Prediction of IBNR and RBNS Liabilities using Estimated Delay Probabilities by a Zero-Inflated Gamma Mixture Model

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***Abstract*—Consider the problem of predicting the amount/number of claims that have been incurred but not reported (say IBNR) and reported but not settled (say RBNS). To predict both the IBNR's and the RBNS' loss reserves and their corresponding mean square error of prediction, we used the delay probabilities, which are modeled by the Zero-Inflated Gamma Mixture (ZIGM) distribution. The practical application of our findings is applied against a real data set. Moreover, the accuracy of this approach has been compared with the traditional chain ladder method.**

***Index Terms*—IBNR and RBNS loss reserves, Prediction, Mean Square Error of Prediction, Zero-Inflated Gamma Mixture distribution, Maximum likelihood estimator, EM algorithm, Motor third party liability insurance.**

I. Introduction

WHEN a claim occurs, it may be reported to the insurer sometime later, say reporting (or Type 1) delay. Such reported claims may be paid (or settled) after a significant amount of time, say settlement (or Type 2) delay. Each type of delay arrives from different reasons, but for both delays, all insurance companies must predict and hold a sufficient reserve in order to fulfill all possible payments regarding their corresponding claims. According to actuarial science, these two types of claim reserves are well known as: incurred but not reported (IBNR) and reported but not settled (RBNS), respectively.

Since there is not a prespecified prediction method, therefore, prediction of the claim reserves is an important and challenging task for an actuary. On the other hand, any appropriate prediction method will be varied by a line of business, an insurer's approach to risk, etc.

In recent papers, the double chain ladder approach is used to estimate the claim reserve through a micro-level approach of the claim development process, based on the number of reported claims and the payments' amount. For example, Verrall et al. (2010) [30] proposed a run-off triangle of paid claims and also the numbers of reported claims to predict both RBNS and IBNR claims. Their approach drove a middle way between the crude methods based on a single triangle and the very detailed methods based on data at the individual claim level. Martinez-Miranda et al. (2012) [20] defined the claims reserving model at the micro-level. They focused on how various claims delays impact severity and how to incorporate this information into the reserve. Martinez-Miranda et al. (2013a) [21] considered a double chain ladder focusing on two specific types of prior knowledge on zero-claims for some underwriting years and the relationship between a claim's development and its severity.

Martinez-Miranda et al. (2013b) [22] used a micro-level approach to predict the number of IBNR claims. Their continuous chain-ladder setting can be applied to data recorded in continuous time, although it is illustrated in the paper on data aggregated at a monthly level. Antonio and Plat (2014) [1] proposed a micro-level model along with some information about: the claim's occurrence, the delay between occurrence and reporting time, payments' time and their sizes, and final settlement. Verrall and Wüthrich (2016) [31] constructed an inhomogeneous marked Poisson process with a monthly piecewise constant intensity and a weekday seasonal occurrence pattern. Badescu et al. (2016) [3] and Avanzi et al. (2016) [2] employed a marked Cox process to model the claim arrival process along with its reporting delays. Such a model allowed them to consider over-dispersion and serial dependency. Hiabu et al. (2016a) [14] employed expert opinion along with observed data to predict RBNS and IBNR claims. A direct link between continuous granular methods and classical aggregate methods has been studied by Hiabu et al. (2016b) [15].

Denuit and Trufin (2017) [7] considered the frequency and severity of claims along with the number and amount of payments for each run-off triangle's cell. They used a 2-component mixture model to describe such payments. Denuit and Trufin (2018) [8] categorized settled claims based on their settlement's time. Then, they employed a zero-augmented Gamma regression model with a specific inflation effect to study such settled claims. Reserving in general insurance under an excess of loss reinsurance treaty has been studied by Margraf et al. (2018) [19]. Verbelen et al. (2018, 2021) [28, 29] modeled the events' occurrence subject to a reporting delay using a flexible regression framework. Moreover, under such a framework, they employed the EM algorithm to estimate the occurrence and reporting's parameters from granular data at a daily level.

Maciak et al. (2018) [18] focused on three synergetic research branches: (1) inventing stochastic methods for loss reserving based on claim-by-claim data, (2) using a dynamic copula framework for modeling dependencies among types of claims, and (3) deriving appropriate statistical inference for these approaches. A marked Cox process has been employed by Badescu et al. (2019) [4] to show some desirable properties of Badescu et al. (2016)'s [3] findings. Moreover,

Manuscript received December 06, 2021; revised October 25, 2022.

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they showed that using an EM algorithm to estimate parameters decreased the computational cost. Kuncoro and Purwono (2019) [17] modeled the IBNR and the RBNS claim reserves for an MSE-credit insurance product using a homogeneous Poisson process. They indicated that the credit period and the credit amount impact on the risk level of each credit insurance policy.

Crevecoeur et al. (2018, 2019) [5,6] employed a granular model to study the delay time between the occurrence and observation of the event for the IBNR claim reserve. Duval and Pigeon (2019) [10] combined a traditional approach and gradient boosting algorithm to introduce a new method for modeling loss reserving for a non-life insurance company. They showed that the total paid amount of each claim can be predicted, with reasonable accuracy, with their method. The Archimedean copula has been used by Noviyanti et al. (2019) [25] to predict the outstanding IBNR claim reserving. They claimed their method provides a more suitable prediction for fire insurance compared to prediction arrives from the chain ladder method.

