\varpi\odot(\varrho\oplus\varsigma) = (\varpi\odot\varrho)\oplus(\varpi\odot\varsigma)$ and $(\varrho\oplus\varsigma)\odot\varpi = (\varrho \odot \varpi) \oplus (\varsigma \odot \varpi)$.

A quasi-ring $\mathbf{\Psi} = (\Psi; \oplus, \odot, \ominus, 0, 1)$ is called a *quasi-ring with quasi-identity* if the following condition is satisfied for any $\varpi, \varrho \in \Psi$: (QR5) $\varpi \oplus \varrho = (\varpi \oplus \varrho) \odot 1$ and $1 \odot 1 = 1$.

In the following, a quasi-ring $\mathbf{\Psi} = (\Psi; \oplus, \odot, \ominus, 0, 1)$ always means a quasi-ring with quasi-identity. In addition, we shall consider that the operation $\odot$ has priority to the operation $\oplus$.

Proposition 7: Let $\mathbf{\Psi} = (\Psi; \oplus, \odot, \ominus, 0, 1)$ be a quasi-ring. Then the following hold for any $\varpi, \varrho, \varsigma \in \Psi$:

(1) $0 \odot \varpi = \varpi \odot 0 = 0$;
(2) $(\ominus\varpi) \odot \varrho = \varpi \odot (\ominus\varrho) = \ominus(\varpi \odot \varrho)$;
(3) $(\ominus\varpi) \odot (\ominus\varrho) = \varpi \odot \varrho$;
(4) $\varsigma\odot(\varpi\ominus\varrho) = \varsigma\odot\varpi\ominus\varsigma\odot\varrho$ and $(\varpi\ominus\varrho)\odot\varsigma = \varpi\odot\varsigma\ominus\varrho\odot\varsigma$ where $\varpi \ominus \varrho = \varpi \oplus (\ominus\varrho)$;
(5) $\varpi \odot 1 = \varpi \oplus 0$ and $\varpi \odot 1 = 1 \odot \varpi$;
(6) $\varpi \odot \varrho \odot 1 = \varpi \odot \varrho$;
(7) $1 \oplus 0 = 1$.

Proof: (1) Since $0\odot\varpi=(0\oplus 0)\odot\varpi=0\odot\varpi\oplus 0\odot\varpi$, we have $0\odot\varpi\oplus 0=(0\odot\varpi\oplus 0)\oplus(0\odot\varpi\oplus 0)$. Note that $0\odot\varpi\oplus 0\in\Lambda\oplus 0$, it turns out that $0\odot\varpi=0\odot\varpi\oplus 0=0$. Similarly, we have $\varpi\odot 0=0$.

(2) Since $(\ominus\varpi)\odot\varrho\oplus\varpi\odot\varrho=(\ominus\varpi\oplus\varpi)\odot\varrho=0\odot\varrho=0$ by (QR4) and (1), we have $(\ominus\varpi)\odot\varrho\oplus 0=\ominus(\varpi\odot\varrho)\oplus 0$ by Lemma 5, so $(\ominus\varpi)\odot\varrho=\ominus(\varpi\odot\varrho)$. Similarly, we have $\varpi\odot(\ominus\varrho)=\ominus(\varpi\odot\varrho)$.

(3) We have $(\ominus\varpi)\odot(\ominus\varrho)=\ominus(\varpi\odot(\ominus\varrho))=\ominus(\ominus(\varpi\odot\varrho))=\varpi\odot\varrho$ by (2) and (QG2).

(4) We have $\varsigma\odot(\varpi\ominus\varrho)=\varsigma\odot(\varpi\oplus(\ominus\varrho))=\varsigma\odot\varpi\oplus\varsigma\odot(\ominus\varrho)=\varsigma\odot\varpi\oplus(\ominus(\varsigma\odot\varrho))=\varsigma\odot\varpi\ominus\varsigma\odot\varrho$. Similarly, we have $(\varpi\ominus\varrho)\odot\varsigma=\varpi\odot\varsigma\ominus\varrho\odot\varsigma$.

(5) We have $\varpi\oplus 0=(\varpi\oplus 0)\odot 1=\varpi\odot 1\oplus 0\odot 1=\varpi\odot 1\oplus 0=\varpi\odot 1$. Similarly, $0\oplus\varpi=1\odot\varpi$. Since $\varpi\oplus 0=0\oplus\varpi$ by Lemma 5, we have $\varpi\odot 1=1\odot\varpi$.

