A Nonlinear Prey–Predator Model with Holling Type III Functional Response: An Analytical Approach with Stability and Sensitivity Analyses

T. Vijayalakshmi and R. Senthamarai *Member, IAENG,*

***Abstract*—In this paper, we investigated a model of a two-species prey predator with a Holling type III interaction. The interior equilibrium point's behaviour was explored. The theory was used to investigate the presence and stability of equilibriums, as well as the system's sensitivity behaviour. We derived an analytical expression of the prey and the predator that appeared in the system. Finally, our analytical results are interpreted ecologically and compared to the numerical outputs using MATLAB programme. We found a very good agreement between the numerical and analytical results.**

***Index Terms*—Prey-predator; Stability analysis; HPM; Numerical simulation; Sensitivity analysis;**

I. Introduction

ECOLOGY is concerned with the stability of ecological systems and the persistence of species within them are of interest to ecologists. Mathematical models of ecological systems have been used to explore the stability of a range of systems, reflecting these concerns. Due to its worldwide existence and importance, the dynamic connection between predator and prey has long been and will continue to be one of the major issues in mathematical ecology. For the Lotka–Volterra type predator–prey system, much great work has been done. Holling postulated that the predator has three functional responses, which are referred to as Holling type I, Holling type II, and Holling type III. He proposed the Holling type II form as follows:

$$\phi(x) = \frac{gx}{a+x} \tag{1}$$

where $\phi(x)$ is the number of prey consumed by single predator, x is the prey density, g is the time available for searching and a is a constant of proportionality, termed the 'discovery rate' by Holling. The absorption of substrate by microorganisms in microbial dynamics kinetics is commonly described by a Holling type II response function. This is always true when the predator is an invertebrate. In addition, he proposed the Holling type III response function, which takes the form [1, 2].

$$\phi(x) = \frac{gx^2}{a^2+x^2} \tag{2}$$

Manuscript received March 18, 2022; revised October 26, 2022.
T.Vijayalakshmi is a Ph.D Scholar in the Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur, Chennai-603203, India. E-mail:vt1785@srmist.edu.in

R. Senthamarai is an Assistant Professor in the Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, Chennai-603203, India. (Corresponding author) E-mail: senthamr@srmist.edu.in

II. MATHEMATICAL MODEL

Continuous time models with differential equations and discrete time models with difference equations are the two forms of mathematical models used in population theory. As shown in several studies, research on interspecific interactions has primarily focused on continuous prey-predator models of two variables, where the dynamics consist only of stable equilibrium. However, discrete-time prey-predator models may provide a significantly wider set of dynamics than continuous-time models. We consider predator-prey systems of the Lotka-Volterra type [3,4].

$$\frac{dx}{dt} = rx(1-\frac{x}{k}) - \frac{\alpha x^2 y}{a^2+x^2} \tag{3}$$

$$\frac{dy}{dt} = \frac{m\alpha x^2 y}{a^2+x^2} - dy - \delta y^2 \tag{4}$$

with the initial conditions,
$x(0) = l_1$ and $y(0) = l_2, \forall\ l_1 l_2 \geq 0$,
where x and y are the prey and predator densities respectively. The intrinsic growth rate, the carrying capacity, the conversion factor, and the natural death rate are represented by the parameters $r,\ k,\ m$, and d. The intra-specific competition rate is represented by δ and the predation rate is denoted by α. The existence of equilibrium points and the local stability of the fixed points are discussed in the third and fourth sections. In the fifth section, we have discussed the sensitivity analysis. We studied the dynamical behaviour of the system numerically and analytically in the sixth section. Numerical simulation is covered in the seventh section. The results and discussion in the eighth section and our work's conclusion is in the ninth section.

