Parameter Estimation for Ornstein-Uhlenbeck Process with Small Fractional Lévy Noises

Fang Xu, Yongfei Zhao and Chao Wei

Abstract—**This paper is concerned with least squares estimation for Ornstein-Uhlenbeck process with small fractional Lévy noise from discrete observations. The contrast function is given to obtain the least squares estimators. The consistency and asymptotic distribution of estimators are derived when a small dispersion coefficient $\varepsilon \to 0$ and $n \to \infty$ simultaneously. Some simulations are made to verify the effectiveness of the estimators.**

Index Terms—**least squares estimation; Ornstein-Uhlenbeck process; small fractional Lévy noises; consistency; asymptotic distribution.**

I. Introduction

In the real world, almost all systems are affected by noise and exhibit certain random characteristics. Therefore, it is reasonable and interesting to use random systems to model actual systems. When modeling or optimizing a stochastic system, due to the complexity of the internal structure and the uncertainty of the external environment, parameters of the system are unknown. It is necessary to use theoretical tools to estimate the parameters of the system. In the past few decades, some authors studied the parameter estimation problem for stochastic models ([7], [11], [28], [29]). For example, Barczy and Pap ([1]) analyzed the consistency of the maximum likelihood estimators for nonlinear time inhomogeneous stochastic process, and provided sufficient conditions for the estimation error to satisfy the asymptotic normality. Deck ([3]) and Kan ([9]) used Bayes method to investigate the parameter estimation for linear stochastic system. Wei and Shu ([24]) studied the existence, consistency and asymptotic normality of the maximum likelihood estimator for the nonlinear stochastic differential equation. When the system is observed discretely, Guy et al. ([4])studied the weak convergence of the minimal contrast estimator for multidimensional stochastic differential equations with small diffusion coefficients. Uchida and Yoshida ([22]) constructed a comparison function through a local Gaussian approximation of the transfer density and discussed the convergence rate of the estimator. When the system is observed partially, Singer ([21]) used extended Kalman filtering and local linear filtering to study the state and parameter estimation of nonlinear continuous-discrete time state models. Wei ([25])analyzed state and parameter estimation for nonlinear stochastic systems by extended Kalman filtering. The parameter estimation for diffusion processes with small noise is well developed as well ([15], [26], [27]). However, in practical applications, most of the system noise are non-Gaussian. Non-Gaussian noise can more accurately reflect the practical random perturbation. Therefore, fractional Lévy noise, as a kind of important non-Gaussian noise, has attracted many authors' attention ([2], [20]).

The parameter estimation of Ornstein-Uhlenbeck processes has been an extremely active topic of research recently, see e.g. ([23]) and references therein. For example, Hu and Long ([5], [6]) discussed parameter estimator problem for Ornstein-Uhlenbeck processes driven by Lévy motions. Mai ([17]) investigated efficient maximum likelihood estimation for Lévy-driven Ornstein-Uhlenbeck processes. Zhang and Zhang ([30]) studied a least squares estimator for discretely observed Ornstein-Uhlenbeck processes driven by symmetric α-stable motions. However, the Ornstein-Uhlenbeck process discussed in above literature is not driven by fractional Lévy noise and a common denominator in all these works is assumed that the equation admits only one unknown parameter. The fractional Lévy process has non stationary increments, the increments over non overlapping intervals are more weakly correlated and their covariance decays polynomially at a higher rate, which makes the fractional Lévy process a possible candidate for models involving long-range dependence, self-similarity and non-stationary. Since the fractional Lévy process is not a martingale, methods of stochastic analysis are more sophisticated. In this paper, we consider the parameter estimation problem for Ornstein-Uhlenbeck process with two unknown parameters driven by fractional Lévy noise from discrete observations. The contrast function is introduced to obtain the least squares estimators. The consistency and asymptotic distribution of the estimators are derived by Markov inequality, Cauchy-Schwarz inequality and Gronwall's inequality. Some numerical calculus and simulations are given to verify the effectiveness of estimators.