(6) We have $(\varpi\odot\varrho)\odot 1=(\varpi\odot\varrho)\oplus 0=\varpi\odot\varrho$ by (5).

(7) We have $1\oplus 0=1\odot 1=1$. ■

Definition 7: A quasi-ring $\mathbf{\Psi}=(\Psi;\oplus,\odot,\ominus,0,1)$ is *Boolean*, if Ψ satisfies $\varpi^2=\varpi\odot 1$ for any $\varpi\in\Psi$.

Lemma 6: Let $\mathbf{\Psi}=(\Psi;\oplus,\odot,\ominus,0,1)$ be a Boolean quasi-ring. For any $\varpi,\varrho\in\Psi$, we have $\varpi\oplus\varpi=0$ and $\varpi\odot\varrho=\varrho\odot\varpi$.

Proof: For any $\varpi\in\Psi$, we have $(\varpi\oplus\varpi)^2=(\varpi\oplus\varpi)\odot 1=\varpi\oplus\varpi$ and $(\varpi\oplus\varpi)^2=\varpi^2\oplus\varpi^2\oplus\varpi^2\oplus\varpi^2=\varpi\odot 1\oplus\varpi\odot 1\oplus\varpi\odot 1\oplus\varpi\odot 1=(\varpi\oplus\varpi\oplus\varpi\oplus\varpi)\odot 1=\varpi\oplus\varpi\oplus\varpi\oplus\varpi$, it turns out that $\varpi\oplus\varpi\oplus\varpi\oplus\varpi=\varpi\oplus\varpi$, so $\varpi\oplus\varpi=0$. For any $\varpi,\varrho\in\Psi$, on the one hand, we have $(\varpi\oplus\varrho)^2=(\varpi\oplus\varrho)\odot 1=\varpi\oplus\varrho$, on the other hand, we have $(\varpi\oplus\varrho)^2=\varpi^2\oplus\varpi\odot\varrho\oplus\varrho\odot\varpi\oplus\varrho^2=\varpi\odot 1\oplus\varpi\odot\varrho\oplus\varrho\odot\varpi\oplus\varrho\odot 1=\varpi\oplus 0\oplus\varpi\odot\varrho\oplus\varrho\odot\varpi\oplus\varrho\oplus 0=\varpi\oplus\varpi\odot\varrho\oplus\varrho\odot\varpi\oplus\varrho$, it turns out that $\varpi\oplus\varpi\odot\varrho\oplus\varrho\odot\varpi\oplus\varrho=\varpi\oplus\varrho$, so $\varpi\odot\varrho\oplus\varrho\odot\varpi=0$. Since $\varpi\odot\varrho\oplus\varpi\odot\varrho=0=\varpi\odot\varrho\oplus\varrho\odot\varpi$, we have $\varpi\odot\varrho\oplus 0=\varrho\odot\varpi\oplus 0$ by Lemma 5, so $\varpi\odot\varrho=\varrho\odot\varpi$. ■

Theorem 2: Let $\mathbf{\Omega}=(\Omega;\sqcup,\sqcap,^\star,0,1)$ be a quasi-Boolean algebra. Define $\mathbf{\Omega}^{\otimes}$ to be the algebra $(\Omega;\oplus,\odot,\ominus,0,1)$ where for any $\varpi,\varrho\in\Omega$, $\varpi\oplus\varrho=(\varpi\sqcap\varrho^\star)\sqcup(\varpi^\star\sqcap\varrho)$, $\varpi\odot\varrho=\varpi\sqcap\varrho$ and $\ominus\varpi=\varpi$. Then $\mathbf{\Omega}^{\otimes}$ is a Boolean quasi-ring.

