A New Fast Numerical Solution Method for Dynamical Differential Equations with Irreversible Mass Matrix

Yuewei Yu, Yunpeng Song, Leilei Zhao and Changcheng Zhou

Abstract—**For the large-scale dynamic differential equations with irreversible mass matrix, based on the state space representation method in modern control theory, we proposed a new fast numerical solution method for its vibration response by ingenious mathematical transformation and iteration iterative calculation. Compared with the traditional solution method of MATLAB/Simulink module, the process of the modeling and calculation of this method is very simple and fast. As long as the high-order coefficient matrix corresponding to each block equation is written, it can be solved iteratively quickly. When it is applied to the analysis of large-scale system dynamics problems in engineering, its computational efficiency and economy can be significantly improved.**

Index Terms—**dynamic differential equations, irreversible mass matrix, state space, numerical solution, fast solution**

I. INTRODUCTION

In the study of large-scale system dynamics, in order to facilitate the engineering application, the method of solving its vibration differential equations must have the characteristics of high efficiency and practicability [1]. Because the mathematical models of the large-scale system dynamics can ultimately be reduced to second-order differential equations with multiple degrees of freedom, in order to improve their computational efficiency, they are usually first expressed as the matrix equations (i.e., $[\boldsymbol{M}]\{\ddot{\boldsymbol{X}}\}+[\boldsymbol{C}]\{\dot{\boldsymbol{X}}\}+[\boldsymbol{K}]\{\boldsymbol{X}\}=\{\boldsymbol{P}\}$, in which [$\boldsymbol{M}$], [$\boldsymbol{C}$], [$\boldsymbol{K}$] are the mass, damping, and stiffness matrices of the system; $\{\boldsymbol{X}\}$, $\{\dot{\boldsymbol{X}}\}$, $\{\ddot{\boldsymbol{X}}\}$ are the generalized displacement, velocity, and acceleration vectors of the system; $\{\boldsymbol{P}\}$ is the generalized load vector of the system), and then the corresponding numerical method is used to solve its numerically. Among the existing numerical methods, the Newmark-β method [2], Wilson-θ method [3], Houbolt method [4], Hiber-Hughes α method and β-θ collocation method [5], Park method [6], and Zhai method [7] are the most common ones. However, when these methods are used to analyze the dynamics of large-scale systems, the coefficient matrix corresponding to the second derivative (i.e., the mass matrix [$\boldsymbol{M}$]) must be diagonally reversible. However, for a large number of practical engineering problems, because the mass of the small components in the system is usually neglected (e.g., the mass of the shock absorber in the railway vehicle [8], the mass of the rubber bushing in the suspension of the truck cab [9]), the mass matrix [$\boldsymbol{M}$] in the vibration differential equations is often irreversible, which we call it the dynamic differential equations with irreversible mass matrix, as follows

$$\begin{cases} a_i\ddot{z}_i+\sum_{j=1}^{m}b_{ij}\dot{z}_j+\sum_{j=1}^{m}c_{ij}z_j+\sum_{j=1}^{n}d_{ij}\Delta z_j=f_i(t), \qquad i=1,2\cdots,m \\ a'_k\Delta\dot{z}_k+\sum_{j=1}^{m}b'_{kj}\dot{z}_j+\sum_{j=1}^{m}c'_{kj}z_j+\sum_{j=1}^{n}d'_{kj}\Delta z_j=f'_k(t), k=1,2\cdots,n \end{cases} \tag{1}$$

where, a_i is the mass coefficient; a'_k , b_{ij}, b'_{kj} are the damping coefficients; c_{ij}, d_{ij}, c'_{kj} , d'_{kj} are the stiffness coefficients, $\ddot{z}_i$ is the acceleration vector; $\dot{z}_j$, $\Delta\dot{z}_k$ are the velocity vectors; z_j, Δz_j are the displacement vectors, $f_i(t)$, $f'_k(t)$ are the acting loads; m is the number of systems with mass, n is the number of systems without mass. Here, the mass matrix of the system for Eq. (1) is $\boldsymbol{M}$=diag(a_1,a_2,…,a_m,0, …,0), it can be seen that $\boldsymbol{M}$ is irreversible.