The main aim of this article is to obtain a computationally reasonable expression for predictors of both the IBNR and the RBNS loss reserve and their mean square errors of predictions based on history up to today's time t. To solve the prediction problem, this article decomposes the outstanding claims as IBNR and RBNS. Then, it considers a Zero-Inflated Gamma Mixture distribution for random reporting delay and a discrete random variable for the settlement delay. Using updated observations at time t unknown parameters are estimated under the maximum likelihood approach, then both the IBNR and the RBNS outstanding claims are predicted.

The article is organized as follows: Section 2 presents the theoretical foundation of the article. Section 3 shows how the previous section's findings can be applied in practice. Suggestions and concluding remarks are given in Section 4.

II. Theoretical foundation

A large class of distributions, including several heavy tail distributions such as Weibull and Pareto distributions, can be approximated, arbitrarily closely, by Gamma mixture distributions (Payandeh Najafabadi, 2018) [27]. On the other hand, in many actuarial applications, there is a considerable number of zeros in the collected data set. Therefore, it is reasonable to consider a zero-inflated distribution for such cases, see Jaya et al. (2021) [16], Nwozo and Nkeki (2011) [26], Nkeki and Nwozo (2013) [24], and Fang (2014) [11], among others for more detail.

We consider the Zero-Inflated Gamma Mixture distribution, say ZIGM, as an appropriate distribution for the random reporting delay U which is given by the following definition.

Definition 1: A random variable U has the Zero-Inflated Gamma Mixture, say ZIGM, distribution if its density function is

$$g(u,\psi) = \pi I_{\{0\}}(u) + (1-\pi)\sum_{h=1}^{k} w_h Gamma(\alpha_h,\theta_h) I_{(0,\infty)}(u), \quad (1)$$

where $\psi = (w_1,\cdots,w_k,\alpha_1,\cdots,\alpha_k,\theta_1,\cdots,\theta_k)$ and π is the probability of extra zeros and $0 \le \pi \le 1$. The parameters w_h are called mixing coefficients where satisfy $0 \le w_h \le 1$ and $\sum_{h=1}^{k} w_h = 1$ in order to valid probabilities and $Gamma(\alpha_h,\theta_h)$ stands for the Gamma density function with the scale parameter α_h and the shape parameter θ_h . $I_{\{A\}}$ denotes the indicator function of event A.
See He and Chen (2022) [13] and Gharib (1995) [12] for some properties of the Gamma Mixture distribution.

This article considers reporting delay time at a monthly (30 days) level. Therefore, a considerable number of reporting delay times will be zero, while some of them stand far from others. These two facts justify the implementation of a zero-inflated and heavy distribution. For some practical reasons, we assume that the random reporting delay has been distributed according to the ZIGM distribution.

The following represent assumptions that we consider hereafter now.

Model Assumption 1: Assume that:

- A_1) The total number of claims related to the accident time i, say N_i, follows from a homogeneous Poisson process with finite intensity λ_i;
- A_2) Random reporting delay U has been distributed according to the ZIGM distribution, given by Definition (1);
- A_3) Conditionally on the total number of claims for the accident time i and reported with j unit time delay, $N_{i,j}$, the number of paid claims have been distributed according to a multinomial distribution. In other words, for each given (i,j), the random vector $(N^{paid}_{i,j,0},\cdots,N^{paid}_{i,j,I-1}|N_{i,j}) \sim multinomial(N_{i,j};q_0,\cdots,q_{I-1})$, where claim reporting delay probabilities $q_0,\cdots,q_{I-1}$ satisfy $\sum_{l=0}^{I-1} q_l = 1$ and $0 \le q_l \le 1, \forall l = 0,\cdots,I-1$.
- A_4) Discrete random settlement delay D has probability mass function $q_l = P(D=l)$, for $l = 0,\cdots,d$;
- A_5) The individual payments $Y^{(k)}_{i,j-l,l}$ are iid random variables with $E\left(Y^{(k)}_{i,j-l,l}\right) = \mu < \infty$ and $Var\left(Y^{(k)}_{i,j-l,l}\right) = \sigma^2 < \infty$;
- A_6) $\mathcal{F}_t$ stands for updated filtration based upon past information at observation time t;
- A_7) Claims are settled with a single payment;
- A_8) Settlement delay is independent of the reporting delay and the payments are independent of the reporting and the settlement delays.

As mentioned above, the outstanding claims represent claims which occurred at accident time i and were reported to the insurance company j unit time later. But for some practical reasons, they paid (or settled) l unit time after j.

To model this issue, suppose $N_{i,j}$, for $i = 1,\cdots,I$and $j = 0,\cdots,I-1$, stands for the total number of claims that occurred at accident time i and fully paid before or at time $i+j$. Assume that $N^{paid}_{i,j-l,l}$ denotes the number of the future payment originating from the $N_{i,j}$ claims that occurred at accident time i, reported at $i+j-l$ time and paid at $i+j$ time. The aggregate paid claim, denoted by $N^{paid}_{i,j}$, has the following form

$$N^{paid}_{i,j} = \sum_{l=0}^{min(j,d)} N^{paid}_{i,j-l,l}$$

where d is the maximum delay period to pay the claim (after being reported). Moreover, suppose that, for $k = 1,\cdots,N^{paid}_{i,j-l,l}$, i.i.d random variable $Y^{(k)}_{i,j-l,l}$ stands for size of the k^{th} individual payments that occurred at accident time i, reported at $i+j-l$ time and paid at $i+j$ time.

Therefore, the total payment at time $i+j$ is

$$X_{ij} = \sum_{l=0}^{\min\{j,d\}} \sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)},$$

where the maximum payment delay d can be chosen from the evaluation process for each insurance company.