Proof: Firstly, we show that $(\Omega;\oplus,\ominus,0)$ is a commutative quasi-group. For any $\varpi\in\Omega$, since $\varpi\oplus 0=(\varpi\sqcap 0^\star)\sqcup(\varpi^\star\sqcap 0)=(\varpi\sqcap\varpi)\sqcup 0=\varpi\sqcap\varpi$, it is easy to see that $(\Omega\oplus 0;\oplus,\ominus,0)$ is a group and $0\oplus 0=0\sqcap 0=0$. Meanwhile, (QG2) we have $(\ominus(\ominus\varpi))=\ominus\varpi=\varpi$; (QG3) since $\ominus(\varpi\oplus 0)=\varpi\oplus 0$ and $(\ominus\varpi)\oplus 0=\varpi\oplus 0$, we have $\ominus(\varpi\oplus 0)=(\ominus\varpi)\oplus 0$; (QG4) we have $(\varpi\oplus 0)\oplus(\varrho\oplus 0)=(\varpi\sqcap\varpi)\oplus(\varrho\sqcap\varrho)=((\varpi\sqcap\varpi)\sqcap(\varrho\sqcap\varrho)^\star)\sqcup((\varpi\sqcap\varpi)^\star\sqcap(\varrho\sqcap\varrho))=((\varpi\sqcap\varpi)\sqcap(\varrho^\star\sqcup\varrho^\star))\sqcup((\varpi^\star\sqcup\varpi^\star)\sqcap(\varrho\sqcap\varrho))=((\varpi\sqcap\varpi)\sqcap(\varrho^\star\sqcap\varrho^\star))\sqcup((\varpi^\star\sqcap\varpi^\star)\sqcap(\varrho\sqcap\varrho))=(\varpi\sqcap\varrho^\star)\sqcup(\varpi^\star\sqcap\varrho)=\varpi\oplus\varrho$. Moreover, we have $\varpi\oplus\varrho=(\varpi\sqcap\varrho^\star)\sqcup(\varpi^\star\sqcap\varrho)=(\varrho\sqcap\varpi^\star)\sqcup(\varrho^\star\sqcap\varpi)=\varrho\oplus\varpi$. So $(\Omega;\oplus,\ominus,0)$ is a commutative quasi-group. Secondly, for any $\varpi\in\Omega$, we have $(\varpi\odot\varrho)\odot\varsigma=(\varpi\sqcap\varrho)\sqcap\varsigma=\varpi\sqcap(\varrho\sqcap\varsigma)=\varpi\odot(\varrho\odot\varsigma)$ which implies that $(\Omega;\odot)$ is a semigroup. Moreover, $(\varpi\odot\varrho)\oplus 0=(\varpi\odot\varrho)\sqcap(\varpi\odot\varrho)=(\varpi\sqcap\varrho)\sqcap(\varpi\sqcap\varrho)=\varpi\sqcap\varrho=\varpi\odot\varrho$. Thirdly, on the one hand, we have $\varpi\odot(\varrho\oplus\varsigma)=\varpi\sqcap(\varrho\oplus\varsigma)=\varpi\sqcap((\varrho\sqcap\varsigma^\star)\sqcup(\varrho^\star\sqcap\varsigma))=(\varpi\sqcap\varrho\sqcap\varsigma^\star)\sqcup(\varpi\sqcap\varrho^\star\sqcap\varsigma)$, on the other hand, we have $(\varpi\odot\varrho)\oplus(\varpi\odot\varsigma)=(\varpi\sqcap\varrho)\oplus(\varpi\sqcap\varsigma)=((\varpi\sqcap\varrho)\sqcap(\varpi\sqcap\varsigma)^\star)\sqcup((\varpi\sqcap\varrho)^\star\sqcap(\varpi\sqcap\varsigma))=(\varpi\sqcap\varrho\sqcap(\varpi^\star\sqcup\varsigma^\star))\sqcup((\varpi^\star\sqcup\varrho^\star)\sqcap(\varpi\sqcap\varsigma))=(\varpi\sqcap\varrho\sqcap\varpi^\star)\sqcup(\varpi\sqcap\varrho\sqcap\varsigma^\star)\sqcup(\varpi^\star\sqcap\varpi\sqcap\varsigma)\sqcup(\varrho^\star\sqcap\varpi\sqcap\varsigma)=(\varpi\sqcap\varrho\sqcap\varsigma^\star)\sqcup(\varrho^\star\sqcap\varpi\sqcap\varsigma)$, so $\varpi\odot(\varrho\oplus\varsigma)=(\varpi\odot\varrho)\oplus(\varpi\odot\varsigma)$. Since $\varpi\odot\varrho=\varpi\sqcap\varrho=\varrho\sqcap\varpi=\varrho\odot\varpi$, we have $(\varrho\oplus\varsigma)\odot\varpi=\varrho\odot\varpi\oplus\varsigma\odot\varpi$. Finally, $(\varpi\oplus\varrho)\odot 1=(\varpi\oplus\varrho)\sqcap 1=\varpi\oplus\varrho$ and $1\odot 1=1\sqcap 1=1$. Thus $\mathbf{\Omega}^{\otimes}$ is a quasi-ring with quasi-identity. Note that $\varpi^2=\varpi\odot\varpi=\varpi\sqcap\varpi=\varpi\sqcap 1=\varpi\odot 1$, we have that $\mathbf{\Omega}^{\otimes}$ is a Boolean quasi-ring. ■