At present, the most effective solution method to this kind of dynamic differential equations with irreversible mass matrix (seen in Eq. (1)) is to solve it with the help of MATLAB/Simulink module [10]. Although this solution method is feasible for some small-scale system dynamics problems, in fact, there are often dozens or even hundreds of equations in many large-scale system dynamics, which brings great inconvenience to its modeling and solving process. In order to improve the efficiency and economy of solving this kind of differential equations, based on the state space representation method in modern control theory, we proposed a new fast solution method of its vibration response through ingenious mathematical transformation and iterative solution.

II. NUMERICAL SOLUTION OF THE DYNAMICAL DIFFERENTIAL EQUATIONS WITH IRREVERSIBLE MASS MATRIX

The following is an introduction to the solving process of the dynamic differential equations with irreversible mass matrix. The solving process is divided into four steps.

Step 1. *Equation transformation*

Manuscript received July 18, 2022; revised November 4, 2022. This work was supported in part by the Natural Science Foundation of Shandong Province under Grant ZR2021QE082 and ZR2020ME127.

Yuewei Yu is a lecturer of the School of Transportation and Vehicle Engineering, Shandong University of Technology, Zibo 255000, China (corresponding author, e-mail: yuyuewei2010@163.com).

Yunpeng Song is a postgraduate of the School of Transportation and Vehicle Engineering, Shandong University of Technology, Zibo 255000, China (e-mail: 2376859086@qq.com).

Leilei Zhao is an associate professor of the School of Transportation and Vehicle Engineering, Shandong University of Technology, Zibo 255000, China (e-mail: zhaoleilei611571@163.com).

Changcheng Zhou is a professor of the School of Transportation and Vehicle Engineering, Shandong University of Technology, Zibo 255000, China (e-mail: dyn_keylab@163.com).

According to Eq. (1), dividing the two sides of each equation by the coefficients of its highest derivative, and converting them into the following forms

$$\begin{cases} \ddot{z}_i = -\sum_{j=1}^{m}\frac{b_{ij}}{a_i}\dot{z}_j - \sum_{j=1}^{m}\frac{c_{ij}}{a_i}z_j - \sum_{j=1}^{n}\frac{d_{ij}}{a_i}\Delta z_j + \frac{f_i(t)}{a_i}, \quad i = 1,2\cdots,m \\ \Delta\dot{z}_k = -\sum_{j=1}^{m}\frac{b'_{kj}}{a'_k}\dot{z}_j - \sum_{j=1}^{m}\frac{c'_{kj}}{a'_k}z_j - \sum_{j=1}^{n}\frac{d'_{kj}}{a'_k}\Delta z_j + \frac{f'_k(t)}{a'_k}, k = 1,2\cdots,n \end{cases} \tag{2}$$

Step 2. *Extraction of coefficient matrix from block equation*

Representing Eq. (2) as the following three matrix equation forms, and extract its coefficient matrix:

(1) Composite system equation for systems with and without mass:

$$\Big[\underbrace{\ddot{z}_1\ \ \ddot{z}_2\cdots\ddot{z}_m}_{\text{Mass system}}\ \Big|\ \overbrace{\Delta\dot{z}_1\ \Delta\dot{z}_2\cdots\Delta\dot{z}_n}^{\text{Massless system}}\Big]^{\mathrm{T}}_{1\times(m+n)} =$$

$$[A_1]\Big[\underbrace{\dot{z}_1\ \ \dot{z}_2\cdots\dot{z}_m}_{\text{Mass system}}\ \Big|\ \overbrace{\Delta z_1\ \Delta z_2\cdots\Delta z_n}^{\text{Massless system}}\Big]^{\mathrm{T}}_{1\times(m+n)} + [\boldsymbol{F}] \tag{3}$$

Taking the velocities and displacement of each subsystem in the composite system as the state variable, the coefficient matrix $[\boldsymbol{A}_1]$ and generalized load vector $[\boldsymbol{F}]$ are extracted according to Eq. (3).

(2) Mass system equation:

$$\Big[\underbrace{\ddot{z}_1\ \ \ddot{z}_2\cdots\ddot{z}_m}_{\text{Mass system}}\Big]^{\mathrm{T}}_{1\times m} = [A_2]\Big[\underbrace{z_1\ \ z_2\cdots z_m}_{\text{Mass system}}\Big]^{\mathrm{T}}_{1\times m} \tag{4}$$

Taking the displacement of each subsystem in the mass system as the state variable, the coefficient matrix $[\boldsymbol{A}_2]$ is extracted from Eq. (4).