For the upper triangle $i+j \leq I$, the total payments X_{ij} are known. But for the lower triangle $i+j > t > I$, the total payments X_{ij} are unknown and one has to predict. For such a situation, there are two types of unknown claims, one has not been reported yet, say X_{ij}^{IBNR}, another one has been reported but not fully paid, say X_{ij}^{RBNS}. Taking this fact into account, one may decompose the total payments at time $i+j$ as

$$\begin{aligned} X_{ij} &= \sum_{l=0}^{i+j-I-1} \sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)} + \sum_{l=i+j-I}^{\min\{j,d\}} \sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)} \\ &= X_{ij}^{IBNR} + X_{ij}^{RBNS}, \quad \forall\, i+j > I. \end{aligned} \tag{2}$$

Remark 1: It is worthwhile mentioning that in a situation where one would like to make an inference about the number of outstanding claims rather than the size of outstanding claims, he/she can consider $Y_{i,j-l,l}^{(k)} = 1$.

Under Model Assumption (1) and using the Poisson process properties, one may conclude that: (**1**) the total number of claims that occurred at accident time i and paid at time $i+j$, $N_{i,j}^{paid}$ follows from a homogeneous Poisson process with intensity $\lambda_i p_j$; (**2**) the total number of payments at time $i+j$, related to accident time i, $N_{i,j-l,l}^{paid}$, follows from a homogeneous Poisson process with intensity $\lambda_i p_{j-l} q_l$, where, for $l = 0, \cdots, d$, delay probability p_j is

$$p_j = P(j \leq U \leq j+1) = \int_j^{j+1} dG_{ZIGM}(u, \psi), \tag{3}$$

and $G_{ZIGM}(.,.)$ stands for the cumulative distribution function of the ZIGM distribution.

The following theorem provides the conditional expectation of the total payments X_{ij} given updated filtration $\mathcal{F}_t$, where $i+j > t > I$.

Theorem 1: Suppose filtration $\mathcal{F}_t$ provides all available information about the number of payments at observation time t, $i+j > t > I$. Moreover, besides assumptions A_1 to A_8, given in Model Assumption (1), assume $E(N_{i,j-l}^{paid}) < \infty$, $E(Y_{i,j-l,l}^{k}) < \infty$, $Var(N_{i,j-l}^{paid}) < \infty$ and $Var(Y_{i,j-l,l}^{k}) < \infty$. The best estimation for $X_{i,j}$, say $X_{ij}^{(t)}$, can be restated as

$$X_{ij}^{(t)} = \sum_{l=0}^{i+j-I-1} \lambda_i^{(t)} p_{j-l}^{(t)} q_l^{(t)} \mu^{(t)} + \sum_{l=i+j-I}^{\min\{j,d\}} n_{i,j-l}^{paid} q_l^{(t)} \mu^{(t)}.$$

Proof. It is well-known that the conditional expectation of $E(X_{ij}|\mathcal{F}_t)$ plays an essential role in predicting future loss liabilities, see Wüthrich and Merz (2008) [32] and Taylor (2012), among others for more details. By conditioning on $\mathcal{F}_t$, one may conclude that

$$\begin{aligned} X_{ij}^{(t)} &= E(X_{ij}|\mathcal{F}_t) \\ &= E(\sum_{l=0}^{i+j-I-1} \sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)}|\mathcal{F}_t) \\ &\quad + E(\sum_{l=i+j-I}^{\min\{j,d\}} \sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)}|\mathcal{F}_t) \\ &= \sum_{l=0}^{i+j-I-1} E(N_{i,j-l,l}^{paid}|\mathcal{F}_t) E(Y_{i,j-l,l}^{(k)}|\mathcal{F}_t) \\ &\quad + \sum_{l=i+j-I}^{\min\{j,d\}} E(N_{i,j-l,l}^{paid}|\mathcal{F}_t) E(Y_{i,j-l,l}^{(k)}|\mathcal{F}_t) \\ &= \sum_{l=0}^{i+j-I-1} \lambda_i^{(t)} p_{j-l}^{(t)} q_l^{(t)} \mu^{(t)} + \sum_{l=i+j-I}^{\min\{j,d\}} n_{i,j-l}^{paid} q_l^{(t)} \mu^{(t)}, \end{aligned}$$

where $E(Y_{i,j-l,l}^{(k)}) = \mu^{(t)}$ and in the first term, $E(N_{i,j-l,l}^{paid}|\mathcal{F}_t) = \lambda_i^{(t)} p_{j-l}^{(t)} q_l^{(t)}$. Because these claims come from the future values of $N_{i,j}$ and, in this article, these are predicted. Note that, the $N_{i,j-l,l}^{paid}$ claims in the second term are known because these claims arise from the claims which have already been reported.

Based upon Equation (2) the above conditional expectation can be decomposed as

$$\begin{aligned} X_{ij}^{IBNR,(t)} &= \sum_{l=0}^{i+j-I-1} \lambda_i^{(t)} p_{j-l}^{(t)} q_l^{(t)} \mu^{(t)} \\ X_{ij}^{RBNS,(t)} &= \sum_{l=i+j-I}^{\min\{j,d\}} n_{i,j-l}^{paid} q_l^{(t)} \mu^{(t)}. \end{aligned}$$

The conditional MSEP can be decomposed as

$$MSEP_{\mathcal{F}_t}(X_{ij}, \hat{X}_{ij}) = Var(X_{ij}|\mathcal{F}_t) + E[(\hat{X}_{ij} - E(X_{ij}|\mathcal{F}_t))^2|\mathcal{F}_t],$$

where the first term is well-known as the process variance and the second term is the estimation error.