Theorem 3: Let $\mathbf{\Psi}=(\Psi;\oplus,\odot,\ominus,0,1)$ be a Boolean quasi-ring. Define $\mathbf{\Psi}^{\otimes}$ to be the algebra $(\Psi;\sqcup,\sqcap,^\star,0,1)$ where for any $\varpi,\varrho\in\Psi$, $\varpi\sqcup\varrho=\varpi\oplus\varrho\oplus\varpi\odot\varrho$, $\varpi\sqcap\varrho=\varpi\odot\varrho$ and $\varpi^\star\in\Psi$ with $\varpi^\star\oplus 0=(\varpi\oplus 0)^\star=1\oplus\varpi$. Then $\mathbf{\Psi}^{\otimes}$ is a generalized quasi-Boolean algebra.

Proof: Firstly, we show that $(\Psi;\sqcup,\sqcap)$ is a q-lattice. For any $\varpi,\varrho,\varsigma\in\Psi$, (QL1) we have $\varpi\sqcup\varrho=\varpi\oplus\varrho\oplus(\varpi\odot\varrho)=\varrho\oplus\varpi\oplus(\varrho\odot\varpi)=\varrho\sqcup\varpi$ and $\varpi\sqcap\varrho=\varpi\odot\varrho=\varrho\odot\varpi=\varrho\sqcap\varpi$ by Lemma 6. (QL2) We have $\varpi\sqcup(\varrho\sqcup\varsigma)=\varpi\oplus(\varrho\sqcup\varsigma)\oplus\varpi\odot(\varrho\sqcup\varsigma)=\varpi\oplus\varrho\oplus\varsigma\oplus\varrho\odot\varsigma\oplus\varpi\odot(\varrho\oplus\varsigma\oplus\varrho\odot\varsigma)=\varpi\oplus\varrho\oplus\varsigma\oplus\varrho\odot\varsigma\oplus\varpi\odot\varrho\oplus\varpi\odot\varsigma\oplus\varpi\odot\varrho\odot\varsigma$. The value of this last expression does not change if we permute ϖ, ϱ and ς, so $\varpi\sqcup(\varrho\sqcup\varsigma)=\varsigma\sqcup(\varpi\sqcup\varrho)$ and then $\varpi\sqcup(\varrho\sqcup\varsigma)=(\varpi\sqcup\varrho)\sqcup\varsigma$. Meanwhile, we have $(\varpi\sqcap\varrho)\sqcap\varsigma=(\varpi\odot\varrho)\odot\varsigma=\varpi\odot(\varrho\odot\varsigma)=\varpi\sqcap(\varrho\sqcap\varsigma)$. (QL3) We have $\varpi\sqcup\varpi=\varpi\oplus\varpi\oplus\varpi\odot\varpi=0\oplus\varpi\odot 1=\varpi\odot 1$ and $\varpi\sqcup(\varrho\sqcap\varpi)=\varpi\oplus(\varrho\sqcap\varpi)\oplus\varpi\odot(\varrho\sqcap\varpi)=\varpi\oplus\varrho\odot\varpi\oplus\varpi\odot(\varrho\odot\varpi)=\varpi\oplus\varrho\odot\varpi\oplus\varpi\odot\varrho\odot\varpi=\varpi\oplus\varrho\odot\varpi\oplus\varpi\odot\varpi\odot\varrho=\varpi\oplus(\varpi\odot\varrho\oplus\varrho\odot\varpi)=\varpi\oplus(\varpi\odot\varrho\oplus\varpi\odot\varrho)=\varpi\oplus 0=\varpi\odot 1$. Thus $\varpi\sqcup(\varrho\sqcap\varpi)=\varpi\sqcup\varpi$. Similarly, we have $\varpi\sqcap(\varrho\sqcup\varpi)=\varpi\sqcap\varpi$. (QL4) We have $\varpi\sqcup(\varrho\sqcup\varrho)=\varpi\sqcup(\varrho\odot 1)=\varpi\oplus\varrho\odot 1\oplus\varpi\odot\varrho\odot 1=\varpi\oplus\varrho\oplus\varpi\odot\varrho=\varpi\sqcup\varrho$ and $\varpi\sqcap(\varrho\sqcap\varrho)=\varpi\odot(\varrho\odot\varrho)=\varpi\odot(\varrho\odot 1)=(\varpi\odot\varrho)\odot 1=\varpi\odot\varrho=\varpi\sqcap\varrho$. (QL5) We have $\varpi\sqcup\varpi=\varpi\odot 1=\varpi\odot\varpi=\varpi\sqcap\varpi$. Hence $(\Psi;\sqcup,\sqcap)$ is a q-lattice. Secondly, since $\varpi\sqcup(\varrho\sqcap\varsigma)=\varpi\sqcup(\varrho\odot\varsigma)=\varpi\oplus\varrho\odot\varsigma\oplus\varpi\odot\varrho\odot\varsigma$ and $(\varpi\sqcup\varrho)\sqcap(\varpi\sqcup\varsigma)=(\varpi\oplus\varrho\oplus\varpi\odot\varrho)\odot(\varpi\oplus\varsigma\oplus\varpi\odot\varsigma)=\varpi\odot\varpi\oplus\varpi\odot\varsigma\oplus\varpi\odot\varpi\odot\varsigma\oplus\varrho\odot\varpi\oplus\varrho\odot\varsigma\oplus\varrho\odot\varpi\odot\varsigma\oplus\varpi\odot\varrho\odot\varpi\oplus\varpi\odot\varrho\odot\varsigma\oplus\varpi\odot\varrho\odot\varpi\odot\varsigma=\varpi\odot 1\oplus(\varpi\odot\varsigma\oplus\varpi\odot\varpi\odot\varsigma)\oplus(\varrho\odot\varpi\oplus\varpi\odot\varrho\odot\varpi)\oplus(\varrho\odot\varpi\odot\varsigma\oplus\varpi\odot\varrho\odot\varpi\odot\varsigma)\oplus\varrho\odot\varsigma\oplus\varpi\odot\varrho\odot\varsigma=\varpi\oplus\varrho\odot\varsigma\oplus\varpi\odot\varrho\odot\varsigma$, we have $\varpi\sqcup(\varrho\sqcap\varsigma)=(\varpi\sqcup\varrho)\sqcap(\varpi\sqcup\varsigma)$. Similarly, we have $\varpi\sqcap(\varrho\sqcup\varsigma)=(\varpi\sqcap\varrho)\sqcup(\varpi\sqcap\varsigma)$. Moreover, we have $\varpi\sqcap 0=\varpi\odot 0=0$ and $\varpi\sqcup 1=\varpi\oplus 1\oplus\varpi\odot 1=\varpi\oplus 1\oplus\varpi\oplus 0=1$. Finally, we have $\varpi\sqcup\varpi^\star=\varpi\oplus\varpi^\star\oplus\varpi\odot\varpi^\star=\varpi\oplus(\varpi^\star\oplus 0)\oplus\varpi\odot(\varpi^\star\oplus 0)=\varpi\oplus(1\oplus\varpi)\oplus\varpi\odot(1\oplus\varpi)=\varpi\oplus 1\oplus\varpi\oplus\varpi\odot 1\oplus\varpi\odot\varpi=1$ and $\varpi\sqcap\varpi^\star=\varpi\odot\varpi^\star=\varpi\odot(\varpi^\star\oplus 0)=\varpi\odot(1\oplus\varpi)=\varpi\odot 1\oplus\varpi\odot\varpi=\varpi\odot 1\oplus\varpi\odot 1=0$. Hence $(\Psi;\sqcup,\sqcap,^\star,0,1)$ is a complemented distributive q-lattice. Note that for any $\varpi\in\Psi$, $\varpi^\star\sqcap\varpi^\star=\varpi^\star\odot\varpi^\star=(\varpi^\star\oplus 0)\odot(\varpi^\star\oplus 0)=(1\oplus\varpi)\odot(1\oplus\varpi)=(1\oplus\varpi)\odot 1=1\oplus\varpi$ and $(\varpi\sqcup\varpi)^\star=(\varpi\odot 1)^\star=(\varpi\oplus 0)^\star=1\oplus\varpi$, we get $(\varpi\sqcup\varpi)^\star=\varpi^\star\sqcap\varpi^\star$ which implies that $\mathbf{\Psi}^{\otimes}$ is a generalized quasi-Boolean algebra. ■

Corollary 1: Let $\mathbf{\Psi}=(\Psi;\oplus,\odot,\ominus,0,1)$ be a Boolean quasi-ring and $\mathbf{\Psi}^{\otimes}$ be defined in Theorem 3. If the unary operation * satisfies an additional condition $\varpi^{\star\star}=\varpi$ for any $\varpi\in\Psi$, then $\mathbf{\Psi}^{\otimes}$ is a quasi-Boolean algebra. Moreover, $\mathbf{\Psi}^{\otimes\otimes}=\mathbf{\Psi}$ under this case.