The paper is organized as follows. In Section 2, we give the contrast function to obtain the least squares estimators. In Section 3, we obtain the consistency and asymptotic distribution of the estimators. In Section 4, some simulation studies are provided. The conclusion is given in Section 5.

II. Problem Formulation and Preliminaries

Definition 1: ([19]) Let $L = (L(t))_{t\in\mathbb{R}}$ be a zero-mean two sided Lévy process with $E[L(1)^2] < \infty$ and without a Brownian component. For fractional integration parameter $d \in (0, \frac{1}{2})$, a stochastic process

$$L_t^d := \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} [(t-s)_+^d - (-s)_+^d] L(ds), \quad t \in \mathbb{R},$$

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is called a fractional Lévy process, where $x_+ = x \vee 0$.

In this paper, we study the parametric estimation problem for Ornstein-Uhlenbeck process with small fractional Lévy noises described by the following stochastic differential equation:

$$\begin{cases} dX_t =(\alpha - \beta X_t)dt + \varepsilon dL_t^d, \quad t \in [0,1], d \in (0, \frac{1}{2}) \\ X_0 = x_0, \end{cases} \tag{1}$$

where α and β are unknown parameters, L_t^d is a fractional Lévy process. Assume that this process is observed at n regularly spaced time points $\{t_i = \frac{i}{n}, i = 1, 2, \cdots, n\}$, $\varepsilon \in (0, 1]$.

Consider the following contrast function

$$\rho_{n,\varepsilon}(\alpha, \beta) = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}} - (\alpha - \beta X_{t_{i-1}})\Delta t_{i-1}|^2, \tag{2}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

It is easy to obtain the least square estimators

$$\begin{cases} \widehat{\alpha}_{n,\varepsilon} = \frac{n\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})X_{t_{i-1}} \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ \quad - \frac{n\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^n X_{t_{i-1}}^2}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ \widehat{\beta}_{n,\varepsilon} = \frac{n^2\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ \quad - \frac{n\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2}. \end{cases} \tag{3}$$

III. Main Results and Proofs

Let $X^0 = (X_t^0, t \geq 0)$ be the solution to the underlying ordinary differential equation under the true value of the parameters:

$$dX_t^0 = (\alpha - \beta X_t^0)dt, \quad X_0^0 = x_0. \tag{4}$$

Note that

$$X_{t_i} - X_{t_{i-1}} = \frac{1}{n}\alpha - \beta \int_{t_{i-1}}^{t_i} X_s ds + \varepsilon \int_{t_{i-1}}^{t_i} dL_s^d. \tag{5}$$

Then, we can give a more explicit decomposition for $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ as follows

$$\begin{aligned} \widehat{\alpha}_{n,\varepsilon} &= \alpha + \frac{n\beta \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \sum_{i=1}^n X_{t_{i-1}}^2}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n\beta \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{n\varepsilon \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dL_s^d \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} dL_s^d \sum_{i=1}^n X_{t_{i-1}}^2}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &= \alpha + \frac{\beta \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\beta \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{\varepsilon \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dL_s^d \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} dL_s^d \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}. \end{aligned}$$

$$\begin{aligned} \widehat{\beta}_{n,\varepsilon} &= \frac{n\beta \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n^2\beta \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{n^2\varepsilon \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dL_s^d}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} dL_s^d \sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &= \frac{\beta \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\beta \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{\varepsilon \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dL_s^d}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} dL_s^d \frac{1}{n} \sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}. \end{aligned}$$

Before giving the theorems, we need to establish some preliminary results.