It is worthwhile to recall that both $\hat{X}_{ij}$ and $E(X_{ij}|\mathcal{F}_t)$ are $\mathcal{F}_t$-measurable, therefore the expectation in the outer term of the above equation is redundant, i.e.,

$$E[(\hat{X}_{ij} - E(X_{ij}|\mathcal{F}_t))^2|\mathcal{F}_t] = (\hat{X}_{ij} - E(X_{ij}|\mathcal{F}_t))^2.$$

On the other hand, we know that $\hat{X}_{ij} = E(X_{ij}|\mathcal{F}_t)$, therefore, the estimation error reduces to zero and consequently, the conditional MSEP can be written as

$$MSEP_{\mathcal{F}_t}(X_{ij}, \hat{X}_{ij}) = Var(X_{ij}|\mathcal{F}_t) \tag{4}$$

Lemma 1: Suppose all Theorem (1)'s assumptions hold. Then, the results of the conditional MSEP for $X_{i,j}^{IBNR}$ and $X_{i,j}^{RBNS}$ given for updated filtration $\mathcal{F}_t$, for t where $i+j > t > I$, are

$$MSEP(X_{ij}^{IBNR,(t)}|\mathcal{F}_t) = \mu^{(t)} \phi^{(t)} \lambda_i^{(t)} \sum_{l=0}^{i+j-I-1} p_{j-l}^{(t)} q_l^{(t)}$$

$$MSEP(X_{ij}^{RBNS,(t)}|\mathcal{F}_t) \approx \mu^{(t)} \phi^{(t)} \lambda_i^{(t)} \sum_{l=i+j-I}^{\min\{j,d\}} n_{i,j-l}^{paid} q_l^{(t)}$$

Proof. By using Equation (4), we calculate the prediction error for $X_{i,j}^{IBNR}$ and $X_{i,j}^{RBNS}$ by

$$\begin{aligned} V_1 &= Var(X_{ij}^{IBNR}|\mathcal{F}_t) \\ &= Var(\sum_{l=0}^{i+j-I-1} \sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)}|\mathcal{F}_t) \\ &= \sum_{l=0}^{i+j-I-1} \left(E(Var(\sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)}|N_{i,j-l,l}^{paid})|\mathcal{F}_t) \right. \\ &\quad \left. + \sum_{l=0}^{i+j-I-1} \left(Var(E(\sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)}|N_{i,j-l,l}^{paid})|\mathcal{F}_t) \right) \right. \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{i+j-I-1} \left(\sigma^{2,(t)} E(N_{i,j-l,l}^{paid}|\mathcal{F}_t) + \mu^{2,(t)} Var(N_{i,j-l,l}^{paid}|\mathcal{F}_t)\right) \\
&= \mu^{(t)}\phi^{(t)}\lambda_i^{(t)} \sum_{l=0}^{i+j-I-1} p_{j-l}^{(t)} q_l^{(t)}
\end{aligned}
$$

and

$$
\begin{aligned}
V_2 &= Var(X_{ij}^{RBNS}|\mathcal{F}_t) \\
&= Var\Big(\sum_{l=i+j-I}^{\min\{j,d\}} \sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)}|\mathcal{F}_t\Big) \\
&= \sum_{l=i+j-I}^{\min\{j,d\}} \left(E(Var(\sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)}|N_{i,j-l,l}^{paid})|\mathcal{F}_t) \right. \\
&\quad \left. + \sum_{l=i+j-I}^{\min\{j,d\}} \left(Var(E(\sum_{k=1}^{N_{i,j-l,l}^{paid}} Y_{i,j-l,l}^{(k)}|N_{i,j-l,l}^{paid})|\mathcal{F}_t) \right) \right. \\
&= \sum_{l=i+j-I}^{\min\{j,d\}} \left(\sigma^{2,(t)} E(N_{i,j-l,l}^{paid}|\mathcal{F}_t) + \mu^{2,(t)} Var(N_{i,j-l,l}^{paid}|\mathcal{F}_t)\right) \\
&= \sum_{l=i+j-I}^{\min\{j,d\}} \left(\sigma^{2,(t)} n_{i,j-l}^{paid} q_l + \mu^{2,(t)} n_{i,j-l}^{paid} q_l^{(t)}(1-q_l^{(t)})\right) \\
&= \sum_{l=i+j-I}^{\min\{j,d\}} n_{i,j-l}^{paid} q_l^{(t)} \left(\sigma^{2,(t)} + \mu^{2,(t)}(1-q_l^{(t)})\right) \\
&\approx \mu^{(t)}\phi^{(t)}\lambda_i^{(t)} \sum_{l=i+j-I}^{\min\{j,d\}} n_{i,j-l}^{paid} q_l^{(t)},
\end{aligned}
$$

where $\phi^{(t)} = (\sigma^{2,(t)} + \mu^{2,(t)})/\mu^{(t)}$ is the over-dispersion parameter.

Since $X_{ij}^{IBNR,(t)} = E(X_{ij}^{IBNR}|\mathcal{F}_t)$ and $X_{ij}^{RBNS,(t)} = E(X_{ij}^{RBNS}|\mathcal{F}_t)$, the conditional mean square error of prediction for these two predictions are $MSEP(X_{ij}^{IBNR,(t)}|\mathcal{F}_t) = \mu^{(t)}\phi^{(t)}\lambda_i^{(t)} \sum_{l=0}^{i+j-I-1} p_{j-l}^{(t)} q_l^{(t)}$ and $MSEP(X_{ij}^{RBNS,(t)}|\mathcal{F}_t) \approx \mu^{(t)}\phi^{(t)}\lambda_i^{(t)} \sum_{l=i+j-I}^{\min\{j,d\}} n_{i,j-l}^{paid} q_l^{(t)}$, respectively.