Corollary 2: Let $\mathbf{\Omega}=(\Omega;\sqcup,\sqcap,^\star,0,1)$ be a quasi-Boolean algebra and $\mathbf{\Omega}^{\otimes}$ be defined in Theorem 2. If the unary operation * defined in $\mathbf{\Omega}^{\otimes\otimes}$ is same to $\mathbf{\Omega}$, then $\mathbf{\Omega}^{\otimes\otimes}=\mathbf{\Omega}$.

Given a quasi-ring $\mathbf{\Psi}=(\Psi;\oplus,\odot,\ominus,0,1)$, we define an *ideal* Σ of $\mathbf{\Psi}$, if the following conditions are satisfied for any $\varpi,\varrho,\varepsilon\in\Psi$, $\varpi,\varrho\in\Sigma$ imply $\varpi\ominus\varrho\in\Sigma$, $\varepsilon\odot\varpi\in\Sigma$ and $\varpi\odot\varepsilon\in\Sigma$.

Theorem 4: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra and Σ be a weak ideal of $\mathbf{\Omega}$. Then Σ is an ideal of $\mathbf{\Omega}^{\otimes}$.

Proof: Let Σ be a weak ideal of $\mathbf{\Omega}$. For any $\varpi, \varrho \in \Sigma$, since $\varpi \sqcap \varrho^\star \preceq \varpi$ and $\varpi^\star \sqcap \varrho \preceq \varrho$ and Σ is a weak ideal of $\mathbf{\Omega}$, we have $\varpi \sqcap \varrho^\star \in \Sigma$ and $\varpi^\star \sqcap \varrho \in \Sigma$, it turns out that $\varpi \oplus \varrho = (\varpi \sqcap \varrho^\star) \sqcup (\varpi^\star \sqcap \varrho) \in \Sigma$. Meanwhile, we have $\ominus\varrho = \varrho \in \Sigma$, so $\varpi \ominus \varrho \in \Sigma$. Since $\varepsilon \odot \varpi = \varepsilon \sqcap \varpi = \varpi \odot \varepsilon \preceq \varpi$, we have $\varepsilon \odot \varpi \in \Sigma$ and $\varpi \odot \varepsilon \in \Sigma$. Hence Σ is an ideal of $\mathbf{\Omega}^{\otimes}$. ■

Theorem 5: Let $\mathbf{\Psi} = (\Psi; \oplus, \odot, \ominus, 0, 1)$ be a Boolean quasi-ring and Σ be an ideal of $\mathbf{\Psi}$. Then Σ is a weak ideal of $\mathbf{\Psi}^{\otimes}$.

Proof: Let Σ be an ideal of $\mathbf{\Psi}$. For any $\varpi \in \Sigma$, $0 = 0 \odot \varpi \in \Sigma$. If $\varpi, \varrho \in \Sigma$, then $\varpi \oplus \varrho \in \Sigma$ and $\varpi \odot \varrho \in \Sigma$, it follows that $\varpi \sqcup \varrho = \varpi \oplus \varrho \oplus \varpi \odot \varrho \in \Sigma$. Finally, if $\varpi \in \Sigma$ and $\varrho \in \Psi$ with $\varrho \preceq \varpi$, then $\varrho \sqcap \varrho = \varpi \sqcap \varrho = \varpi \odot \varrho \in \Sigma$. Hence Σ is a weak ideal of $\mathbf{\Psi}^{\otimes}$. ■

Corollary 3: Let $\mathbf{\Omega} = (\Omega; \sqcup, \sqcap, ^\star, 0, 1)$ be a quasi-Boolean algebra and $\mathbf{\Omega} = \mathbf{\Omega}^{\otimes\otimes}$. Then Σ is a weak ideal of $\mathbf{\Omega}$ if and only if Σ is an ideal of $\mathbf{\Omega}^{\otimes}$.

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