III. EQUILIBRIUM POINTS

To find the equilibrium points of the system, we have considered the following equations

$$rx(1-\frac{x}{k}) - \frac{\alpha x^2 y}{a^2+x^2} = 0 \tag{5}$$

$$\frac{m\alpha x^2 y}{a^2+x^2} - dy - \delta y^2 = 0 \tag{6}$$

(i) The trivial equilibrium is E_0 = $(0, 0)$ always exists.
(ii) The predator free equilibrium is E_1 = $(k, 0)$
(iii) The coexistence equilibrium is $E_* = (x^*, y^*)$
Now (5) becomes

$$r(1-\frac{x^*}{k}) - \frac{\alpha x^* y^*}{a^2+x^{2*}} = 0 \tag{7}$$

$$\frac{m\alpha x^{*2}}{a^2+x^{*2}} - d - \delta y^* = 0 \quad (8)$$

$$\frac{m\alpha x^{*2}}{a^2+x^{*2}} - d = \delta y^* \quad (9)$$

$$\frac{m\alpha x^{*2} - d(a^2+x^{*2})}{\delta(a^2+x^{*2})} = y^* \quad (10)$$

By substituting the equation (10) in (7) we get,

$$r(1-\frac{x^*}{k}) - \frac{\alpha x^*}{a^2+x^{*2}}[\frac{m\alpha x^{*2} - d(a^2+x^{*2})}{\delta(a^2+x^*2)}] = 0 \quad (11)$$

$$\begin{aligned} & rk\delta\alpha^4 + 2a^2x^{*2}rk\delta + rk\delta x^{*4} - r\delta a^4x^* \\ & -2a^2r\delta x^{3*} - r\delta x^{*5} - m\alpha^2kx^{*3} + k\alpha da^2x^* \\ & +d\alpha kx^{*3} = 0 \end{aligned} \quad (12)$$

$$\begin{aligned} & -r\delta x^{*5} + rk\delta x^{*4} + (d\alpha k - 2a^2r\delta - m\alpha^2k)x^{*3} \\ & +2a^2rk\delta x^{*2} + (k\alpha da^2 - r\delta a^4)x^* \\ & +rk\delta a^4 = 0 \end{aligned} \quad (13)$$

$$A_1x^{*5} + A_2x^{*4} - A_3x^{*3} + A_4x^{*2} - A_5x^* + A_6 = 0 \quad (14)$$

where
$A_1 = -r\delta;\ A_2 = rk\delta;\ A_3 = -(2a^2r\delta + m\alpha^2k - d\alpha k);\ A_4 = 2a^2rk\delta;\ A_5 = (r\delta a^4 - k\alpha da^2);\ A_6 = rka^4\delta$.
There are three different sign alterations. There are no noticeable changes when we substitute x by $-x$. So, according to Descarte's sign rule, there are exactly three positive roots. Because of this, the roots are real, negative or complex with a real negative part, and the point of equilibrium is a node.

IV. STABILITY ANALYSIS

Around the prey - predator equilibrium point, the variational matrix of the system [5-11, 20, 21] is as follows:

$$v(x,y) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (15)$$

where $a_{11} = r(1-\frac{2x}{k}) - \frac{(a^2+x^2)2\alpha xy - 2\alpha x^3y}{(a+x)^2}$
$a_{12} = -\frac{\alpha x^2}{a^2+x^2}$
$a_{21} = \frac{(a^2+x^2)2\alpha xmy - 2\alpha mx^2y}{(a^2+x^2)^2}$
$a_{22} = \frac{m\alpha x^2}{(a^2+x^2)} - d - 2y\delta$

(i) the trivial equilibrium point $E_0 = (0,0)$
The Jacobian matrix of system at E_0 is

$$v(0,0) = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix} \quad (16)$$

The characteristic equation is

$$\begin{vmatrix} r-\lambda & 0 \\ 0 & -d-\lambda \end{vmatrix} = 0 \quad (17)$$

Therefore the eigenvalues are $\lambda = r, -d$, so the equilibrium point $E_0(0,0)$ is a saddle point.
(ii) the predator free equilibrium point E_1=$(k,0)$
The Jacobian matrix of system at E_1 is

$$v(k,0) = \begin{bmatrix} -r & -\frac{\alpha k^2}{(a^2+k^2)} \\ 0 & \frac{m\alpha k^2}{(a^2+k^2)} - d \end{bmatrix} \quad (18)$$