(3) Massless system equation:

$$\Big[\underbrace{\Delta\dot{z}_1\ \Delta\dot{z}_2\cdots\Delta\dot{z}_n}_{\text{Massless system}}\Big]^{\mathrm{T}}_{1\times n} = [A_3]\Big[\underbrace{z_1\ \ z_2\cdots z_m}_{\text{Mass system}}\Big]^{\mathrm{T}}_{1\times m} \tag{5}$$

Taking the displacement of each subsystem in the massless system as the state variable, the coefficient matrix $[\boldsymbol{A}_3]$ is extracted from Eq. (5).

Step 3. *State space equation*

According to the velocity and displacement variables of each degree of freedom in the composite system with and without mass, taking the displacement and velocities of each subsystem in the composite system as the components of the state vector, the state vector can be obtained, that is

$$\{\boldsymbol{X}\} = \Big[\underbrace{\dot{z}_1\ \ \dot{z}_2\cdots\dot{z}_m}_{\text{Mass system}}\ \Big|\ \overbrace{\Delta z_1\ \Delta z_2\cdots\Delta z_n}^{\text{Massless system}}\ \Big|\ \underbrace{z_1\ \ z_2\cdots z_m}_{\text{Mass system}}\Big]^{\mathrm{T}}_{1\times(2m+n)} \tag{6}$$

Therefore, according to the coefficient matrix of each block equation extracted in step 2, the following state space equation can be obtained.

$$\{\dot{\boldsymbol{X}}\} = [\boldsymbol{A}]\{\boldsymbol{X}\} + [\boldsymbol{B}]\{\boldsymbol{U}\} \tag{7}$$

where, $[\boldsymbol{A}] = \left[\begin{array}{c|c} [A_1] & \begin{array}{c}[A_2]\\ \\ [A_3]\end{array} \\ \hline [\boldsymbol{I}]_{m\times m} & [\boldsymbol{O}]_{m\times(m+n)} \end{array}\right]$, $\{\boldsymbol{U}\} = \left[\begin{array}{c} [\boldsymbol{F}] \\ \hline [\boldsymbol{O}]_{m\times 1} \end{array}\right]$,

$[\boldsymbol{B}] = \left[\begin{array}{c|c} [\boldsymbol{I}]_{(m+n)\times(m+n)} & [\boldsymbol{O}]_{(m+n)\times m} \\ \hline \multicolumn{2}{c}{[\boldsymbol{O}]_{m\times(2m+n)}} \end{array}\right]$. Here, $[\boldsymbol{O}]$ is the zero matrix, $[\boldsymbol{I}]$ is the identity matrix.

Step 4. *Iterative solution*

According to the state space Eq. (7), when Δt is small enough, the following relationship can be obtained:

$$\{\boldsymbol{X}\}_k = \{\boldsymbol{X}\}_{k-1} + \Delta t\{\dot{\boldsymbol{X}}\}_k \tag{8}$$

where, Δt is the time integral step; subscripts k and k-1 represent the current step $t=k\Delta t$ and the previous step $t=(k-1)\Delta t$, respectively.

Then, substituting Eq. (8) into Eq. (7), the following relationship can be obtained.

$$\{\dot{\boldsymbol{X}}\}_k = [\boldsymbol{A}]\big(\{\boldsymbol{X}\}_{k-1} + \Delta t\{\dot{\boldsymbol{X}}\}_k\big) + [\boldsymbol{B}]\{\boldsymbol{U}\}_k \tag{9}$$

Thus, Eq. (9) can be transformed into the following form

$$\big([\boldsymbol{I}] - [\boldsymbol{A}]\Delta t\big)\{\dot{\boldsymbol{X}}\}_k = [\boldsymbol{A}]\{\boldsymbol{X}\}_{k-1} + [\boldsymbol{B}]\{\boldsymbol{U}\}_k \tag{10}$$

According to Eq. (10), the following expression can be obtained.

$$\{\dot{\boldsymbol{X}}\}_k = \big(\boldsymbol{I} - [\boldsymbol{A}]\Delta t\big)^{-1}\big([\boldsymbol{A}]\{\boldsymbol{X}\}_{k-1} + [\boldsymbol{B}]\{\boldsymbol{U}\}_k\big) \tag{11}$$

