Exact Run Length Evaluation on Extended EWMA Control Chart for Seasonal Autoregressive Process

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Abstract— **Herein, we derive explicit formulas for the average run length (*ARL*) on an extended exponentially weighted moving average (EWMA) chart running a seasonal autoregressive process of order *p* ($SAR(p)_L$) with exponential white noise. The observations are from $SAR(p)_L$ process. The accuracy of the explicit formulas-derived *ARL* the numerical integral equation method. Although their accuracies were hardly different, the explicit formulas method required a much shorter CPU time to perform the calculation. Furthermore, the efficiency of the extended EWMA control chart was also compared with that of the conventional EWMA control chart utilizing the explicit formulas technique for the ARL. The results show that for a small shift size in the process mean, detection on the extended EWMA control chart was much earlier than on the standard EWMA control chart. Moreover, the proposed method was applied to the $SAR(p)_L$ process with real-world data running on the extended EWMA control chart to demonstrate its efficacy.**

Index Terms— **average run length, explicit formula, extended EWMA control chart, seasonal autoregressive process**

I. INTRODUCTION

STATISTICAL process control (SPC) is a significant and effective technique for monitoring and enhancing processes in many fields. Control charts, one of the SPC techniques, have been used in a variety of fields, including economics [1], business [2], and health and medical [3]. The Shewhart control chart, which can detect major changes in process parameters, was created by Shewhart [4]. Later, the cumulative sum (CUSUM) [5] and the exponentially weighted moving average (EWMA) control charts [6] became widely used for monitoring small-to-moderate shifts in the mean of a process since they have excellent performance.

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Moreover, Khan et al. [7] further extended the modified exponentially weighted moving average (modified EWMA) control chart, which was first proposed by Patel and Divecha [8] for processes where the observations are both autocorrelated and from an independently normal distribution. Naveed et al. [9] presented the extended exponentially weighted moving average (extended EWMA) control chart, which is capable of accurately detecting minute changes in a process parameter. Evaluating the capability of a control chart is accomplished by using a well-known measurement, the average run length (*ARL*), comprises two components: ARL_0, should be as high as feasible, is the mean number of times the observations fall in the control region before they fall out of the control region, and ARL_1, should be as low as feasible, refers to the mean number of times the observations fall out-of-control region control. Several approaches have been used to estimate the *ARL*, including Markov chain [10], Monte Carlo simulation [11], Martingale [12], and the numerical integral equation (NIE) method [13]. Many researchers have computed the *ARL* with explicit formulas and checked their accuracy by using these methods. For instance, Sukparungsee and Areepong [14] provided explicit formulas for the *ARL* on the EWMA control chart and compared the accuracy of the numerical results via Monte Carlo simulations. Suriyakat et al. [15],[16] derived the *ARL* on the EWMA control chart running the first-order autoregressive process with and without trend with exponential white noise distribution. Petcharat et al. [17] constructed explicit formulas for the *ARL* on the CUSUM control chart running the first-order moving average process and compared it with the NIE method. Later, Petcharat [18] derived the *ARL* based on explicit formulas on the EWMA control chart running a seasonal moving average process of order *q* ($SMA(q)_L$) with exponential white noise. Zhang and Busababodhin [19] derived explicit formulas for the *ARL* on the CUSUM control chart running an ARIMA process with exponential white noise. Peerajit et al. [20] compared the efficiency of *ARL* methods based on explicit formulas and the NIE method for the CUSUM control chart running a seasonal ARFIMA process. Next, Supharakonsakun et al. [21] proposed the exact solution for the *ARL* on the modified EWMA control chart running a MA(1) process. Phanthuna et al. [22] proposed explicit formulas for evaluating the *ARL* on a two-sided modified EWMA control chart running an AR(1) process with exponential white noise. In the same year, Phanthuna et al. [23] provided the analytical expression of the ARL on the modified EWMA control chart with the AR process containing exponential white noise. Recently, the ARL on explicit formulas on a modified EWMA control chart running a seasonal AR(p) process was recently provided by Phanthuna and Areepong [24]. To our knowledge, no one has ever published explicit formulas for the ARL on an extended EWMA control chart running a $SAR(p)_L$ process with white noise exponential distribution. However, there has been much previous research that used the time series model with seasonality for engineering analysis. The double SARIMA model for predicting electrical power consumption was proposed by Mado et al. [29]. The SARIMA model for forecasting water consumption was published by Oliveira et al. [30]. So we derive them here for the *ARL* on an extended EWMA control chart for both $SAR(1)_{12}$ and $SAR(2)_{12}$ processes. Additionally, for processes using both simulated and real-world data, the efficiency of the explicit formula solutions for the ARL on the extended and standard EWMA control charts is contrasted.

II. Materials and Methods

A. Exponentially Weighted Moving Average Control Chart

Robert [6] was the one who first suggested the exponentially weighted moving average (EWMA) control chart. It's suitable for detecting small process shifts. The following formula is used to calculate the EWMA statistic:

$$Z_t = (1-\lambda)Z_{t-1} + \lambda Y_t, t = 1,2,... \tag{1}$$

where Y_t is the process with the mean, the smoothing constant λ is that $0<\lambda\le 1$. The initial value of the EWMA statistic Z_0 is equal to u. The control limits for the EWMA control chart are as follows:

$$UCL = \mu_0 + W_Z\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \text{ and } LCL = \mu_0 - W_Z\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \tag{2}$$

where μ_0 is the target mean, σ is the process standard deviation, and W_Z is width of the control limits. The stopping time is given by $\tau_Z = \inf\{t\ge 0: Z_t<h, Z_t>k\}$ and then k is UCL and h is LCL.

B. Extended Exponentially Weighted Moving Average Control Chart

The Extended Exponentially Weighted Moving Average (extended EWMA or EEWMA) control chart was suggested by Naveed et al. [9]. It is developed from the EWMA control chart. The EEWMA statistic is calculated as:

$$E_t = \lambda_1 Y_t - \lambda_2 Y_{t-1} + (1-\lambda_1+\lambda_2)E_{t-1}, t = 1,2,... \tag{3}$$

where the smoothing constant λ_1 and λ_2 are that ($0<\lambda_1\le 1$) and ($0\le\lambda_2<\lambda_1$), respectively. The initial value of the extended EWMA statistic E_0 is equal to u. The control limits for the extended EWMA control chart are as follows:

$$UCL = \mu_0 + W_E\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}} \text{ and}$$

$$LCL = \mu_0 - W_E\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}} \tag{4}$$

where W_E is width of the control limits. The stopping time is provided by $\tau_E = \inf\{t\ge 0: E_t<a, E_t>b\}$ and then b is UCL and a is LCL.

III. Exact Solutions of *ARL* on the Extended EWMA Control Chart

When a random variable sequence is used, Y_t is the observation with $SAR(p)_L$ process. A seasonal AR(p) or $SAR(p)_L$ process can be expressed as follows:

$$Y_t = \eta + \phi_1 Y_{t-L} + \phi_2 Y_{t-2L} + ... + \phi_p Y_{t-pL} + \varepsilon_t \text{ or}$$

$$Y_t = \eta + \sum_{i=1}^{p}\phi_i Y_{t-iL} + \varepsilon_t \tag{5}$$

where η is an appropriate constant, ϕ_i is an autoregressive coefficient of autoregressive at $i=1,2,...,p$ or $|\phi_p|<1$, L is a seasonal period time, and ε_t is white noise sequence of exponential ($\varepsilon_t \sim Exp(\alpha)$). The extended EWMA statistic based on $SAR(p)_L$ process can be expression as:

$$E_1 = \lambda_1(\eta + \sum_{i=1}^{p}\phi_i Y_{1-iL} + \varepsilon_1) - \lambda_2 Y_{1-L} + (1-\lambda_1+\lambda_2)E_0$$

$$E_1 = \lambda_1\eta + (1-\lambda_1+\lambda_2)E_0 + (\lambda_1\phi_1-\lambda_2)Y_{1-L}$$

$$+\lambda_1\sum_{i=2}^{p}\phi_i Y_{1-iL} + \lambda_1\varepsilon_1$$

When Y_{t-L} is a initial constant of Y_t observations and $E_0 = u$ is an initial value in the process mean.