The charactertic equation is

$$\begin{vmatrix} -r-\lambda & -\frac{\alpha k^2}{(a^2+k^2)} \\ 0 & (\frac{m\alpha k^2}{(a^2+k^2)} - d) - \lambda \end{vmatrix} = 0 \quad (19)$$

$$-r-\lambda = 0, (\frac{m^2 - d(a^2+k^2)}{a^2+k^2}) - \lambda = 0 \quad (20)$$

If $\lambda = -r < 0$ and $\lambda = -\frac{d(a^2+k^2) - m\alpha k^2}{a^2+k^2} < 0$
The roots are real, distinct and negative, so the system is asympotically stable.
(iii) the coexistentence equilibrium $E^* = (x^*, y^*)$
The charactersic equation is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \quad (21)$$

where $a_{11} = r(1-\frac{2x}{k}) - \frac{(a^2+x^2)2\alpha xy - 2\alpha x^3y}{(a+x)^2}$
$a_{12} = -\frac{\alpha x^{*2}}{a^2+x^{*2}}$
$a_{21} = \frac{(a^2+x^{*2})2m\alpha x^*y^* - 2\alpha mx^*y^*}{(a^2+x^*2)^2}$
$a_{22} = (\frac{m\alpha x^{*2}}{a^2+x^{*2}} - d - 2\delta y^*) - \lambda$

$$i.e.,\ \lambda^2 + B\lambda + C = 0 \quad (22)$$

where

$$B = [r(1-\frac{2x^*}{k}) - \frac{(a^2+x^{*2})2\alpha x^*y^* - 2\alpha x^{*3}y^*}{(a^2+x^*2)^2}] \\ ([\frac{m\alpha x^{*2}}{a^2+x^{*2}} - d - 2\delta y^*]) \quad (23)$$

$$C = [-\frac{\alpha x^{*2}}{a^2+x^{*2}}][\frac{(a^2+x^{*2})2m\alpha x^*y^* - 2\alpha mx^*y^*}{(a^2+x^*2)^2}] \quad (24)$$

When we replace λ by $-\lambda$, there are two sign modifications. So by Descarte's rule of sign, there are two negative roots when

$$r(1-\frac{2x^*}{k}) > \frac{(a^2+x^{*2})2\alpha x^*y^* - 2\alpha x^{*3}y^*}{(a^2+x^{*2})^2} \quad (25)$$

As a result, the roots are real, distinct, negative, or complex with a negative real portion. Therefore, the equilibrium point, is a node. As a result, the system is asymptotically stable.

V. SENSITIVITY ANALYSIS

In Equations (3) and (4), we have sensitivity index solutions that determine the changes in the value of the state variable that doubling the parameters yields [18-20]. In Figures 1 and 2, it is seen that by doubling the intrinsic growth rate of prey species, the population of prey increases and attains its maximum of 600 in 3 days and decreases to $0.01mgl^{-1}$ at the end of 5 days. When the carrying capacity k is doubled, the prey population increases by $10mgl^{-1}$, and it increases slightly when the predation rates α and a are doubled. Increasing the parameter d by a factor of two increases the predator population by 0.001 in five days. The intra-specific competition rate (δ), predation rate (α), and decrease in predator population are negligible. The semi-relative sensitivity as well as logarithmic solutions determine the changes that doubling parameters yields in the value of a state variable. From Figure 3 and 4, doubling