Therefore, according to Eq. (11), the integral recurrence formula of the system response for dynamical differential equations with irreversible mass matrix can be obtained, that is

$$\begin{cases} \{\dot{\boldsymbol{X}}\}_k = \big(\boldsymbol{I} - [\boldsymbol{A}]_k\Delta t\big)^{-1}\big([\boldsymbol{A}]_k\{\boldsymbol{X}\}_{k-1} + [\boldsymbol{B}]_k\{\boldsymbol{U}\}_k\big) \\ \{\boldsymbol{X}\}_k = \{\boldsymbol{X}\}_{k-1} + \Delta t\{\dot{\boldsymbol{X}}\}_k \end{cases} \tag{12}$$

According to the initial conditions, i.e.,

$$\{\boldsymbol{X}(0)\} = \{\boldsymbol{X}\}_0 \tag{13}$$

The discrete values of the displacement, the velocity, and the acceleration corresponding to each step can be calculated successively according to the integral recurrence formula (12).

It is worth noting that, for linear systems, $[\boldsymbol{A}]_k$ and $[\boldsymbol{B}]_k$ in Eq. (12) are constant $[\boldsymbol{A}]$, $[\boldsymbol{B}]$, and their values can be easily obtained from Eq. (7), for nonlinear systems, the dynamic equation is simply written in incremental form, and different $[\boldsymbol{A}]$, $[\boldsymbol{B}]$ matrices are used in different integration periods, and the integration in each period is consistent with the linear situation.

In order to analyze the accuracy of the calculation formula, Eq. (8) is expanded by Taylor, as follows

$$(\{\boldsymbol{X}\}_k) = \{\boldsymbol{X}\}_{k-1} + \Delta t\{\dot{\boldsymbol{X}}\}_k + \frac{1}{2}\Delta t^2\{\ddot{\boldsymbol{X}}\}_k \tag{14}$$

Thus, according to Eq. (8) and Eq. (14), the truncation error can be obtained, that is

$$\{E(\boldsymbol{X})\} = (\{\boldsymbol{X}\}_k) - \{\boldsymbol{X}\}_k = \frac{1}{2}\Delta t^2\{\ddot{\boldsymbol{X}}\}_k \tag{15}$$

That is to say, $\{E(\boldsymbol{X})\}$ has $[O(\Delta t^2)]$ order accuracy, and $\{E(\dot{\boldsymbol{X}})\}$ also has $[O(\Delta t^2)]$ order accuracy.

As stated above, for the dynamical differential equations with irreversible mass matrix, we can easily extract the coefficient matrix of each sub equation by using computer software programming, and then the dynamic responses of the system can be obtained by iterative calculation quickly.

III. CALCULATION EXAMPLE

A. Vibration response solution of the quarter railway vehicle vibration model

Taking the 1/4 vehicle vibration model [11] shown in Fig. 1 as an example, using the established integral recurrence formula (12) to calculate its vibration response, and comparing the results with the Matlab/Simulink method. Here, the masses m_c=14 398 kg, m_t=1 379 kg; the damping coefficients C_p=28 300N·s/m, C_s=60 000 N·s/m; the spring stiffness K_p=2.74×10^6 N/m, K_s=5.68×10^5 N/m; the rubber joint stiffness K_{pd}=40×10^6 N/m, K_{sd}=20×10^6 N/m; the external excitation z_v=0.5sin6t.

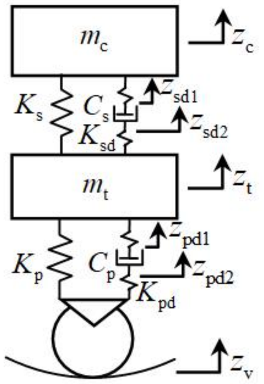

Fig. 1. 1/4 railway vehicle vibration model

According to Newton's second law, the vibration differential equations of the system shown in Fig. 1 can be obtained, as follows

$$\begin{cases} m_c\ddot{z}_c + K_s(z_c - z_t) + K_{sd}(z_c - z_{sd1}) = 0 \\ m_t\ddot{z}_t + K_s(z_t - z_c) + K_p(z_t - z_v) + K_{sd}(z_t - z_{sd2}) \\ \quad + K_{pd}(z_t - z_{pd1}) = 0 \\ C_s(\dot{z}_{sd1} - \dot{z}_{sd2}) + K_{sd}(z_{sd1} - z_c) = 0 \\ C_s(\dot{z}_{sd2} - \dot{z}_{sd1}) + K_{sd}(z_{sd2} - z_t) = 0 \\ C_p(\dot{z}_{pd1} - \dot{z}_{pd2}) + K_{pd}(z_{pd1} - z_t) = 0 \\ C_p(\dot{z}_{pd2} - \dot{z}_{pd1}) + K_{pd}(z_{pd2} - z_v) = 0 \end{cases} \tag{16}$$