Lemma 1: ([19]) Let $|f|, |g| \in H$, H is the completion of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with respect to the norm $||g||_H^2 = \mathbb{E}[L^2(1)] \int_R (I_{-g}^d)^2(u)du$. Then,

$$\begin{aligned} &\mathbb{E}[\int_R f(s)dL_s^d \int_R g(s)dL_s^d] \\ &= \frac{\Gamma(1-2d)\mathbb{E}[L^2(1)]}{\Gamma(d)\Gamma(1-d)} \int_R \int_R f(t)g(s)|t-s|^{2d-1} dsdt. \end{aligned}$$

Lemma 2: ([19]) For any $0 < b_2 \leq b_1 \leq a_1$, $0 < b_2 \leq a_2 \leq a_1$, and $b_1 - b_2 = a_1 - a_2$, there exists a constant C

only depends on r and d and satisfies

$$\begin{aligned}&|\int_{b_2}^{b_1}\int_{a_2}^{a_1} e^{r(u+v)}|u-v|^{2d-1}dudv| \\ &\leq \begin{cases} C|e^{r(a_1+b_1)}-e^{r(a_2+b_2)}||a_1-b_2|^{2d}, & if r\neq 0, \\ C|a_1-b_2|^{2d}, & if r=0,\end{cases}\end{aligned}$$

where r denotes a constant and d is the fractional integration parameter of fractional Lévy process.

Lemma 3: When $\varepsilon \to 0$, we have

$$\sup_{0\leq t\leq 1}|X_t - X_t^0| \xrightarrow{P} 0.$$

Proof: Observe that

$$X_t - X_t^0 = -\beta\int_0^t (X_s - X_s^0)ds + \varepsilon\int_0^t dL_s^d. \tag{6}$$

By using the Cauchy-Schwarz inequality, we find

$$\begin{aligned}&|X_t - X_t^0|^2 \\ \leq\ & 2\beta^2|\int_0^t (X_s - X_s^0)ds|^2 + 2\varepsilon^2|\int_0^t dL_s^d|^2 \\ \leq\ & 2\beta^2 t\int_0^t |X_s - X_s^0|^2 ds + 2\varepsilon^2 \sup_{0\leq t\leq 1}|\int_0^t dL_s^d|^2\end{aligned}$$

.

According to the Gronwall's inequality, we obtain

$$|X_t - X_t^0|^2 \leq 2\varepsilon^2 e^{2\beta^2 t^2}\sup_{0\leq t\leq 1}|\int_0^t dL_s^d|^2. \tag{7}$$

Then, it follows that

$$\sup_{0\leq t\leq 1}|X_t - X_t^0| \leq \sqrt{2}\varepsilon e^{\beta^2}\sup_{0\leq t\leq 1}|\int_0^t dL_s^d|. \tag{8}$$

By the Markov inequality, the results in Lemma 1 and Lemma 2 when $f(s) = g(s) = 1$, $r = 0$, for any given $\delta > 0$, when $\varepsilon \to 0$, we have

$$\begin{aligned}&\mathbb{P}(\sqrt{2}\varepsilon e^{\beta^2}\sup_{0\leq t\leq 1}|\int_0^t dL_s^d| > \delta) \\ &\leq \delta^{-2}2\varepsilon^2 e^{2\beta^2}\mathbb{E}[\sup_{0\leq t\leq 1}|\int_0^t dL_s^d|]^2 \\ &\leq \delta^{-2}2\varepsilon^2 e^{2\beta^2}\mathbb{E}[|\int_0^1 dL_s^d|]^2 \\ &\leq C\delta^{-2}2\varepsilon^2 e^{2\beta^2}\int_0^1\int_0^1 |t-s|^{2d-1}dsdt \\ &\leq C\delta^{-2}2\varepsilon^2 e^{2\beta^2} \\ &\to 0,\end{aligned}$$

where C is a constant.

Therefore, it is easy to check that

$$\sup_{0\leq t\leq 1}|X_t - X_t^0| \xrightarrow{P} 0. \tag{9}$$

The proof is complete. ■

Lemma 4: When $\varepsilon \to 0$ and $n \to \infty$, we have,

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt.$$

Proof: Since

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 = \frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^0)^2 + \frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^2 - (X_{t_{i-1}}^0)^2). \tag{10}$$

It is clear that

$$\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^0)^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt. \tag{11}$$