To use the above finding, all model parameters must be given, or have to be estimated based on available information in $\mathcal{F}_t$. The next section considers this issue.

A. Parameter estimation

The log-likelihood function based upon the observed data up to observation date t is

$$logL = \sum_i \sum_j (n_{i,j} \log(\lambda_i) + n_{i,j} \log(p_j) - \lambda_i p_j - log(n_{i,j}!)). \quad (5)$$

Therefore, the maximum likelihood estimator for λ_i is $\hat{\lambda}_i = \sum_{j=i}^{t} n_{ij} / \sum_{j=i}^{t} p_j$.

Substituting $\hat{\lambda}_i$ in the above log-likelihood function leads to

$$
\begin{aligned}
logL \;\propto\; & \sum_{i=1}^{t}\sum_{j=i}^{t} [n_{ij}(log(\sum_{j=i}^{t} n_{ij}) - log(\sum_{j=i}^{t} p_j) - \sum_{j=i}^{t} n_{ij}] \quad (6) \\
& + \sum_{i=1}^{t}\sum_{j=i}^{t} n_{ij} log(p_j)
\end{aligned}
$$

The above log-likelihood function can be understood as a log-likelihood function for truncated reporting delay random variable where truncation point $(t-i)$ is the maximal observed delay for a claim that was incurred at accident time i.

The log-likelihood given by Equation (6) depends on the parameters of both the Poisson model for claim occurrence and the reporting delay distribution. It is not straightforward and reasonable to calculate the maximum likelihood estimation with respect to p_j. The complications of parameters estimation are simplified by applying the expectation-maximization, say EM, algorithm for delay probability. An overview of the EM algorithm is provided in Algorithm (1).

Algorithm 1: Estimate delay probabilities.

Input: Initial parameters ψ.
Output: Parameters estimation.
Set $k = 0$;
while *the values will be converged* **do**
  Expectation step: Estimate the values of missing or incomplete data by using the observed data. It is used to update the variables. After the first iteration of the EM, the new estimators ψ^1 for ψ are obtained. In this step, we estimate unobserved data;
  Maximization step: Use the data arrived from the expectation step, and compute an updated maximum likelihood estimate of unknown parameters;
  Set $k \leftarrow k+1$.

Please see da Silva and Yongacoglu (2015) [9] or Mouret et al. (2022) [23] for more details about the EM algorithm.

To estimate the p_j, at first one has to estimate the unknown parameters ψ, given by Definition (1).

Consider the random variable U has a mixture of k-Gamma distribution. Now, we introduce the EM algorithm in the context of Gamma mixture models. To find the maximum likelihood estimators with the EM algorithm, we can introduce a sample $\mathbf{v} = (v_1, \cdots, v_m)$ of the random variable V which indicates which of the k component densities was observed for each m; $v_m \in \{1, \cdots, k\}$. We shall call $\{U, V\}$ the complete data set, and we say U is incomplete. Now suppose that $f_{U,V}$ stands for the joint density function of U and V have joint density. Therefore, the log-likelihood is given by

$$log[L(\psi|\mathbf{u}, \mathbf{v})] = log[\prod_{m=1}^{n} f_{U,V}(u_m, v_m; \psi)].$$

And for given v_m, we have

$$log[L(\psi|\mathbf{u}, \mathbf{v})] = \sum_{m=1}^{n} log[w_{v_m} f_{v_m}(v_m; \psi_{v_m})].$$

We guess the parameters $\psi^g = (w_1^g, \cdots, w_m^g, \alpha_1^g, \cdots, \alpha_m^g, \theta_1^g, \cdots, \theta_m^g)$ of the mixing density. Now, we use the EM algorithm to update the parameters at each step, i.e.

$$
\begin{aligned}
Q(\psi, \psi^g) &= E_V\{log[L(\psi|\mathbf{u}, \mathbf{v})]|\mathbf{u}, \psi^g\} \\
&= \sum_V \sum_{m=1}^{n} log[w_{v_m} f_{v_m}(u_m; \theta_{v_m})] \\
&\quad \times \prod_{m=1}^{n} f_{V|U=u_m}(v_m; \psi^g)
\end{aligned}
$$

$Q(\psi, \psi^g)$ is the expectation of the complete-data log-likelihood evaluated for parameter value ψ. From a series of simplifying steps, the objective function is

$$Q(\psi, \psi^g) = \sum_{h=1}^{k}\sum_{m=1}^{n} log(w_h) f_{V|U=u_m}(h, \psi^g) + \sum_{h=1}^{k}\sum_{m=1}^{n} log[f_h(U_m;\theta_h)] f_{V|U=u_m}(h, \psi^g)$$

where $f_{V|U=u_i}(h, \psi^g) = A_h / \sum_{j=1}^{k} A_j$ and $A_h = w_h \theta_{h(g)}^{\alpha_{h(g)}} u^{\alpha_{h(g)}-1} e^{-u\theta_{h(g)}} / \Gamma(\alpha_{h(g)})$.

Moreover, $log[f_h(u_m;\theta_h)] = \alpha_h log(\theta_h) + (\alpha_h - 1) log(u_m) - \theta_h u_m - log(\Gamma(\alpha_h))$.