Let $\Delta z_2 = z_{sd1} - z_{sd2}$, $\Delta z_1 = z_{pd1} - z_{pd2}$, according to Eq. (16), the following expression can be obtained.

$$\begin{cases} z_{sd1} = \frac{1}{2}(z_c + z_t + \Delta z_2) \\ z_{sd2} = \frac{1}{2}(z_c + z_t - \Delta z_2) \end{cases}, \begin{cases} z_{pd1} = \frac{1}{2}(z_t + z_v + \Delta z_1) \\ z_{pd2} = \frac{1}{2}(z_t + z_v - \Delta z_1) \end{cases} \tag{17}$$

Then, substituting Eq. (17) into Eq. (16), the matrix of the high order terms in Eq. (16) can be converted into the standard form as shown in Eq. (1), that is

$$\begin{cases} m_c\ddot{z}_c + K_s(z_c - z_t) + \frac{1}{2}K_{sd}(z_c - z_t - \Delta z_2) = 0 \\ m_t\ddot{z}_t + K_s(z_t - z_c) + K_p(z_t - z_v) + \\ \quad \frac{1}{2}K_{sd}(z_t - z_c + \Delta z_2) + \frac{1}{2}K_{pd}(z_t - z_v - \Delta z_1) = 0 \\ C_s\Delta\dot{z}_2 + \frac{1}{2}K_{sd}(\Delta z_2 - z_c + z_t) = 0 \\ C_p\Delta\dot{z}_1 + \frac{1}{2}K_{pd}(\Delta z_1 - z_t + z_v) = 0 \end{cases} \tag{18}$$

At this time, according to the solution steps as stated in section II, Eq. (18) can be expressed as the following form, that is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{19}$$

where, $\boldsymbol{x}(t) = \left[\dot{z}_c \;\; \dot{z}_t \;\; \Delta z_2 \;\; \Delta z_1 \;\; z_c \;\; z_t\right]^{\mathrm{T}}_{1\times 6}$; $\boldsymbol{A} = \left[\begin{array}{c|c} \boldsymbol{A}_1 & \begin{array}{c} \boldsymbol{A}_2 \\ \boldsymbol{A}_3 \end{array} \\ \hline \boldsymbol{I}_{2\times 2} & \boldsymbol{O}_{2\times 4} \end{array}\right]$,

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 0 & \frac{K_{sd}}{2m_c} & 0 \\ 0 & 0 & -\frac{K_{sd}}{2m_t} & \frac{K_{pd}}{2m_t} \\ 0 & 0 & -\frac{K_{sd}}{2C_s} & 0 \\ 0 & 0 & 0 & -\frac{K_{pd}}{2C_p} \end{bmatrix}, \quad \boldsymbol{A}_3 = \begin{bmatrix} \frac{K_{sd}}{2C_s} & -\frac{K_{sd}}{2C_s} \\ 0 & \frac{K_{pd}}{2C_p} \end{bmatrix},$$

$$\boldsymbol{A}_2 = \begin{bmatrix} -\frac{K_s + \frac{K_{sd}}{2}}{m_c} & \frac{K_s + \frac{K_{sd}}{2}}{m_c} \\ \frac{K_s + \frac{K_{sd}}{2}}{m_t} & -\frac{K_p + \frac{K_{pd}}{2} + K_s + \frac{K_{sd}}{2}}{m_t} \end{bmatrix}; \quad \boldsymbol{F}(t) = \begin{bmatrix} 0 \\ \frac{K_p + \frac{K_{pd}}{2}}{m_t} \\ 0 \\ -\frac{K_{pd}}{2C_p} \end{bmatrix},$$

$$\boldsymbol{u}(t) = \left[\begin{array}{c} \boldsymbol{F}(t) \\ \hline \boldsymbol{O}_{2\times 1} \end{array}\right], \quad \boldsymbol{B} = \left[\begin{array}{c|c} \boldsymbol{I}_{4\times 4} & \boldsymbol{O}_{4\times 2} \\ \hline \multicolumn{2}{c}{\boldsymbol{O}_{2\times 6}} \end{array}\right].$$

Hence, according to the established integral recurrence formula (12), the vibration responses of the system shown in Fig. 1 can be calculated by iteratively solving Eq. (19). The comparison curves of the acceleration and displacement of m_c between the calculation result and the Simulink simulation result are shown in Fig. 2 and Fig. 3.