For $\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^2 - (X_{t_{i-1}}^0)^2)$, according to Lemma 3 and the fact that $\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}^0| \xrightarrow{P} \int_0^1 |X_t^0|dt$, When $\varepsilon \to 0$ and $n \to \infty$, we have

$$\begin{aligned}&|\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^2 - (X_{t_{i-1}}^0)^2)| \\ &= |\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}} + X_{t_{i-1}}^0)(X_{t_{i-1}} - X_{t_{i-1}}^0)| \\ &\leq \frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0|(|X_{t_{i-1}}| + |X_{t_{i-1}}^0|) \\ &\leq \frac{1}{n}\sum_{i=1}^n (|X_{t_{i-1}} - X_{t_{i-1}}^0|^2 \\ &+2|X_{t_{i-1}}^0||X_{t_{i-1}} - X_{t_{i-1}}^0|) \\ &= \frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0|^2 \\ &+2\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}^0||X_{t_{i-1}} - X_{t_{i-1}}^0| \\ &\leq (\sup_{0\leq t\leq 1}|X_t - X_t^0|)^2 \\ &+2\sup_{0\leq t\leq 1}|X_t - X_t^0|\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}^0| \\ &\xrightarrow{P} 0.\end{aligned}$$

Therefore, we obtain

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt. \tag{12}$$

The proof is complete. ■

In the following theorem, the consistency of the least squares estimators are proved.

Theorem 1: When $\varepsilon \to 0$, $n \to \infty$ and $\varepsilon n^{1-d} \to 0$, the least squares estimators $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ are consistent, namely

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha, \quad \widehat{\beta}_{n,\varepsilon} \xrightarrow{P} \beta.$$

Proof: According to Lemma 3 and Lemma 4, it is clear that

$$(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} (\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt. \tag{13}$$

With the results that $\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt$ and $\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} \int_0^1 X_t^0 dt$, when $\varepsilon \to 0$ and $n \to \infty$, it can be checked that

$$\beta\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \beta\int_0^1 X_t^0 dt\int_0^1 (X_t^0)^2 dt, \tag{14}$$

and

$$\beta\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}} \xrightarrow{P} \beta\int_0^1 (X_t^0)^2 dt\int_0^1 X_t^0 dt. \quad (15)$$

According to Lemma 3 and Lemma 4, we have

$$\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2 -\beta\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}} \xrightarrow{P} 0. \quad (16)$$

Since

$$\begin{aligned}
&|\varepsilon\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}dL_s^d| \\
&\le \varepsilon\sum_{i=1}^{n}|X_{t_{i-1}}||\int_{t_{i-1}}^{t_i}dL_s^d| \\
&\le \varepsilon\sum_{i=1}^{n}(|X_{t_{i-1}}^0|+|X_{t_{i-1}}-X_{t_{i-1}}^0|)|\int_{t_{i-1}}^{t_i}dL_s^d| \\
&\le \varepsilon\sum_{i=1}^{n}|X_{t_{i-1}}^0||\int_{t_{i-1}}^{t_i}dL_s^d| \\
&+\varepsilon\sup_{0\le t\le 1}|X_t-X_t^0||\int_{t_{i-1}}^{t_i}dL_s^d|.
\end{aligned}$$

By the Markov inequality, the results of Lemma 1 and Lemma 2 when $f(s)=g(s)=1$, $r=0$ and $\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}^0|\to\int_0^1 X_t^0 dt$, we obtain

$$\begin{aligned}
&P(|\varepsilon\sum_{i=1}^{n}|X_{t_{i-1}}^0||\int_{t_{i-1}}^{t_i}dL_s^d||>\delta) \\
&\le \delta^{-2}\varepsilon^2(\sum_{i=1}^{n}|X_{t_{i-1}}^0|)^2\mathbb{E}|\int_{t_{i-1}}^{t_i}dL_s^d|^2 \\
&\le C\delta^{-2}\varepsilon^2(\sum_{i=1}^{n}|X_{t_{i-1}}^0|)^2\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}|t-s|^{2d-1}dsdt \\
&\le C\delta^{-2}\varepsilon^2(\sum_{i=1}^{n}|X_{t_{i-1}}^0|)^2|t_i-t_{i-1}|^{2d} \\
&= C\delta^{-2}(\varepsilon n^{1-d})^2(\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}^0|)^2 \\
&\to 0,
\end{aligned}$$

which implies that $\varepsilon\sum_{i=1}^{n}|X_{t_{i-1}}^0||\int_{t_{i-1}}^{t_i}dL_s^d|\xrightarrow{P}0$ as $\varepsilon\to 0$, $n\to\infty$ and $\varepsilon n^{1-d}\to 0$.