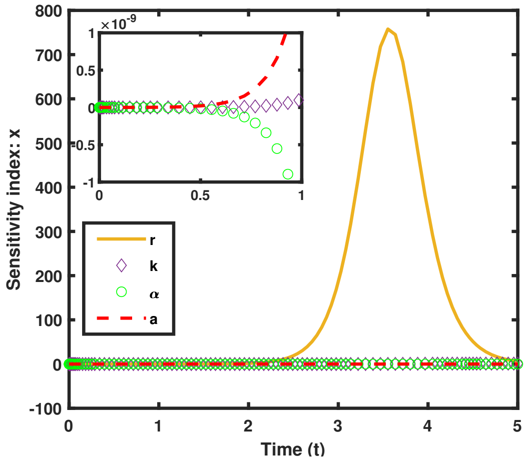

Fig. 1. Sensitivity analysis of the prey population

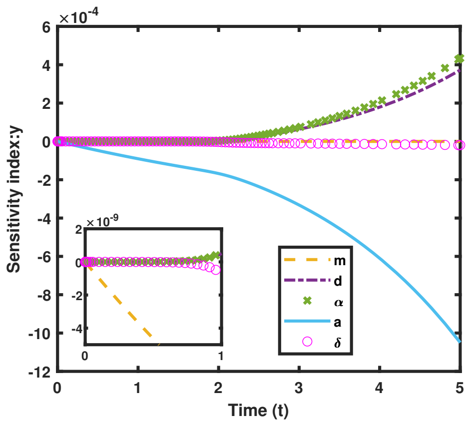

Fig. 2. Sensitivity analysis of the predator population

the parameter r increases the prey population also increases by 2.5 in 3 days and again decreases by 0.0001 in end of 5 days. Also, when the parameters k, α and a are doubled, the prey population grows slightly. When the parameters δ, d and α are doubled, the predator population changes slightly. However, the predator population decreases by 5 in 5 days on doubling the parameter a and increases by 3 in 5 days by the effect of the parameter m. Moreover, logarithmic sensitivity solutions calculate the percentage change in the value of a state variable induced by doubling a parameter. In Figure 5 and 6, it is seen that by doubling the parameter r of prey species, the prey population increases to a maximum of 1000% in 3 days and decreases to 0.01% in the end of 5 days. Also, the population increases by 100% in 5 days by doubling the parameter k. The predation rates indicate a slight increase in the prey population. Furthermore, doubling the parameter d reduces the predator population by 50% in 5 days, and the parameters δ, α, and a result in a minor decrease in the predator population.