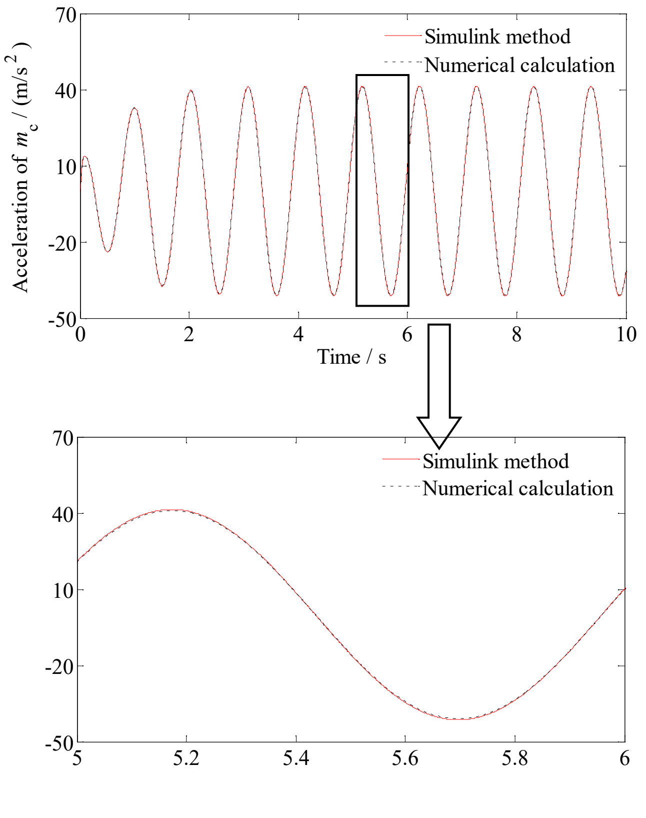

Fig. 2. Calculation results of the acceleration of m_c under two calculation methods

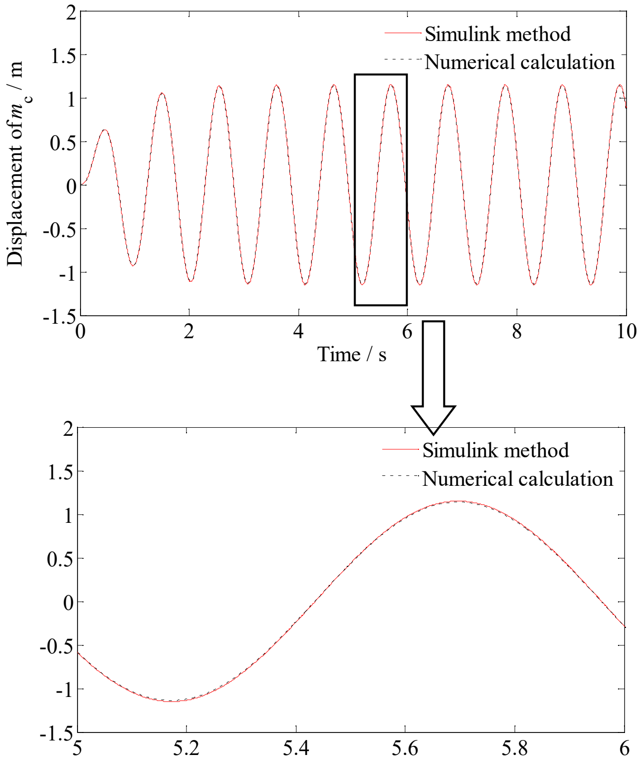

Fig. 3. Calculation results of the displacement of m_c under two calculation methods

It can be seen from Fig. 2 and Fig. 3, the responses of the system obtained by the two calculation methods are consistent, indicating that, the new fast numerical calculation method established is correct. It is worth noting that, the Matlab/Simulink method is only suitable for solving low-degree-of-freedom systems, it is difficult to be used to solve the large-scale structural systems. In addition, the Matlab/Simulink method not only requires an incredible modeling time, but also requires considerable computational memory. Therefore, the method established can effectively solve the problem of the large-scale dynamical differential equations with irreversible mass matrix, and provide reliable technical support for the analysis of dynamic characteristics of the large-scale structural systems.