According to Lemma 3, when $\varepsilon\to 0$ and $n\to\infty$, it is obvious that

$$\varepsilon\sup_{0\le t\le 1}|X_t-X_t^0||\int_{t_{i-1}}^{t_i}dL_s^d|\xrightarrow{P}0. \quad (17)$$

Then, we have

$$\varepsilon\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}dL_s^d\xrightarrow{P}0. \quad (18)$$

With the results of Lemma 4, (13) and (18), we have

$$\frac{\varepsilon\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}dL_s^d\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}\xrightarrow{P}0. \quad (19)$$

Then, when $\varepsilon\to 0$ and $n\to\infty$, it is easy to check that

$$\frac{\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}dL_s^d\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}\xrightarrow{P}0. \quad (20)$$

Moreover, there is no possible singularity in (19) and (20).

Therefore, When $\varepsilon\to 0$, $n\to\infty$ and $\varepsilon n^{1-d}\to 0$, we have

$$\widehat{\alpha}_{n,\varepsilon}\xrightarrow{P}\alpha.$$

Using the same methods, it can be easily to check that When $\varepsilon\to 0$, $n\to\infty$ and $\varepsilon n^{1-d}\to 0$, we have

$$\widehat{\beta}_{n,\varepsilon}\xrightarrow{P}\beta.$$

The proof is complete. ■

Theorem 2: When $\varepsilon\to 0$, $n\to\infty$, $\varepsilon n^{1-d}\to 0$ and $n\varepsilon\to\infty$,

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha) \xrightarrow{d}\to\frac{\int_0^1 X_t^0 dt\int_0^1 X_t^0 dL_t^d-\int_0^1(X_t^0)^2dt\int_0^1 dL_t^d}{(\int_0^1 X_t^0 dt)^2-\int_0^1(X_t^0)^2dt},$$

$$\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon}-\beta) \xrightarrow{d}\frac{\int_0^1 X_t^0 dL_t^d-\int_0^1 dL_t^d\int_0^1 X_t^0 dt}{(\int_0^1 X_t^0 dt)^2-\int_0^1(X_t^0)^2dt}.$$

Proof: According to the explicit decomposition for $\widehat{\alpha}_{n,\varepsilon}$, it is obvious that

$$\begin{aligned}
&\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha) \\
&=\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} \\
&-\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} \\
&+\frac{\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}dL_s^d\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} \\
&-\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}dL_s^d\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}.
\end{aligned}$$

From Lemma 3, when $\varepsilon\to 0$, $n\to\infty$ and $n\varepsilon\to\infty$,

$$\begin{aligned}
&|\varepsilon^{-1}\beta\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds| \\
&\le \varepsilon^{-1}\beta\sum_{i=1}^{n}|X_{t_{i-1}}||\int_{t_{i-1}}^{t_i}X_s ds| \\
&\le \varepsilon^{-1}n^{-1}\beta\sum_{i=1}^{n}(|X_{t_{i-1}}-X_{t_{i-1}}^0|+|X_{t_{i-1}}^0|) \\
&\sup_{t_{i-1}\le t\le t_i}|X_t|\xrightarrow{P}0.
\end{aligned}$$

Then, it is easy to check that

$$\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\xrightarrow{P}0.$$

Thus, we have

$$\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}\xrightarrow{P}0, \quad (21)$$

and

$$\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}\xrightarrow{P}0. \quad (22)$$