In this case, an analytical solution is possible. We differentiate $Q(h, \psi^g)$ with respect to each parameter, set the expressions equal to zero, and solve for the parameter. For each w_h we have the restriction $\sum_{h=1}^{k} w_h = 1$. Thus, we employ a Lagrangian method with a Lagrange multiplier parameter β and obtain

$$\frac{\partial[Q(\psi, \psi^g) + \beta(\sum_{h=1}^{k} w_h - 1)]}{\partial w_h} = 0$$
$$\iff \frac{1}{w_h}\sum_{m=1}^{n} f_{V|U=u_m}(h, \psi^g) + \beta = 0$$
$$\iff w_h = \frac{-\sum_{m=1}^{n} f_{V|U=u_m}(h, \psi^g)}{\beta}.$$

Summing over h leads to $\sum_{h=1}^{k}\sum_{m=1}^{n} f_{V|U=u_m}(h, \psi^g) = n$. Using the fact that $\sum_{h=1}^{k} w_h = 1$, we get $\beta = -n$, therefore, in each iteration g of the algorithm, for each w_h, the MLE is

$$\hat{w}_h = \frac{\sum_{m=1}^{n} f_{V|U=u_m}(h, \psi^g)}{n}$$

and

$$\hat{\theta}_h = \frac{\sum_{m=1}^{n} u_m f_{V|U=u_m}(h, \psi^g)}{\hat{\alpha}_h^{MLE} \sum_{m=1}^{n} u_m f_{V|U=u_m}(h, \psi^g)}.$$

The MLE for each α_h does not have an explicit solution, therefore it has to be found numerically.

Now, we estimate other parameters, q_l and ϕ. Based on Model Assumption (1), the mass function of N_{ijl}^{paid} given N_{ij} follows a multinomial distribution with probabilities q_l. The settlement delay probabilities, q_l, can be found through an MLE method. The likelihood of observed data is

$$P(N_{i,j,0}^{paid} = n_{i,j,0}, \cdots, N_{i,j-l,l}^{paid} = n_{i,j-l,l} | N_{ij} = n_{ij}) = \prod_{i=1}^{I}\prod_{j=0}^{I-1}\prod_{l=0}^{a_{j,I}} \frac{n_{ij}!}{n_{i,j,0}! n_{i,j-1,1}! \cdots n_{i,j-l,l}!} q_0^{n_{i,j,0}} \cdots q_l^{n_{i,j-l,l}},$$

where $a_{j,I} = min\{j, I-1\}$.

Given observed values, the log-likelihood function, denoted by L, is

$$L = \sum_{i=0}^{I}\sum_{j=0}^{I-1}\sum_{l=0}^{a_{j,I}} log(n_{ij}!) - \sum_{i=0}^{I}\sum_{j=0}^{I-1}\sum_{l=0}^{a_{j,I}} log(n_{i,j-l,l}!) + \sum_{i=0}^{I}\sum_{j=0}^{I-1}\sum_{l=0}^{a_{j,I}} n_{i,j-l,l} log(q_l).$$

Using the fact that $\sum_{l=0}^{I-1} q_l = 1$, the above log-likelihood function can be restated in the context of the Lagrange method with a Lagrange multiplier parameter β as the following,

$$L^* = \sum_{i=0}^{I}\sum_{j=0}^{I-1}\sum_{l=0}^{a_{j,I}} log(n_{ij}!) - \sum_{i=0}^{I}\sum_{j=0}^{I-1}\sum_{l=0}^{a_{j,I}} log(n_{i,j-l,l}!) + \sum_{i=0}^{I}\sum_{j=0}^{I-1}\sum_{l=0}^{a_{j,I}} n_{i,j-l,l} log(q_l) - \beta(1 - \sum_{l=0}^{I-1} q_l),$$

where $a_{j,I} = min\{j, I-1\}$.

Taking a partial derivative with respect to β and q_l lead to

$$\frac{\partial L^*}{\partial \beta} = 1 - \sum_{l=0}^{I-1} q_l$$
$$\frac{\partial L^*}{\partial q_l} = \frac{\sum_{i=1}^{I}\sum_{j=0}^{I-1} n_{i,j-l,l}}{q_l} - \beta.$$

A straightforward calculation along with the fact that $\sum_{l=0}^{I-1} q_l = 1$ lead to

$$\hat{\beta} = \sum_{i=1}^{I}\sum_{j=l}^{I-1}\sum_{l=0}^{min\{j,I-1\}} n_{i,j-l,l}$$
$$\hat{q}_l = \frac{\sum_{i=1}^{I}\sum_{j=l}^{I-1} n_{i,j-l,l}}{\sum_{i=1}^{I}\sum_{j=0}^{I-1}\sum_{l=0}^{min\{j,I-1\}} n_{i,j-l,l}}.$$

III. A Practical Application

This section considers material damage motor third-party liability insurance claim portfolio from a private insurance company in Iran. We have 85649 claims in our data set. In this study, the time period is from 20-March-2012 to 19-March-2019. After a primary investigation and removing illogical events, such as transactions before the occurrence, recovery of claims, and allocated loss adjuster expenses, we just trusted information about 66346 claims.