The error term ε_1 can be reformed as an in-control process and refers to E_1 that is written as:

$$\frac{a-(1-\lambda_1+\lambda_2)u-(\lambda_1\phi_1-\lambda_2)Y_{1-L}}{\lambda_1} - \sum_{i=2}^{p}\phi_i Y_{1-iL} - \eta < \varepsilon_1$$
$$< \frac{b-(1-\lambda_1+\lambda_2)u-(\lambda_1\phi_1-\lambda_2)Y_{1-L}}{\lambda_1} - \sum_{i=2}^{p}\phi_i Y_{1-iL} - \eta \tag{6}$$

On the extended EWMA control chart, the ARL for the SAR(p)L process is determined by an initial value as follows.

$$ARL = ARL_E(u) = E_\infty(\tau_E) \ge T \tag{7}$$

Let $ARL_E(u)$ denote the *ARL* on the extended EWMA control chart for $SAR(p)_L$ process. The Fredholm integral equation of the second kind can be used to construct the function ARL [25]. ARL is characterized as follows:

$$ARL_E(u) = 1 + \int ARL_E(E_1) f(\varepsilon_1) d\varepsilon_1 \tag{8}$$

If

$$s = (1-\lambda_1+\lambda_2)u + (\lambda_1\phi_1-\lambda_2)Y_{1-L} + \lambda_1\sum_{i=2}^{p}\phi_i Y_{1-iL} + \lambda_1\eta + \lambda_1\varepsilon_1$$

The function $ARL_E(u)$ is rearranged as follows when the integration variable is changed:

$$ARL_E(u) = 1 + \frac{1}{\lambda_1}\int_a^b ARL_E(s) f\left(\begin{array}{c}\frac{s-(1-\lambda_1+\lambda_2)u-(\lambda_1\phi_1-\lambda_2)Y_{1-L}}{\lambda_1} \\ -\sum_{i=2}^{p}\phi_i Y_{1-iL} - \eta\end{array}\right) ds \tag{9}$$

when $\varepsilon_1 \sim Exp(\alpha)$, then

$$1+\frac{1}{\lambda_1\alpha}\cdot e^{\left(\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha}+\frac{(\lambda_1\phi_1-\lambda_2)Y_{1-L}}{\lambda_1\alpha}+\frac{\sum_{i=2}^{p}\phi_i Y_{1-iL}+\eta}{\alpha}\right)}\int_a^b ARL_E(s)e^{-\frac{s}{\lambda_1\alpha}}ds \qquad (10)$$

In next step, Equation (9) is demonstrated to be accurate by using Banach's fixed point theorem [26] to confirm the consistency and uniqueness of the ARL solution on the extended EWMA control chart for the SAR(p)L process. The procedure is presented as follows.

By using Banach's fixed-point theorem, the *ARL* solution demonstrates that the integral equation for explicit formulas exists only once. Assume that T belongs to the class of all continuous functions.

$$T(ARL_E(u))=1+\frac{1}{\lambda_1}\int_a^b ARL_E(s)f\left(\frac{s-(1-\lambda_1+\lambda_2)u-(\lambda_1\phi_1-\lambda_2)Y_{1-L}}{\lambda_1}-\sum_{i=2}^{p}\phi_i Y_{1-iL}-\eta\right)ds \qquad (11)$$

A single solution exists for the fixed-point problem $T(ARL_E(u))=ARL_E(u)$ if operator T is a contraction. Theorem can be applied as shown in the examples below to prove that (11) exists and has a unique solution.

Theorem 1 Banach's Fixed-point Theorem: Let's assume that Y represents a complete metric space and $T:Y\to Y$ is a mapping of contractions with $r\in[0,1)$ being a continuous contraction so that $\|T(ARL_{E1})-T(ARL_{E2})\|\le r\|ARL_{E1}-ARL_{E2}\|$, $\forall ARL_{E1},ARL_{E2}\in Y$.

Then an exclusive $ARL_E(\cdot)\in Y$ exists such that $T(ARL_E(u))=ARL_E(u)$, i.e., a unique fixed-point in Y.

Proof: Let *T*, as specified in (13) be a mapping for contractions for $ARL_{E1},ARL_{E2}\in u[a,b]$.

$$\|T(ARL_{E1})-T(ARL_{E2})\|_\infty=\sup_{u\in[a,b]}|ARL_{E1}(u)-ARL_{E2}(u)|$$

$$=\sup_{u\in[a,b]}\left|\frac{1}{\lambda_1\alpha}e^{\left\{\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\phi_1-\lambda_2)Y_{1-L}}{\lambda_1\alpha}+\frac{\sum_{i=2}^{p}\phi_i Y_{1-iL}+\eta}{\alpha}\right\}}\int_a^b (ARL_{E1}(s)-ARL_{E2}(s))e^{-\frac{s}{\lambda_1\alpha}}ds\right|$$

$$\le\sup_{u\in[a,b]}\left|\|ARL_{E1}-ARL_{E2}\|\frac{1}{\lambda_1\alpha}\cdot(-\lambda_1\alpha)(e^{-\frac{b}{\lambda_1\alpha}}-e^{-\frac{a}{\lambda_1\alpha}})e^{\left\{\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\phi_1-\lambda_2)Y_{1-L}}{\lambda_1\alpha}+\frac{\sum_{i=2}^{p}\phi_i Y_{1-iL}+\eta}{\alpha}\right\}}\right|$$

$$=\|ARL_{E1}-ARL_{E2}\|_\infty\sup_{u\in[a,b]}\left|e^{\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\phi_1-\lambda_2)Y_{1-L}}{\lambda_1\alpha}+\frac{\sum_{i=2}^{p}\phi_i Y_{1-iL}+\eta}{\alpha}}\right|\left|e^{-\frac{a}{\lambda_1\alpha}}-e^{-\frac{b}{\lambda_1\alpha}}\right|$$

$$\le r\|ARL_{E1}-ARL_{E2}\|_\infty$$

A. The Explicit Formulas of the ARL

After checking the uniqueness of the *ARL*, Equation (9) can be changed by adding different variables with the following values that are shown as below:

$$ARL_E(u)=1+\frac{Q(u)}{\lambda_1\alpha}\cdot H \qquad (12)$$

where

$$Q(u)=e^{\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\phi_1-\lambda_2)Y_{1-L}}{\lambda_1\alpha}+\frac{\sum_{i=2}^{p}\phi_i Y_{1-iL}+\eta}{\alpha}},$$

$$H=\int_a^b ARL_E(s)e^{-\frac{s}{\lambda_1\alpha}}ds$$

Taking the constant H and turning $ARL_E(s)$ with (12),

$$H=\frac{-\lambda_1\alpha(e^{-\frac{b}{\lambda_1\alpha}}-e^{-\frac{a}{\lambda_1\alpha}})}{1+\frac{1}{\lambda_1-\lambda_2}\cdot\left\{e^{\frac{(\lambda_1\phi_1-\lambda_2)Y_{1-L}}{\lambda_1\alpha}+\frac{\sum_{i=2}^{p}\phi_i Y_{1-iL}+\eta}{\alpha}}\cdot(e^{-\frac{(\lambda_1-\lambda_2)b}{\lambda_1\alpha}}-e^{-\frac{(\lambda_1-\lambda_2)a}{\lambda_1\alpha}})\right\}} \qquad (13)$$

Substituting constant H from (13) into (12), then $ARL_E(u)$ can be rearranged follows (14) as

$$ARL_E(u)=1-\frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha}}\cdot(e^{-\frac{b}{\lambda_1\alpha}}-e^{-\frac{a}{\lambda_1\alpha}})}{(\lambda_1-\lambda_2)e^{-\left\{\frac{(\lambda_1\phi_1-\lambda_2)Y_{1-L}}{\lambda_1\alpha}+\frac{\sum_{i=2}^{p}\phi_i Y_{1-iL}+\eta}{\alpha}\right\}}+(e^{-\frac{(\lambda_1-\lambda_2)b}{\lambda_1\alpha}}-e^{-\frac{(\lambda_1-\lambda_2)a}{\lambda_1\alpha}})} \qquad (14)$$

Finally, the solution to (14) is an explicit formula of ARL for the $SAR(p)_L$ process on the extended EWMA control chart.