Since

$$\begin{aligned}&\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}dL_s^d\\&=\sum_{i=1}^{n}(X_{t_{i-1}}-X_{t_{i-1}}^0+X_{t_{i-1}}^0)\int_{t_{i-1}}^{t_i}dL_s^d\\&=\sum_{i=1}^{n}(X_{t_{i-1}}-X_{t_{i-1}}^0)\int_{t_{i-1}}^{t_i}dL_s^d\\&+\sum_{i=1}^{n}X_{t_{i-1}}^0\int_{t_{i-1}}^{t_i}dL_s^d.\end{aligned}$$

According to Lemma 3, we have

$$\sum_{i=1}^{n}(X_{t_{i-1}}-X_{t_{i-1}}^0)\int_{t_{i-1}}^{t_i}dL_s^d\xrightarrow{P}0. \quad (23)$$

Moreover,

$$\begin{aligned}&|\sum_{i=1}^{n}X_{t_{i-1}}^0\int_{t_{i-1}}^{t_i}dL_s^d-\int_0^1X_s^0dL_s^d|\\&\leq\int_0^1|X_{t_{i-1}}^0-X_s^0|dL_s^d\\&\leq\sup_{0\leq s\leq 1}|X_{t_{i-1}}^0-X_s^0||\int_0^1dL_s^d|\\&\xrightarrow{P}0.\end{aligned}$$

We obtain that

$$\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}dL_s^d\xrightarrow{P}\int_0^1X_s^0dL_s^d. \quad (24)$$

Then, we have

$$\begin{aligned}&\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha)\\&\xrightarrow{d}\frac{\int_0^1X_t^0dt\int_0^1X_t^0dL_t^d-\int_0^1(X_t^0)^2dt\int_0^1dL_t^d}{(\int_0^1X_t^0dt)^2-\int_0^1(X_t^0)^2dt}. \quad (25)\end{aligned}$$

As

$$\begin{aligned}&\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon}-\beta)\\&=\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}\\&-\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}-\varepsilon^{-1}\beta\\&+\frac{\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}dL_s^d}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}\\&-\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}dL_s^d\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}.\end{aligned}$$

It is obvious that

$$\begin{aligned}&\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}\\&-\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}\\&-\varepsilon^{-1}\beta\xrightarrow{P}0, \quad (26)\end{aligned}$$

and

$$\begin{aligned}&\frac{\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}dL_s^d}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}\\&-\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}dL_s^d\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}\\&\xrightarrow{d}\frac{\int_0^1X_t^0dL_t^d-\int_0^1dL_t^d\int_0^1X_t^0dt}{(\int_0^1X_t^0dt)^2-\int_0^1(X_t^0)^2dt}. \quad (27)\end{aligned}$$

Then, we have

$$\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon}-\beta)\xrightarrow{d}\frac{\int_0^1X_t^0dL_t^d-\int_0^1dL_t^d\int_0^1X_t^0dt}{(\int_0^1X_t^0dt)^2-\int_0^1(X_t^0)^2dt}. \quad (28)$$

The proof is complete. ■

Remark 1: The fractional Lévy process has non stationary increments, the increments over non overlapping intervals are more weakly correlated and their covariance decays polynomially at a higher rate, which makes the fractional Lévy process a possible candidate for models involving long-range dependence, self-similarity and non-stationary. Since the fractional Lévy process is not a martingale, methods of stochastic analysis are more sophisticated. We have applied Markov inequality, Cauchy-Schwarz inequality and Gronwall's inequality to derive the consistency and asymptotic distribution of estimators.

IV. Simulation

In this experiment, we use iterative approach to generate a discrete sample $(X_{t_{i-1}})_{i=1,\ldots,n}$ and compute $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ from the sample. We let $x_0=0.01$. For every given true value of the parameters-(α,β), the size of the sample is represented as"Size n" and given in the first column of the table. In Tables 1 and 3, $\varepsilon=0.1$, the size is increasing from 1000 to 5000. In Tables 2 and 4, $\varepsilon=0.01$, the size is increasing from 10000 to 50000. In Tables 1 and 2, $d=0.02$, In Tables 3 and 4, $d=0.3$. These tables list the value of least squares estimators "$\widehat{\alpha}_{n,\varepsilon}$","$\widehat{\beta}_{n,\varepsilon}$" and the absolute errors (AE) "$|\widehat{\alpha}_{n,\varepsilon}-\alpha|$","$|\widehat{\beta}_{n,\varepsilon}-\beta|$".