Our data set is extracted from a heterogeneous population consisting of the number of claims with no reporting delay and the number of claims above 30 days. Figures (1.a and 1.b) illustrates such reporting delays in two cases, all claims, and claims that were reported for more than 30 days. As Figure (1.a) illustrates, there is a considerable number of zeros for claims which were reported in the first 30 days. The reporting delay is an important driver in the risk management strategy of the insurer, whose core business is underwriting risks. Figure 2 shows the claim reporting delay in the training set. We can see that our reporting delay data sets are right-skewed with two peaks and have a significant fraction of zeros. This indicates that the distribution might be a mixture distribution. One of the common elements of the finite mixture model is in the count data with excess zeros.

Now, we employ two well-known tests, the Kolmogorov-Smirnov and the Cramér-von Mises test to make a decision about the following hypothesis test.

H_0 : The random reporting delay has been distributed according to a ZIGM distribution

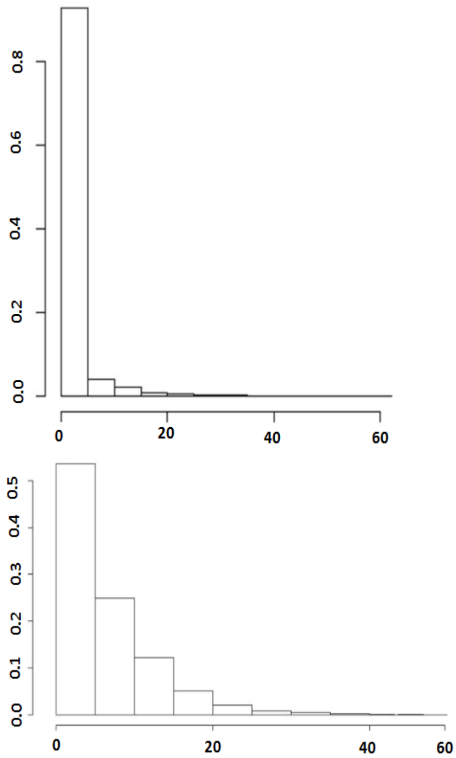

Fig. 1. Reporting delay relative frequency histogram for material damage observed between 2012-03-20 and 2019-03-19: Panel (a) represents all claims and Panel (b) represents claims that were reported to the insurance company more than 30 days after occurrence time.

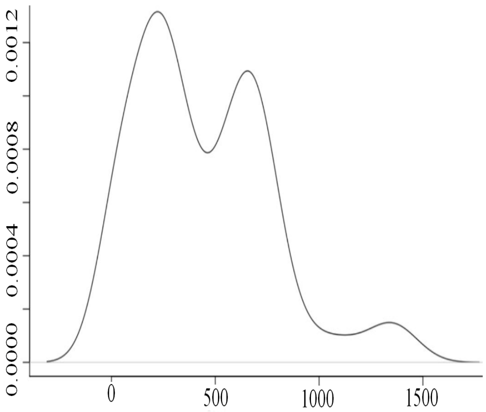

Fig. 2. Fitted distribution to reporting delay in daily scale for material damage observed between 2012-03-20 and 2019-03-19.

The p-value of these two tests are 0.1695 and 0.1458, respectively. Therefore, we failed to reject the null hypothesis at the confidence level of 0.95. Using *gammamixEM* function in *mixtools* package of the R software, releases that a zero-inflated two-Gamma mixture distribution is an appropriate distribution for the random reporting delay.

Note that to determine which distribution is the best to fit the data (the part of claims where the reporting delay is above 30 days), we fit various distributions to the data set. Based on the results, it appears that the two-Gamma mixture distribution fits the data pretty well.

More precisely, the density function for the random reporting delay U is

$$f_U(u) = 0.561 * I_{\{0\}}(u) + 0.093 * Gamma(1.241, 0.385)I_{(0,\infty)}(u) +0.346 * Gamma(1.412, 5.535))I_{(0,\infty)}(u).$$

Table (1) represents an estimation of the claim reporting delay probabilities p_k, homogeneous Poisson intensity λ_k and settlement delay probabilities q_{k-1}.

TABLE I
ESTIMATION OF CLAIM REPORTING DELAY PROBABILITIES p_k, NON-HOMOGENEOUS POISSON INTENSITY λ_k AND SETTLEMENT DELAY PROBABILITIES q_{k-1}.