B. The NIE Method of the ARL

The composite midpoint quadrature rule [13] on the interval $[a,b]$ is used to estimate the *ARL* of the NIE technique ($ARL_N(u)$) based on (9). So, the solution of the numerical integral equation can be explained as

$$ARL_N(u) = 1 + \frac{1}{\lambda_1} \sum_{j=1}^{m} w_j ARL_E(s_j) f\left(\frac{s_j - (1-\lambda_1+\lambda_2)u - (\lambda_1\phi_1 - \lambda_2)Y_{1-L}}{\lambda_1} - \sum_{i=2}^{p} \phi_i Y_{1-iL} - \eta \right) \quad (15)$$

where s_j is a set of the division point on the interval $[a,b]$ as $s_j = \left(j - \frac{1}{2}\right) w_j + a, j = 1, 2, ..., m$ and w_j represents a weight of the composite midpoint formula $w_j = \frac{b-a}{m}$.

IV. Numerical Results

The *ARL* solutions, which are the NIE method with m=500 and the explicit formula, are compared on the extended EWMA control chart by using computation time (CPU time). For the $SAR(p)_L$ process, the initial parameter values are examined at ARL_0 = 370 on the extended EWMA control chart. These are the $SAR(1)_{12}$ and the $SAR(2)_{12}$ processes. $\alpha = \alpha_0$ with shift size ($\delta = 0$) was the parameter value that was supplied to the in-control procedure. On the other hand, the parameter values for the out-of-control process were displayed as $\alpha_1 = (1+\delta)\alpha_0$ with shift sizes (δ) equal to 0.001, 0.002, 0.003, 0.005, 0.010, 0.030, 0.050, 0.100, 0.500, and 1.000. Following is a synopsis of the process:

Step 1: Input parameters $\lambda_1, \lambda_2, \phi_i, \eta, Y_{1-iL}, \alpha_0$.

Step 2: In order to calculate UCL, determine the beginning values ARL_0 = 370.

Step 3: Calculate UCL using the NIE method (15) or the explicit formula (14).

Step 4: From the UCL solution in Step 3, compute ARL_1 by shifting mean $\alpha_1 = (1+\delta)\alpha_0$.

Additionally, the CPU time (PC System: windows10, 64-bit, Intel® Core™ i5-8250U 1.60 GHz 1.80 GHz, RAM 4 GB) was also supplied to compute the speed test results in seconds. MATHEMATICA© was used to compute the analytical outcomes.

In Table. I and Table. II, the explicit formula and the NIE method with varie λ_1 and ϕ_i for the $SAR(1)_{12}$ and $SAR(2)_{12}$ processes, respectively, are used to compute the *ARL* in case of the extended EWMA control chart at $a = 0$. The findings demonstrate that, with an APRE (%) of less than 0.000014%, the analytical findings are consistent with the NIE approximations. The explicit formulas take very little CPU time compared to the 2-3 seconds required by the NIE method.

V. Performance Comparing the *ARL* Results

The absolute percentage relative error (APRE), which can be determined as follows,

$$\text{APRE}(\%) = \frac{|ARL_E(u) - ARL_N(u)|}{ARL_E(u)} \times 100, \quad (16)$$

allows for a performance comparison of the ARL between the explicit formula and the NIE approach.

Moreover, the relative mean index (*RMI*) [28] is used to test the performance of the *ARL* on control charts under different conditions, which can be computed as

$$RMI = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{ARL_i(c) - ARL_i(s)}{ARL_i(s)} \right] \quad (17)$$

where $ARL_i(c)$ is the control chart's ARL for the shift size of row i and $ARL_i(s)$ is the control chart's lowest ARL overall. The *RMI* value for the control chart was the lowest. The results indicate that the control performed the best at detecting changes.

The efficiencies of the control charts were compared by calculating their *ARL*s for specific processes. The ARL results for a $SAR(1)_1$ process (Table III and Fig.1) and a $SAR(2)_{12}$ process (Table IV and Fig.2) show that the proposed method detected a process change more rapidly on the extended EWMA control chart than on the standard EWMA control chart for minute changes in the process mean. Moreover, the *RMI* for the extended EWMA control chart was lower than that for the standard EWMA control chart even when λ_2 was increased.

The outcomes show that, in descending order, the control charts' performances were the extended EWMA control chart with $\lambda_2 = 0.04$ (EEWMA04), $\lambda_2 = 0.03$ (EEWMA03), $\lambda_2 = 0.02$ (EEWMA02), or $\lambda_2 = 0.01$ (EEWMA01), and the standard EWMA control chart, respectively (Figs. 1 and 2). In addition, the outcome of using three different values of λ_1 (0.05, 0.10, or 0.20) for ARL_1 of the extended EWMA control chart shows that $\lambda_2 = 0.04$ provided the lowest λ_1 for detecting process shifts in $SAR(1)_{12}$ and $SAR(2)_{12}$ processes quickly (Fig. 3).

VI. Application for Real Data

On the extended EWMA control chart, the efficiency of the ARL was calculated using explicit formulas. The ARL_0 of 370 was determined with different λ_1 (0.05, 0.10, 0.20) and λ_2 (0.01, 0.02, 0.03, 0.04) values that were compared with that of the standard EWMA ($\lambda_2 = 0$) control chart. The percentages of Internet Explorer 8 and Firefox users in Thailand are determined using two real-world datasets. Those are observations made monthly from January 2013 to December 2019 and from January 2014 to December 2020, respectively. By examining the autocorrelation function (ACF) and partial autocorrelation function, it can be determined that this data represents a stationary time series (PACF). The researchers confirmed that an exponential distribution follows white noise.

For the $SAR(1)_{12}$ process, dataset 1 is the percentages of web browser users by Internet Explorer 8. The $SAR(1)_{12}$ process can be written as $Y_t = 4.639 + 0.829Y_{t-L} + \varepsilon_t$ where $\varepsilon_t \sim Exp(7.1064)$.

For the $SAR(2)_{12}$ process, dataset 2 is the percentages of web browser users by Firefox. The $SAR(2)_{12}$ process can be written as $Y_t = 7.999 + 0.985Y_{t-L} - 0.391Y_{t-2L} + \varepsilon_t$ where $\varepsilon_t \sim Exp(4.4307)$.

Dataset 1, the *ARL* is analyzed in Table. V, and then dataset 2 is evaluated in Table. VI. The results of two datasets appear in the results similar to simulated data. The extended EWMA control chart's outcomes enable faster detection of tiny shift sizes than the EWMA control chart and provide better performance when λ_2 is enlarged.

VII. DISCUSSIONS AND CONCLUSIONS

Using the ARL, control chart was evaluated. For both the $SAR(1)_{12}$ and $SAR(2)_{12}$ processes, the explicit formulas are a feasible alternative to be used to evaluate the *ARL*. Although the *ARL* results of the explicit formulas agree with the NIE method that has an APRE (%) of less than 0.000014%, the explicit formula is effective in reducing CPU time. For small shift sizes, the *ARL* using the explicit formulas on the extended EWMA control chart with different λ_2 values ($\lambda_2 = 0.01, 0.02, 0.03, 0.04$) performed better than on the standard EWMA control chart for both $SAR(1)_{12}$ and $SAR(2)_{12}$ processes. Moreover, the performance of the proposed method on an extended EWMA control chart improved when λ_2 was increased. Following that, different λ_1 values (0.05, 0.10, and 0.20) were used to test ARL_1 on the extended EWMA control chart was tested; the *ARL* results indicate its excellent performance for a low λ_1 value. The *RMI* was used to test the performance of the *ARL* on the two control charts for various λ_1 settings. Moreover, the explicit formulas method was applied to a $SAR(p)_L$ process with real data, the results of which are in agreement with those using simulated data.