B. Vibration response solution of the railway vehicle vertical vibration model

Taking the vertical dynamic model of railway vehicles [12] shown in Fig. 4 as an example, the vibration responses of the system are calculated according to the integral recurrence formula (12). In Fig. 4, z_c, z_{t1}, and z_{t2} are the vertical displacement of the car body, the front bogie frame, and the rear bogie frame; β_c, β_{t1}, and β_{t2} are the pitching displacement of the car body, the front bogie frame, and the rear bogie frame; z_{w1}~z_{w4} are the vertical displacement of the wheelset; z_{pd1}~z_{pd8} and z_{sd1}~z_{sd4} are the vertical displacement of the two ends of the primary and secondary vertical dampers; z_{v1}~z_{v4} are the random input of the track irregularities. The parameters values of the vehicle are as follows: the masses of the car body, the bogie frame, and the wheelset are M_c=3.4×10^3 kg, M_t=3 000 kg, M_w=1 400 kg; the moments of inertia of the car body and the bogie frame are J_c=2.277×10^6 kg·m^2, J_t=2 710 kg·m^2; the vertical stiffness of the primary and secondary suspension are K_p=5.5×10^5 N/m, K_s=4×10^5 N/m; the vertical damping of the primary and secondary suspension are C_p=60 000 N·s/m, C_s=80 000 N·s/m; the rubber joint stiffness of the primary and secondary vertical dampers are K_{pd}=5×10^6 N/m, K_{sd}=5×10^6 N/m; the equivalent linear contact stiffness between the wheel and the rail is K_H=8×10^7 N/m; the half distances of the vehicle and the bogie wheelbase are L_c=9 m, L_t=1.2 m; the vehicle running speed v=250 km/h; the external excitation z_{v1}=1.5sin(20t).

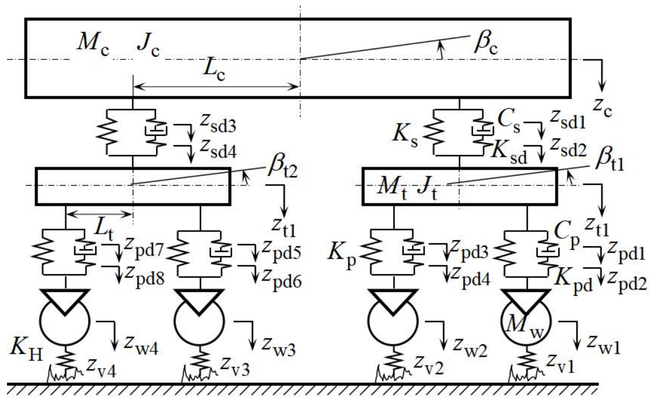

Fig. 4. Railway vehicle vertical vibration model

According to the solution steps shown in Example 1, the vibration responses of the system shown in Fig. 4 can be obtained, as shown in Fig. 5.