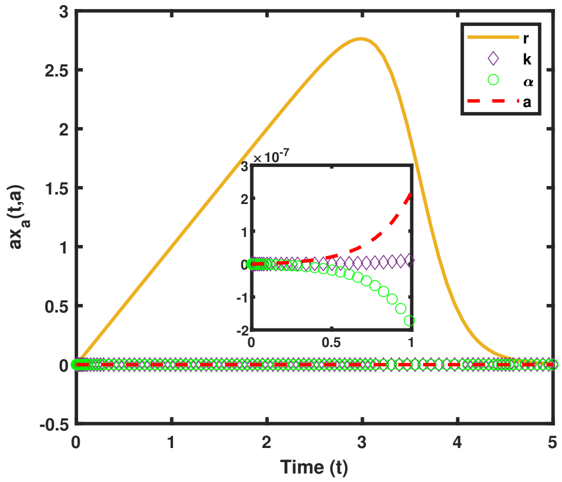

Fig. 3. Semi-relative sensitivity of the prey population

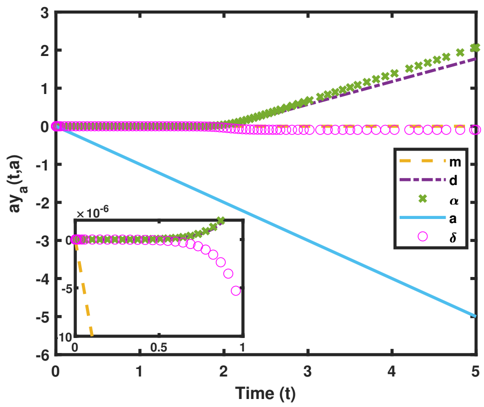

Fig. 4. Semi-relative sensitivity of the predator population

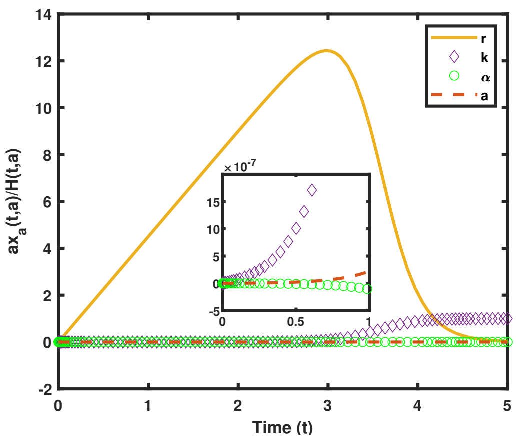

Fig. 5. The sensitivity of the prey population on a logarithmic scale

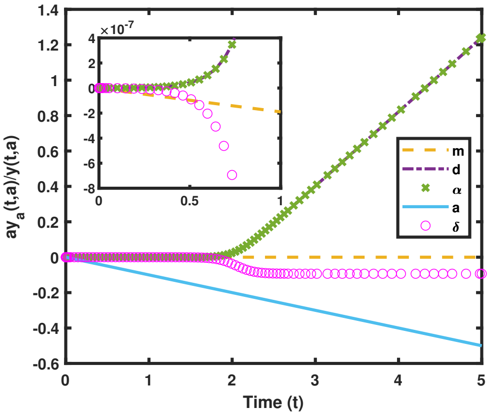

Fig. 6. The sensitivity of the predator population on a logarithmic scale

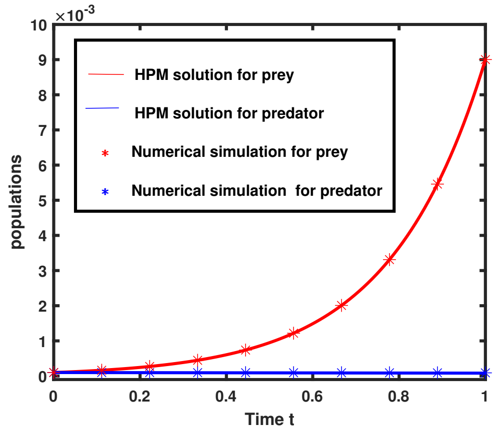

Fig. 7. Comparison of the analytical and the numerical result in prey-predator model (3) and(4)

VI. ANALYTICAL EXPRESSIONS OF NONLINEAR EQUATIONS. (3) AND (4) USING HPM

Estimated analytical solutions of the system of equations (3) and (4) using the HPM [12-16]

$$(1-p)[\frac{dx}{dt}-rx]+p[\frac{dx}{dt}-rx+\frac{rx^2}{k}+\frac{\alpha x^2 y}{a^2} -\frac{\alpha x^4 y}{a^4}+\frac{\alpha x^6 y}{a^6}]=0 \quad (26)$$

$$(1-p)[\frac{dy}{dt}+dy]+p[\frac{dy}{dt}+dy-\frac{m\alpha x^2 y}{a^2} +\frac{m\alpha x^4 y}{a^4}-\frac{m\alpha x^6 y}{a^6}+\delta y^2]=0 \quad (27)$$

with initial conditions as follows $x_0(0)=l_1$ and $y_0(0)=l_2$ According to the HPM, we obtain the approximate analytical expressions for prey and predator as follow:

$$\begin{aligned} x(t) = \frac{e^{(-2td+5rt)}}{a^4k(r-d)(3r-d)}&[a^4(-d^2)(l_1^2)e^{2dt-4rt} \\ &+a^4(d^2)(l_1^2)e^{2dt-3rt}]-3a^4l_1^2r^2e^{(2dt-4rt)} \\ &+3a^4l_1^2r^2e^{(2dt-3rt)}4a^4dl_1^2re^{2dt-4rt} \\ &-a^2d\alpha kl_1^2l_2e^{(td-3rt)}]+a^2d\alpha kl_1^2l_2e^{(2td-4rt)} \\ &+3a^2\alpha kl_1^2l_2re^{(td-3rt)}-3a^2\alpha kl_1^2l_2re^{(2td-4rt)} \\ &+d\alpha kl_1^4l_2e^{(td-rt)}-d\alpha kl^4l_2e^{(2dt-4rt)} \\ &-\alpha kl_1^4l_2re^{(td-rt)}+\alpha kl_1^4l_2re^{(2td-3rt)} \\ &-4a^4dl_1^2re^{(2dt-3rt)} \end{aligned} \quad (28)$$

$$\begin{aligned} y(t) = &\frac{\alpha l_1^2ml_2^2e^{(2rt-1)}e^{-dt}}{2a^2r}-\frac{\alpha l_1^4ml_2^2e^{(4rt-1)}e^{-dt}}{4a^4r} \\ &+l_2e^{-dt}+\frac{\alpha l_1^6ml_2^2e^{(6rt-1)}e^{-dt}}{6a^6r}-\frac{ml_2^2\alpha(e^{dt-1})e^{-2dt}}{\delta} \end{aligned} \quad (29)$$

VII. NUMERICAL SIMULATION

We have obtained very new and closed expressions for the prey and predator populations presented in equations (28) and (29). We compared our analytical solutions obtained by HPM with the numerical simulation. To test the accuracy of our approximate analytical expressions, the system was also solved by using Matlab software for all possible values of the parameters and we received a satisfactory agreement between them.