TABLE I
ARL VALUES OF EXPLICIT FORMULA VALUES ON THE EXTENDED EWMA CONTROL CHART FOR THE $SAR(1)_{12}$ PROCESS COMPARED TO ARL VALUES THE NIE METHOD WITH
$\lambda_1 = 0.05, \lambda_2 = 0.01, \eta = 0, a = 0, ARL_0 = 370$

δ	$\phi_1 = 0.1 (b = 0.03390497)$			$\phi_1 = -0.1 (b = 0.01511539)$		
	Explicit	NIE (CPU time)	APRE(%)	Explicit	NIE (CPU time)	APRE(%)
0.000	370.009431	370.009380 (2.172)	0.000014	370.078681	370.078671 (2.234)	0.000003
0.001	218.573326	218.573301 (2.172)	0.000011	191.750044	191.750040 (2.218)	0.000002
0.002	155.286039	155.286023 (2.171)	0.000010	129.593802	129.593800 (2.344)	0.000002
0.003	120.532733	120.532722 (2.219)	0.000010	97.9806684	97.9806667 (2.375)	0.000002
0.005	83.4254265	83.4254191 (2.187)	0.000009	66.0026596	66.0026585 (2.218)	0.000002
0.010	47.4409181	47.4409142 (2.249)	0.000008	36.6098897	36.6098891 (2.265)	0.000002
0.030	17.9707248	17.9707236 (2.250)	0.000007	13.6411885	13.6411883 (2.219)	0.000001
0.050	11.4170395	11.4170387 (2.233)	0.000006	8.66321803	8.66321792 (2.313)	0.000001
0.100	6.33636188	6.33636154 (2.219)	0.000005	4.83914687	4.83914682 (2.234)	0.000001
0.500	2.14394338	2.14394334 (2.344)	0.000002	1.73422155	1.73422154 (2.281)	0.000000
1.000	1.59662030	1.59662029 (2.234)	0.000001	1.35237320	1.35237320 (2.234)	0.000000

TABLE II
ARL VALUES OF EXPLICIT FORMULA VALUES ON THE EXTENDED EWMA CONTROL CHART FOR THE $SAR(1)_{12}$ PROCESS COMPARED TO ARL VALUES THE NIE METHOD WITH
$\lambda_1 = 0.05, \lambda_2 = 0.01, \eta = 0, a = 0, ARL_0 = 370$

δ	$\phi_1 = \phi_2 = 0.1 (b = 0.03322422)$			$\phi_1 = 0.1, \phi_2 = -0.1 (b = 0.03459988)$		
	Explicit	NIE (CPU time)	APRE(%)	Explicit	NIE (CPU time)	APRE(%)
0.000	370.035004	370.034955 (2.328)	0.000013	370.051674	370.051621 (2.375)	0.000014
0.001	217.798894	217.798871 (2.422)	0.000011	219.380291	219.380265 (2.375)	0.000012
0.002	154.503209	154.503194 (2.344)	0.000010	156.092110	156.092093 (2.360)	0.000011
0.003	119.826529	119.826519 (2.359)	0.000009	121.257483	121.257471 (2.391)	0.000010
0.005	82.8644127	82.8644057 (2.375)	0.000008	83.9998818	83.9998740 (2.344)	0.000009
0.010	47.0830573	47.0830537 (2.468)	0.000008	47.8068607	47.8068567 (2.374)	0.000008
0.030	17.8249041	17.8249029 (2.391)	0.000007	18.1197143	18.1197129 (2.328)	0.000007
0.050	11.3239945	11.3239938 (2.390)	0.000006	11.5120773	11.5120765 (2.375)	0.000007
0.100	6.28572724	6.28572691 (2.407)	0.000005	6.38805435	6.38805399 (2.437)	0.000006
0.500	2.13006251	2.13006247 (2.452)	0.000002	2.15809279	2.15809275 (2.438)	0.000002
1.000	1.58826827	1.58826826 (2.501)	0.000001	1.60513193	1.60513192 (2.390)	0.000001

TABLE III
COMPARING ARL VALUES ON THE EWMA AND THE EXTENDED EWMA CONTROL CHARTS FOR THE SAR(1)$_{12}$ PROCESS GIVEN $\eta = 0.2, \phi_1 = 0.2, a = 0.05, ARL_0 = 370$

λ_1	δ	EWMA ($\lambda_2 = 0$)	Extended EWMA $\lambda_2 = 0.01$	$\lambda_2 = 0.02$	$\lambda_2 = 0.03$	$\lambda_2 = 0.04$
		$b = 0.15062911$	$b = 0.09331915$	$b = 0.0690463$	$b = 0.05843824$	$b = 0.053748771$
0.05	0.000	370	370	370	370	370
	0.001	227.127	142.067	113.340	94.108	79.500
	0.002	164.075	88.285	67.301	54.292	44.924
	0.003	128.561	64.238	48.053	38.340	31.495
	0.005	89.922	41.829	30.798	24.370	19.926
	0.010	51.720	22.741	16.592	13.095	10.716
	0.030	19.886	8.731	6.440	5.160	4.301
	0.050	12.740	5.791	4.338	3.529	2.990
	0.100	7.188	3.551	2.743	2.295	2.000
	0.500	2.602	1.706	1.434	1.287	1.197
	1.000	1.980	1.441	1.251	1.152	1.095
RMI (λ_1)		0	0	0	0	0
RMI (λ_2)		2.399	0.772	0.382	0.155	0
		$b = 0.26245334$	$b = 0.18557995$	$b = 0.13815057$	$b = 0.10791962$	$b = 0.08829093$
0.10	0.000	370	370	370	370	370
	0.001	282.584	218.774	189.741	170.921	156.257
	0.002	228.624	155.542	127.864	111.414	99.340
	0.003	192.014	120.813	96.579	82.802	72.975
	0.005	145.518	83.726	65.059	54.911	47.872
	0.010	90.829	47.757	36.199	30.160	26.072
	0.030	36.718	18.295	13.716	11.378	9.820
	0.050	23.293	11.740	8.849	7.368	6.382
	0.100	12.521	6.655	5.105	4.297	3.757
	0.500	3.530	2.420	2.007	1.770	1.609
	1.000	2.389	1.830	1.578	1.427	1.324
RMI (λ_1)		0.499	0.725	0.779	0.851	0.925
RMI (λ_2)		1.635	0.597	0.287	0.117	0
		$b = 0.532407$	$b = 0.420767$	$b = 0.339926$	$b = 0.2793545$	$b = 0.2329197$
0.20	0.000	370	370	370	370	370
	0.001	330.198	279.850	251.107	231.853	217.607
	0.002	297.995	225.113	190.182	169.017	154.332
	0.003	271.429	188.353	153.157	133.101	119.695
	0.005	230.172	142.096	110.402	93.586	82.798
	0.010	166.244	88.296	65.358	54.061	47.100
	0.030	77.193	35.686	25.508	20.767	17.925
	0.050	49.403	22.716	16.251	13.245	11.443
	0.100	25.303	12.310	8.968	7.380	6.417
	0.500	5.183	3.519	2.861	2.493	2.248
	1.000	3.029	2.362	2.032	1.828	1.685
RMI (λ_1)		1.451	1.810	1.850	1.958	2.104
RMI (λ_2)		1.700	0.615	0.282	0.110	0

TABLE IV
COMPARING ARL VALUES ON THE EWMA AND THE EXTENDED EWMA CONTROL CHARTS FOR THE $SAR(2)_{12}$ PROCESS GIVEN $\eta = 0.2, \phi_1 = 0.2, \phi_2 = -0.2, a = 0.05, ARL_0 = 370$