These tables illustrate that when n is large enough and ε is small enough, the obtained estimators are very close to the true parameter value. If we let n converge to the infinity and ε converge to zero, the estimator will converge to the true value.

V. Conclusion

The aim of this paper is to study the parameter estimation problem for Ornstein-Uhlenbeck process driven by small fractional Lévy noises from discrete observations. The contrast function has been introduced to obtain the explicit formula of the least squares estimators and the error of

TABLE I
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF α AND β

True		Aver		AE	
(α,β)	Size n	$\widehat{\alpha}_{n,\varepsilon}$	$\widehat{\beta}_{n,\varepsilon}$	$\lvert\widehat{\alpha}_{n,\varepsilon}-\alpha\rvert$	$\lvert\widehat{\beta}_{n,\varepsilon}-\beta\rvert$
(1,1)	1000	1.2652	1.2179	0.2652	0.2179
	2000	1.1263	1.1428	0.1263	0.1428
	5000	1.0541	1.0365	0.0541	0.0365
(2,3)	1000	2.2587	3.2649	0.2587	0.2649
	2000	2.1436	3.1368	0.1436	0.1368
	5000	2.0625	3.0451	0.0625	0.0451

TABLE II
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF α AND β

True		Aver		AE	
(α,β)	Size n	$\widehat{\alpha}_{n,\varepsilon}$	$\widehat{\beta}_{n,\varepsilon}$	$\lvert\widehat{\alpha}_{n,\varepsilon}-\alpha\rvert$	$\lvert\widehat{\beta}_{n,\varepsilon}-\beta\rvert$
(1,1)	10000	1.1352	0.8659	0.1352	0.1341
	20000	1.0571	1.0627	0.0571	0.0627
	50000	1.0013	1.0018	0.0013	0.0018
(2,3)	10000	1.8546	3.1571	0.1454	0.1571
	20000	2.0632	3.0528	0.0632	0.0528
	50000	2.0027	3.0041	0.0027	0.0041

TABLE III
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF α AND β

True		Aver		AE	
(α,β)	Size n	$\widehat{\alpha}_{n,\varepsilon}$	$\widehat{\beta}_{n,\varepsilon}$	$\lvert\widehat{\alpha}_{n,\varepsilon}-\alpha\rvert$	$\lvert\widehat{\beta}_{n,\varepsilon}-\beta\rvert$
(1,1)	1000	1.2106	1.1982	0.2106	0.1982
	2000	1.1035	1.1241	0.1035	0.1241
	5000	1.0394	1.0227	0.0394	0.0227
(2,3)	1000	2.1973	3.2061	0.1973	0.2061
	2000	2.0945	3.1158	0.0945	0.1158
	5000	2.0269	3.0371	0.0269	0.0371

TABLE IV
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF α AND β

True		Aver		AE	
(α,β)	Size n	$\widehat{\alpha}_{n,\varepsilon}$	$\widehat{\beta}_{n,\varepsilon}$	$\lvert\widehat{\alpha}_{n,\varepsilon}-\alpha\rvert$	$\lvert\widehat{\beta}_{n,\varepsilon}-\beta\rvert$
(1,1)	10000	1.0852	0.9236	0.0852	0.0764
	20000	1.0187	1.0269	0.0187	0.0269
	50000	1.0010	1.0008	0.0010	0.0008
(2,3)	10000	2.0927	3.0810	0.0927	0.0810
	20000	2.0158	3.0121	0.0158	0.0121
	50000	2.0009	3.0011	0.0009	0.0011

estimation has been given as well. The consistency and asymptotic distribution of the estimators have been derived by Markov inequality, Cauchy-Schwarz inequality and Gronwall's inequality. Further research topics will include parameter estimation for partially observed Ornstein-Uhlenbeck process driven by fractional Lévy noises.

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