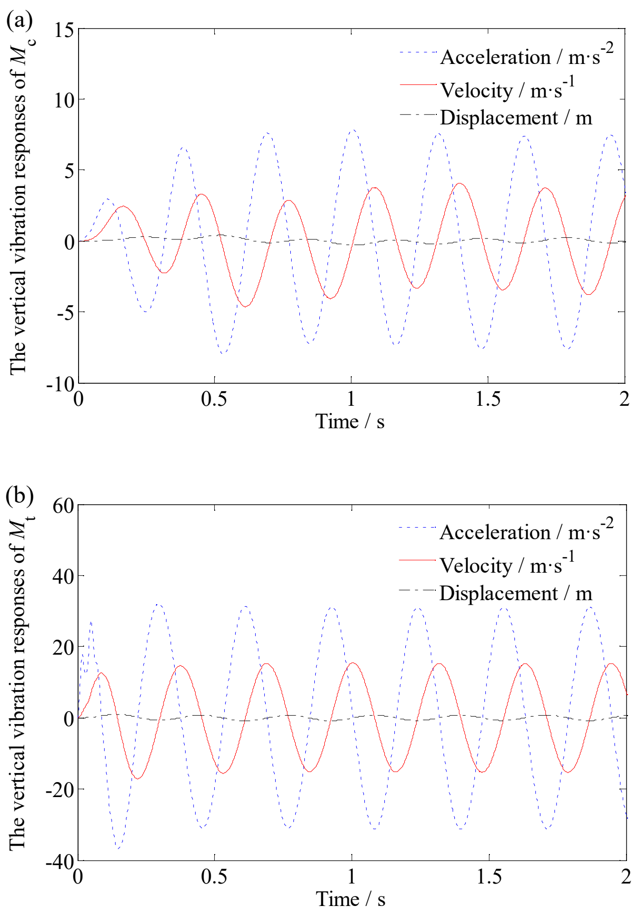

Fig. 5. The vertical vibration responses of the railway vehicle system: (a) the vertical vibration responses of M_c; (b) the vertical vibration responses of M_t

It should be noted that, for complex systems, using the method established in this paper, only a few simple steps of transformation and iterative solution are needed to calculate the vibration response results. In addition, the higher the degree of freedom of the system is, the more advantageous the method is.

IV. Conclusions

In this paper, we have proposed a new fast numerical solution method to solve the large-scale dynamic differential equations with irreversible mass matrix. Compared with the traditional solution method of the MATLAB/Simulink module, the method proposed in this paper can effectively shorten the modeling time, and significantly improve the solving speed of dynamic differential equations with irreversible mass matrix. When it is applied to the analysis of large-scale system dynamics problems in engineering, its computational efficiency and economy can be significantly improved.

This study provides a new numerical method for solving the large-scale dynamic differential equations with irreversible mass matrix.

References

[1] W. M. Zhai, *Vehicle-track coupled dynamics, Fourth edition*. Science Press, Beijing, 2015.

[2] N. M. Newmark, "A method of computation for structural dynamics," *Journal of the Engineering Mechanics Division*, vol. 85, no. 1, pp. 67 – 94, 1959.

[3] K. J. Bathe and E. L, Wilson, *Numerical methods in finite element analysis*. Prentice-Hall, Inc., New Jersey, 1976.

[4] J. C. Houbolt, "A recurrence matrix solution for the dynamic response of elastic aircraft," *Journal of the Aeronautical Sciences*, vol. 17, no. 9, pp. 540 – 550, 1950.

[5] M. H. Hans, J. H. Thomas and L. T. Rober, "Improved numerical dissipation for time integration algorithms in structural dynamics," *Earthquake Engineering and Structural Dynamics*, vol. 5, no. 3, pp. 283 – 292, 1977.

[6] K. C. Park, "An improved stiffly stable method for direct integration of nonlinear structural dynamic equations," *Journal of Applied Mechanics*, vol. 42, no. 3, pp. 464 – 470, 1975.

[7] W. M. Zhai, "Two simple fast integration methods for large-scale dynamic problems in engineering," *International Journal for Numerical Methods in Engineering*, vol. 39, no. 24, pp. 4199 – 4214, 1996.

[8] D. B. Cui, L. Li, X. S. Jin, X. B. Xiao and J. J. Ding, "Influence of vehicle parameters on critical hunting speed based on Ruzicka model," *Chinese Journal of Mechanical Engineering*, vol. 25, no. 3, pp. 536 – 542, 2012.

[9] Z. L. Cao, C. Han, Y. Lin, H. Li and M. Gao, "Research on nonlinear dynamics simulation of heavy truck in time domain," *Proceedings of SAE-China Congress 2015*, Selected Papers. Springer Singapore, 2016.

[10] J. Vojtesek and P. Dostal, *Using simulink in simulation of dynamic behaviour of nonlinear process*. Springer International Publishing, Switzerland, 2015.

[11] C. C. Zhou, Y. W. Yu and L. L. Zhao, "Analytical formulae of secondary vertical suspension system design for high-speed train," *Journal of Mechanical Engineering*, vol. 52, no. 19, pp. 53 – 60, 2016.

[12] J. J. Meng, Z. Q. Yang, G. H. Pu and R. X. Xu, "Vertical vibration response analysis of railway vehicle based on pseudo excitation method," *China Railway Science*, vol. 33, no. 2, pp. 89 – 94, 2012.