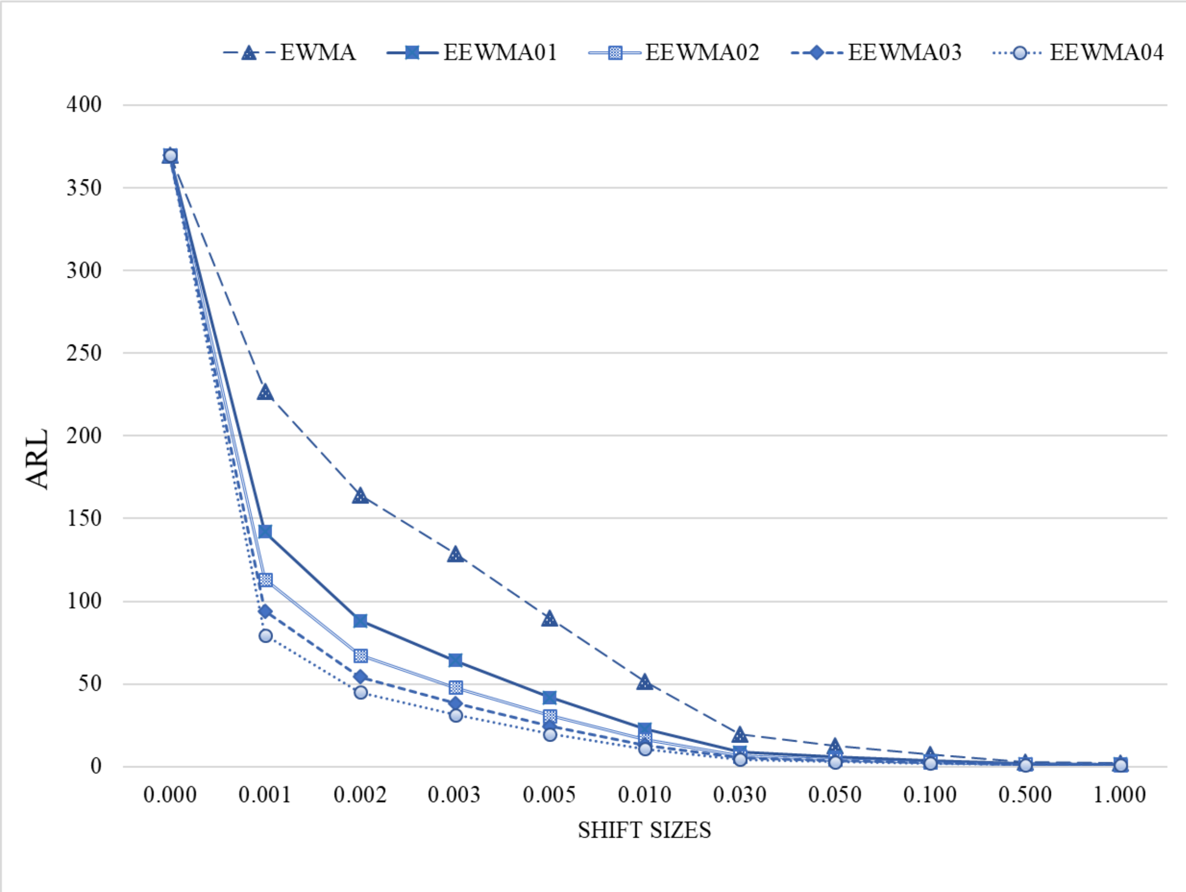
λ_1	δ	EWMA ($\lambda_2 = 0$)	Extended EWMA $\lambda_2 = 0.01$	Extended EWMA $\lambda_2 = 0.02$	Extended EWMA $\lambda_2 = 0.03$	Extended EWMA $\lambda_2 = 0.04$
		$b = 0.1549604$	$b = 0.0951198$	$b = 0.06982836$	$b = 0.05878324$	$b = 0.05390183$
0.05	0.000	370	370	370	370	370
	0.001	237.881	143.966	114.416	94.884	80.119
	0.002	175.426	89.740	68.050	54.806	45.308
	0.003	139.064	65.384	48.619	38.720	31.772
	0.005	98.485	42.625	31.177	24.619	20.104
	0.010	57.286	23.192	16.800	13.229	10.810
	0.030	22.104	8.900	6.517	5.209	4.335
	0.050	14.101	5.898	4.387	3.560	3.012
	0.100	7.860	3.610	2.770	2.312	2.011
	0.500	2.720	1.725	1.443	1.293	1.201
	1.000	2.038	1.454	1.257	1.156	1.097
RMI (λ_1)		0	0	0	0	0
RMI (λ_2)		2.657	0.788	0.386	0.157	0
		$b = 0.272166$	$b = 0.1914835$	$b = 0.1418852$	$b = 0.1103348$	$b = 0.088987285$
0.10	0.000	370	370	370	370	370
	0.001	293.571	222.605	191.947	172.474	157.520
	0.002	243.241	159.417	129.833	112.724	100.355
	0.003	207.657	124.315	98.250	83.881	73.792
	0.005	160.679	86.514	66.307	55.694	48.452
	0.010	102.700	49.535	36.950	30.619	26.405
	0.030	42.228	19.011	14.010	11.555	9.946
	0.050	26.734	12.192	9.035	7.480	6.462
	0.100	14.184	6.894	5.206	4.358	3.801
	0.500	3.759	2.478	2.036	1.788	1.622
	1.000	2.485	1.863	1.596	1.439	1.332
RMI (λ_1)		0.523	0.747	0.791	0.859	0.932
RMI (λ_2)		1.873	0.626	0.295	0.119	0
		$b = 0.5580535$	$b = 0.439007$	$b = 0.3535373$	$b = 0.2897414$	$b = 0.240989$
0.20	0.000	370	370	370	370	370
	0.001	344.289	286.997	255.474	234.960	219.966
	0.002	321.680	234.473	195.239	172.314	156.711
	0.003	301.696	198.253	158.092	136.163	121.840
	0.005	267.975	151.552	114.678	96.097	84.501
	0.010	208.142	95.594	68.325	55.711	48.185
	0.030	105.336	39.091	26.783	21.450	18.365
	0.050	67.826	24.883	17.060	13.678	11.722
	0.100	33.610	13.411	9.391	7.611	6.567
	0.500	5.841	3.707	2.953	2.549	2.288
	1.000	3.241	2.449	2.081	1.861	1.709
RMI (λ_1)		1.763	1.952	1.926	2.008	2.142
RMI (λ_2)		2.243	0.687	0.303	0.116	0

TABLE V
COMPARING ARL VALUES ON THE EWMA AND THE EXTENDED EWMA CONTROL CHARTS FOR THE $SAR(1)_{12}$ PROCESS WITH THE PERCENTAGES OF WEB BROWSER USERS BY INTERNET EXPLORER 8 GIVEN
$\eta = 4.639, \phi_1 = 0.829, a = 0.05, ARL_0 = 370$

λ_1	δ	EWMA ($\lambda_2 = 0$)	Extended EWMA			
			$\lambda_2 = 0.01$	$\lambda_2 = 0.02$	$\lambda_2 = 0.03$	$\lambda_2 = 0.04$
		$b = 0.7133693$	$b = 0.5334708$	$b = 0.4048662$	$b = 0.3116668$	$b = 0.02435206$
0.05	0.000	370	370	370	370	370
	0.001	277.896	243.954	223.553	208.937	197.177
	0.002	222.601	182.129	160.357	145.783	134.617
	0.003	185.729	145.420	125.143	112.075	102.323
	0.005	139.627	103.816	87.136	76.817	69.323
	0.010	86.326	60.798	49.811	43.273	38.643
	0.030	34.780	23.588	19.029	16.381	14.537
	0.050	22.133	15.037	12.142	10.457	9.284
	0.100	12.003	8.333	6.789	5.877	5.238
	0.500	3.462	2.724	2.341	2.093	1.912
	1.000	2.339	1.960	1.739	1.588	1.477
RMI (λ_1)		0	0	0	0	0
RMI (λ_2)		0.872	0.424	0.219	0.092	0
		$b = 1.4449415$	$b = 1.2249445$	$b = 1.0446555$	$b = 0.8953217$	$b = 0.770601$
0.10	0.000	370	370	370	370	370
	0.001	292.680	270.247	254.064	241.725	231.785
	0.002	242.138	212.952	193.602	179.665	168.933
	0.003	206.522	175.784	156.487	143.073	133.014
	0.005	159.651	130.435	113.276	101.831	93.498
	0.010	101.929	79.498	67.259	59.442	53.915
	0.030	42.144	31.692	26.313	22.983	20.675
	0.050	26.837	20.171	16.749	14.631	13.164
	0.100	14.400	10.995	9.210	8.091	7.307
	0.500	3.860	3.259	2.887	2.629	2.434
	1.000	2.511	2.228	2.035	1.892	1.780
RMI (λ_1)		0.126	0.217	0.252	0.272	0.290
RMI (λ_2)		0.626	0.343	0.183	0.077	0
		$b = 3.18193$	$b = 2.873164$	$b = 2.605533$	$b = 2.371305$	$b = 2.164673$
0.20	0.000	370	370	370	370	370
	0.001	317.962	300.716	286.933	275.639	266.119
	0.002	278.689	253.297	234.387	219.709	207.901
	0.003	248.030	218.823	198.162	182.722	170.670
	0.005	203.260	172.053	151.469	136.820	125.801
	0.010	139.829	112.151	95.443	84.220	76.120
	0.030	62.022	47.254	39.057	33.821	30.169
	0.050	39.740	30.151	24.876	21.522	19.190
	0.100	20.904	16.107	13.417	11.685	10.471
	0.500	4.810	4.139	3.693	3.372	3.129
	1.000	2.897	2.625	2.425	2.271	2.149
RMI (λ_1)		0.432	0.585	0.639	0.667	0.689
RMI (λ_2)		0.586	0.336	0.182	0.077	0

TABLE VI
COMPARING ARL VALUES ON THE EWMA AND THE EXTENDED EWMA CONTROL CHARTS FOR THE $SAR(2)_{12}$ PROCESS WITH THE PERCENTAGES OF WEB BROWSER USERS BY FIREFOX GIVEN
$\eta = 7.999, \phi_1 = 0.985, \phi_2 = -0.391, a = 0.05, ARL_0 = 370$