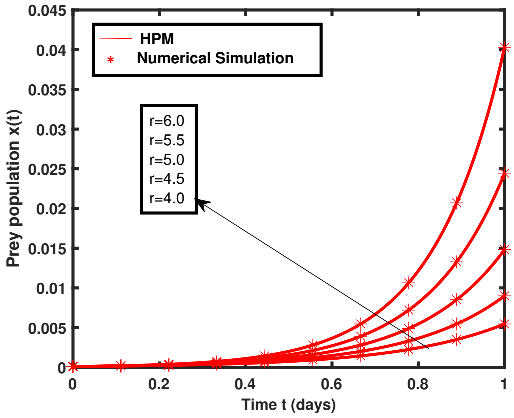

Fig. 8. Prey population series with varying values of the intrinsic growth rate r from 4 to 6.Here the other parameters k, α, d, m are all fixed.

VIII. RESULT AND DISCUSSION

Equations are very new expressions for prey-predator populations. We compared and reported the prey populations by HPM with numerical simulation using MATLAB software for all possible values of the parameters $r, k, \alpha, d, m, \delta$, and a. In Figure 7, the findings are matched in both analytical and numerical simulation. Figure 8 shows that, when the parameter (intrinsic growth rate) r increases, the population of prey also increases. Figure 9 shows that, as the parameter (natural mortality rate) d increases, so does the predator population. Figure 10 depicts that, prey population increases over time t as intrinsic growth rate r increases. Figure 11 depicts that, as natural death rate d increases, the predator population decreases over time t. Other parameters k, α, δ and m donot affect the state variables. As seen in all the figures, our approximate analytical expressions are a better way to measure the populations of prey and predator. We can also use the same method to get approximate analytical solutions for other nonlinear systems.

IX. CONCLUSION

In this paper, we have investigated the dynamics of the prey-predator model. The stability analysis has been carried out For all possible values of the system's parameters,

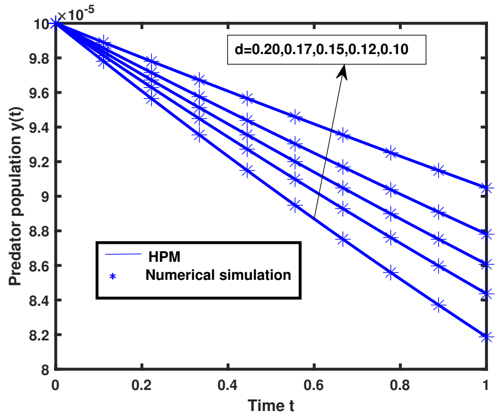

Fig. 9. Predator population series with varying parameter values, when the natural death rate d =0.10-0.20 and the other parameters k, α, r, m are all fixed.

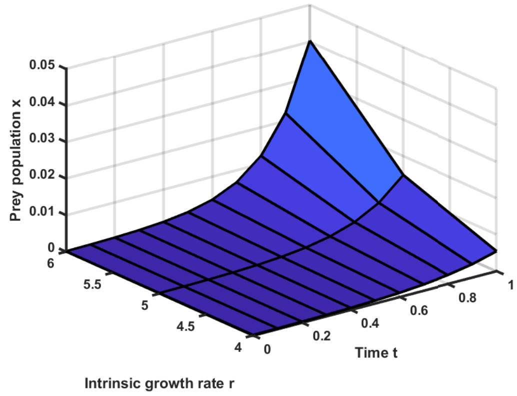

Fig. 10. Prey populations with varying values of the parameters r and time t

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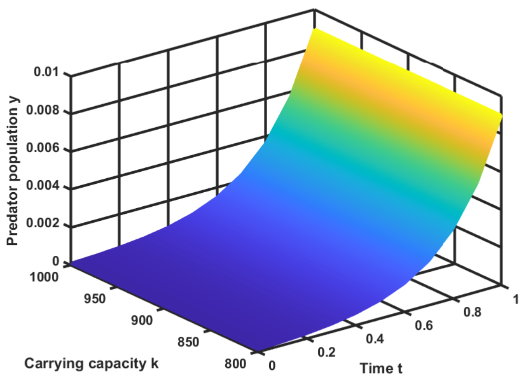

Fig. 11. Predator populations with varying values of the parameters k and time t.

we have provided a new and closed form of approximate analytical solutions to the prey-predator system in this paper. The proposed model is examined by using HPM. Analytical and numerical techniques are extensively utilised for solving nonlinear differential equations. Our logical and numerical results are adequate to study the proposed model. With a few iterations, we have achieved good results. So, the HPM is preferable for solving other nonlinear differential equations.

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