λ_1	δ	EWMA	Extended EWMA			
		($\lambda_2 = 0$)	$\lambda_2 = 0.01$	$\lambda_2 = 0.02$	$\lambda_2 = 0.03$	$\lambda_2 = 0.04$
		$b = 0.4545152$	$b = 0.2954128$	$b = 0.2005925$	$b = 0.1429614$	$b = 0.1075663$
0.05	0.000	370	370	370	370	370
	0.001	269.450	223.307	199.200	181.775	167.158
	0.002	210.970	159.106	135.552	119.796	107.354
	0.003	174.121	124.327	103.422	89.980	79.667
	0.005	128.803	86.400	70.093	60.030	52.529
	0.010	78.532	49.516	39.269	33.176	28.751
	0.030	31.252	18.936	14.820	12.432	10.731
	0.050	19.894	12.103	9.488	7.971	6.895
	0.100	10.859	6.793	5.377	4.549	3.964
	0.500	3.264	2.377	1.988	1.750	1.585
	1.000	2.253	1.773	1.535	1.387	1.287
RMI (λ_1)		0	0	0	0	0
RMI (λ_2)		1.209	0.532	0.274	0.116	0
		$b = 0.8995705$	$b = 0.700295$	$b = 0.552112$	$b = 0.4400333$	$b = 0.3542569$
0.10	0.000	370	370	370	370	370
	0.001	287.161	256.127	236.411	222.132	210.796
	0.002	233.734	194.954	172.835	157.885	146.589
	0.003	197.811	158.146	136.968	123.201	113.074
	0.005	150.992	114.601	96.647	85.475	77.493
	0.010	95.312	68.387	56.167	48.881	43.817
	0.030	38.954	26.798	21.616	18.615	16.566
	0.050	24.794	17.061	13.769	11.861	10.558
	0.100	13.359	9.381	7.640	6.617	5.912
	0.500	3.691	2.936	2.533	2.270	2.079
	1.000	2.438	2.066	1.842	1.688	1.573
RMI (λ_1)		0.148	0.269	0.309	0.342	0.377
RMI (λ_2)		0.829	0.412	0.212	0.088	0
		$b = 1.951367$	$b = 1.675326$	$b = 1.4495531$	$b = 1.2619535$	$b = 1.104174$
0.20	0.000	370	370	370	370	370
	0.001	312.738	289.158	272.059	258.6440	247.947
	0.002	269.982	236.418	214.130	197.901	185.539
	0.003	238.143	200.689	177.310	161.038	148.999
	0.005	192.268	153.795	131.686	117.118	106.712
	0.010	130.039	97.537	80.630	70.135	62.912
	0.030	56.587	40.042	32.176	27.523	24.412
	0.050	36.188	25.507	20.470	17.503	15.521
	0.100	19.129	13.739	11.138	9.586	8.540
	0.500	4.575	3.752	3.267	2.942	2.705
	1.000	2.806	2.453	2.220	2.052	1.923
RMI (λ_1)		0.454	0.668	0.735	0.782	0.835
RMI (λ_2)		0.748	0.394	0.205	0.085	0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EEWMA03
EEWMA04
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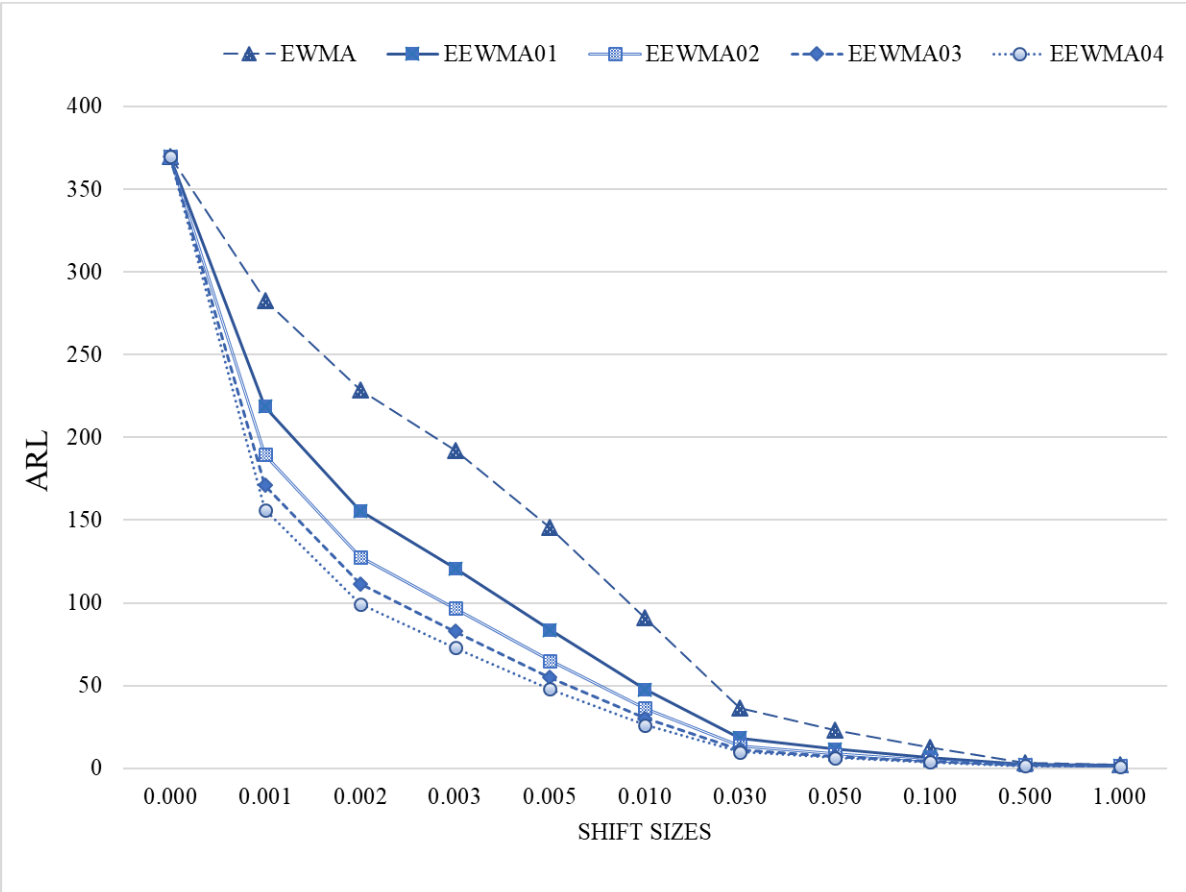
(a)


EWMA
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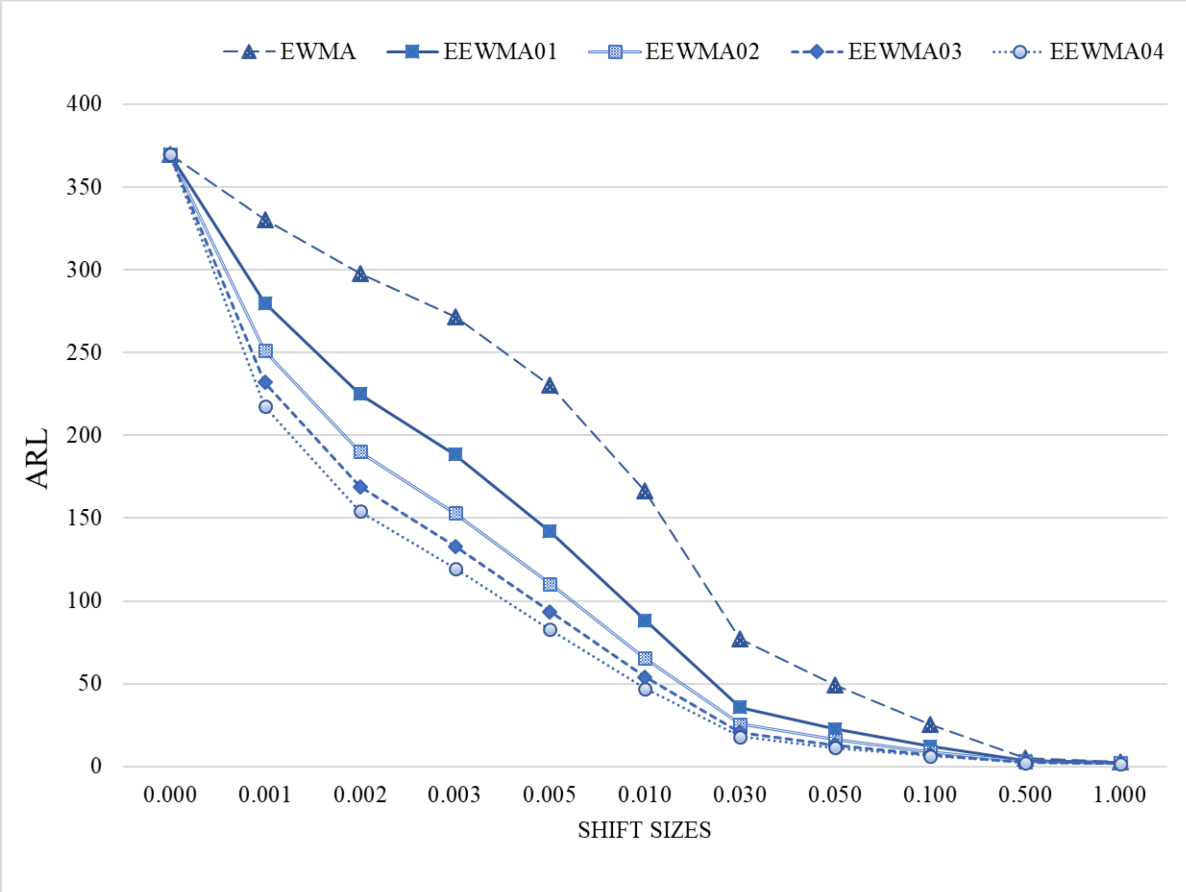
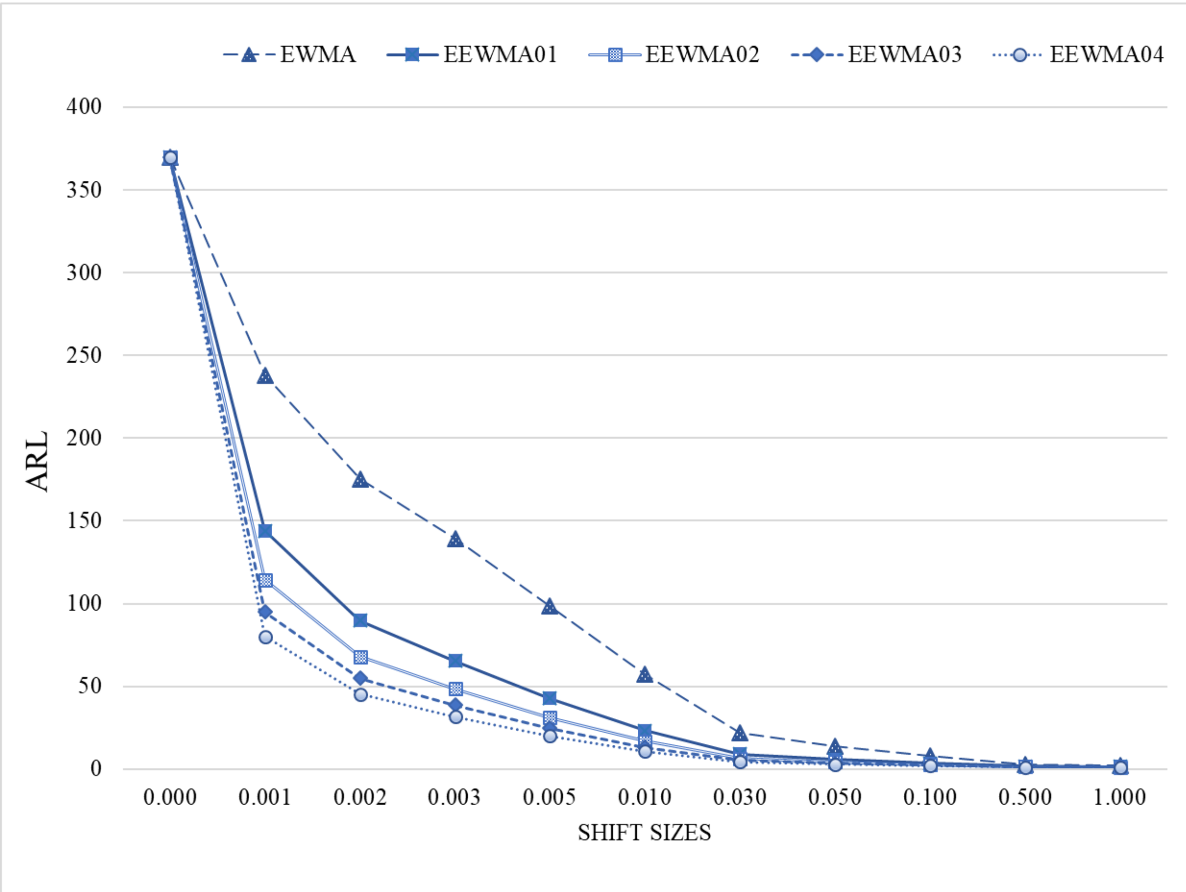
SHIFT SIZES

(b)

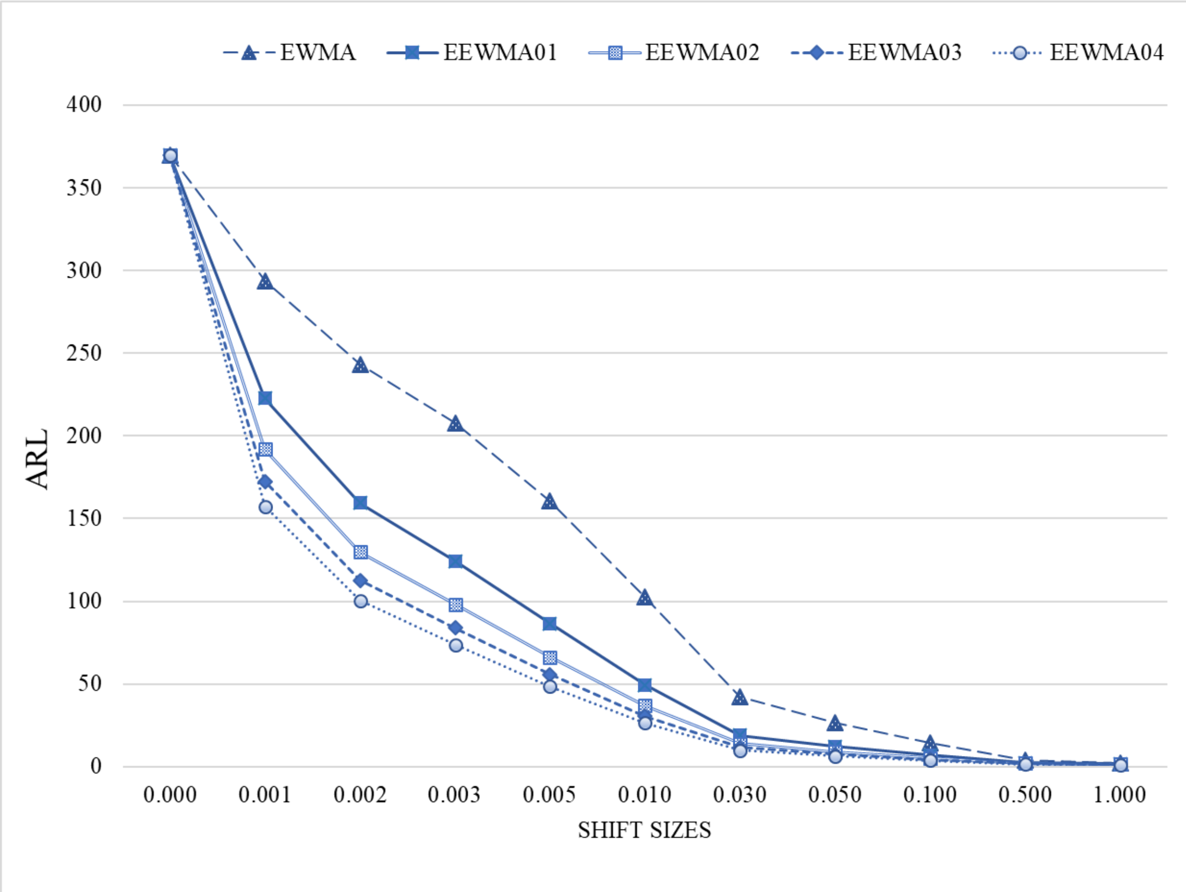


(c)

Fig. 1. *ARL* values on the EWMA and the extended EWMA control charts for the $SAR(1)_{12}$ process with ARL_0 = 370 and (a) : $\lambda_1 = 0.05$, (b) : $\lambda_1 = 0.10$, (c) : $\lambda_1 = 0.20$.



(a)



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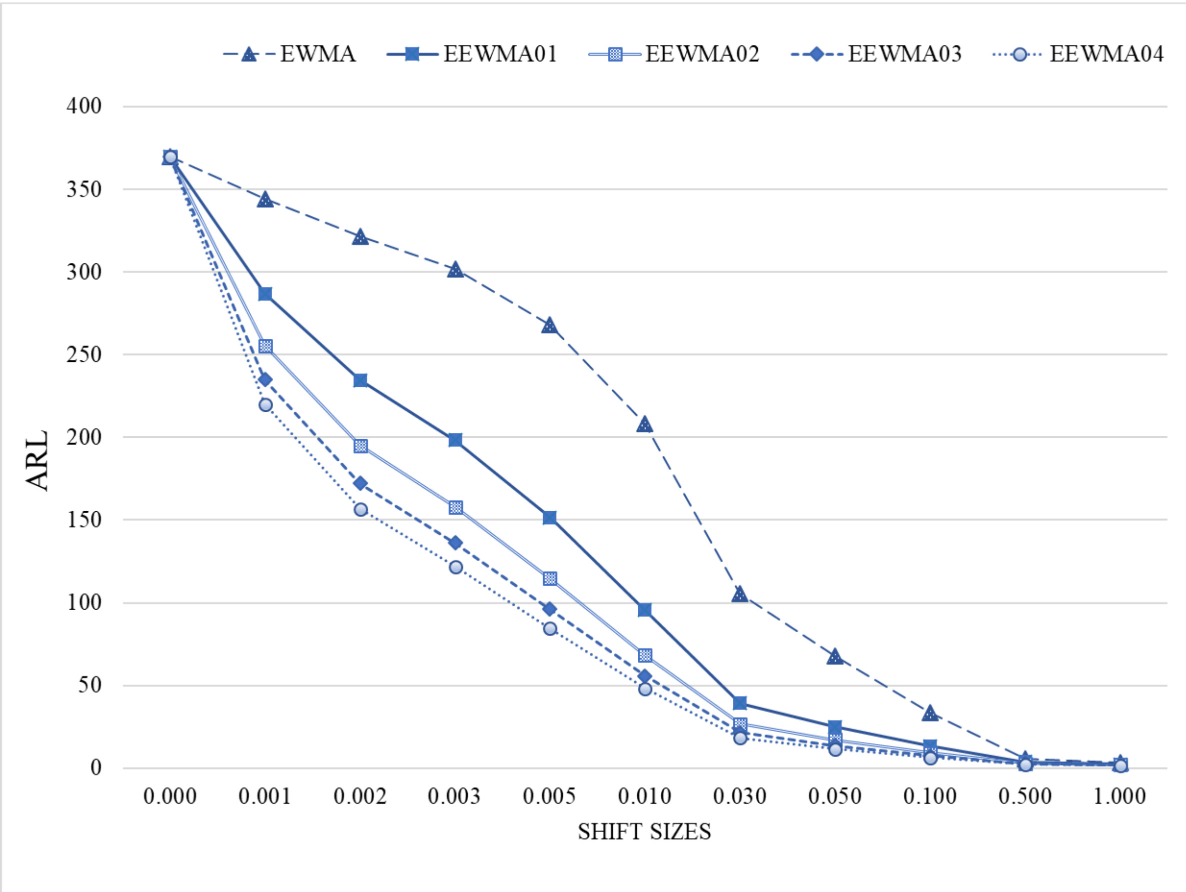
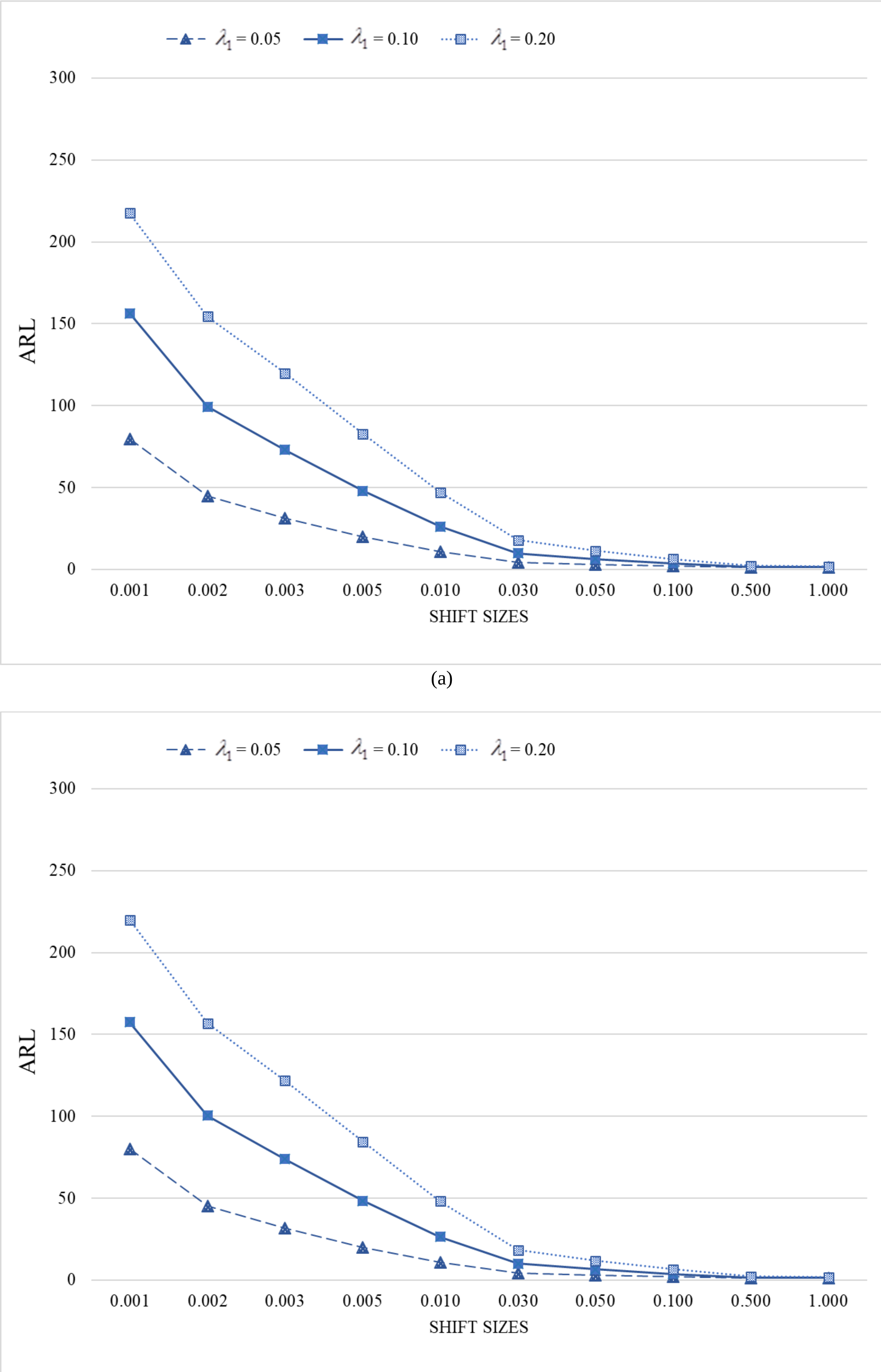


(c)

Fig. 2. *ARL* values on the EWMA and the extended EWMA control charts for the SAR(2)$_{12}$ process with ARL_0 = 370 and (a) : $\lambda_1 = 0.05$, (b) : $\lambda_1 = 0.10$, (c) : $\lambda_1 = 0.20$.



(a)

(b)

Fig. 3. *ARL* values on the EWMA and the extended EWMA control charts ($\lambda_2 = 0.04$) with ARL_0 = 370 for the SAR(p)$_L$ processes (a) : SAR(1)$_{12}$, (b) : SAR(2)$_{12}$.

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