k	p_k	λ_k	q_{k-1}	k	p_k	λ_k	q_{k-1}
1	0.5608	3.9437	0.6736	33	0	1704.662	0.0015
2	0.0435	51.268	0.0602	34	0	1616.915	0.0014
3	0.0287	105.4938	0.0222	35	0	1680.014	0.0014
4	0.0212	222.8187	0.0142	36	0	1396.068	0.0009
5	0.0153	194.227	0.0111	37	0	1353.673	0.0009
6	0.011	275.0727	0.0114	38	0	1218.602	0.0007
7	0.0078	335.214	0.0116	39	0	1161.418	0.0007
8	0.0055	416.0598	0.0115	40	0	962.2615	0.0005
9	0.0038	403.2428	0.0124	41	0	1147.615	0.0007
10	0.0027	437.7501	0.0127	42	0	879.1205	0.0003
11	0.0019	503.807	0.0124	43	0	1006.628	0.0003
12	0.0013	627.0474	0.0117	44	0	905.0779	0.0004
13	0.0009	509.7225	0.0116	45	0	914.9372	0.0003
14	0.0006	601.4134	0.0108	46	0	887.3313	0.0002
15	0.0004	589.5824	0.0101	47	0	909.0216	0.0002
16	0.0003	681.2733	0.0106	48	0	926.7682	0.0003
17	0.0002	656.6252	0.0087	49	0	799.5841	0.0004
18	0.0001	766.0627	0.0082	50	0	738.4568	0.0001
19	0.0001	770.9923	0.0075	51	0	866.6269	0.0001
20	0.0001	855.7817	0.0068	52	0	820.2885	0.0002
21	0	904.092	0.0064	53	0	871.5565	0.0001
22	0	979.0222	0.0058	54	0	862.6832	0.0001
23	0	1059.868	0.0051	55	0	915.9231	0
24	0	1335.927	0.0052	56	0	845.9226	0.0001
25	0	1221.559	0.0047	57	0	926.7683	0.0001
26	0	1369.448	0.0037	58	0	935.6417	0.0001
27	0	1548.886	0.0034	59	0	929.7262	0
28	0	1727.338	0.003	60	0	863.6694	0.0001
29	0	1730.296	0.003	61	0	944.5153	0
30	0	1879.171	0.0025	62	0	814.3734	0
31	0	1948.185	0.0025	63	0	933.6705	0
32	0	1708.606	0.0023	64	0	943.5302	0
65	0	712.8507	0	81	0	815.0098	0
66	0	1051.983	0	82	0	700.7997	0
67	0	1097.336	0	83	0	656.7171	0
68	0	936.6313	0	84	0	636.4467	0
69	0	961.2813	0.0002	85	0	620.3047	0
70	0	946.4949	0	86	0	669.8068	0
71	0	918.8925	0	87	0	636.5304	0
72	0	967.209	0	88	0	716.256	0
73	0	935.6667	0	89	0	736.5403	0
74	0	826.2512	0	90	0	799.5841	0
75	0	738.5181	0	91	0	738.4568	0
76	0	842.0805	0	92	0	866.6269	0
77	0	840.1545	0	93	0	820.2885	0
78	0	869.8067	0	94	0	871.5565	0
79	0	766.3469	0	95	0	862.6832	0
80	0	781.2723	0	96	0	915.9231	0

The policyholder's behavior in reporting claims after their occurrence has a significant effect on the costs of the insurance company. As mentioned before, in our data set, most policyholders tend to report the claim to the insurance company in less than 30 days. As a result, based on the above estimators, the number of IBNR claims for the next 12 months will be $\hat{N}^{IBNR} = \sum_{i,j} \hat{\lambda}_i \hat{p}_{ij} = 192$. We know the actual count for IBNR claims in the calendar year 20-March-2012 till 19-March-2020 is 248.

The estimates of the mean and variance of an individual payment are $\hat{\mu} = 13$ million IRR and $\hat{\sigma}^2 = 181$. Our estimation is distribution-free, i.e., it does not assume the distribution of the payments' data set. By using $\hat{\mu}$ and $\hat{\sigma}^2$, the over-dispersion parameter is $\phi = 27$.

In the third column in Table (1), we have the maximum likelihood estimator for the settlement delay probability, q_l (the numbers round to four decimals). It shows that 67% of automobile material damage claims are settled in the one month after they are reported to the insurance company. These are the cheapest claims based on their average cost calculations.

Using these parameter estimates, the IBNR reserve for the next 12 months is 2270 million IRR and the actual amount is 6150 million IRR. This difference is because of large economic inflation in claim amount severity. The RBNS reserve for the next 12 months, is 21507 million IRR and the actual amount is 18165 million IRR. As a result, the total reserve is 23777 million IRR, close to the actual amount, 24315 million IRR. We compared our findings with the usual chain ladder (CL) method under Mack's conditions, see Wüthrich and Merz (2008, chapter 3) [32] for more details. We use this method on the monthly run-off triangle. The number of run-off triangle's rows/columns is 96. We employed the `ChainLadder` package (in the statistical software `R`) against claim counts/payments of the run-off triangle to predict the corresponding loss reserves under the CL framework. We start by doing a regularity check of the data. The standardized residual values versus fitted values and calendar period, given in Figure 3, don't show any trends, therefore, one cannot reject the CL's assumptions.

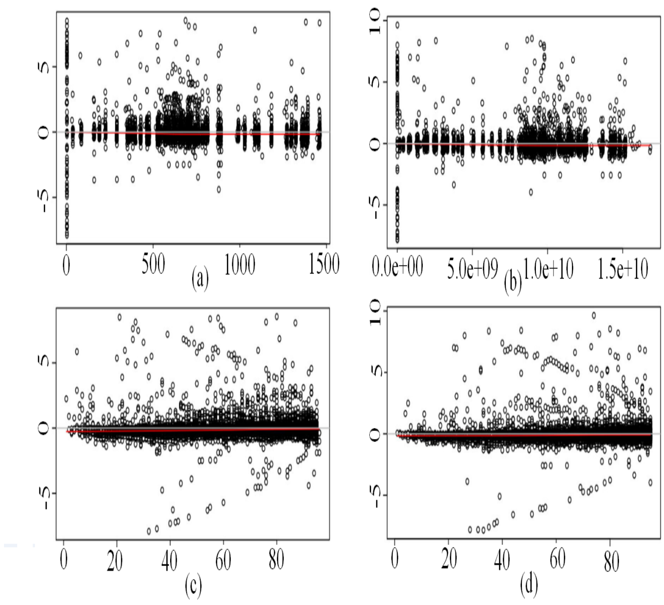

Fig. 3. The standardized residual values versus fitted values (Panel, a for claim counts and Panel b for claim payments) and the standardized residual values versus the calendar period (Panel, c for claim counts, and Panel d for claim payments), under the CL framework.

In Figure 4, we plot the estimated chain ladder parameters and the development factors in a monthly period.

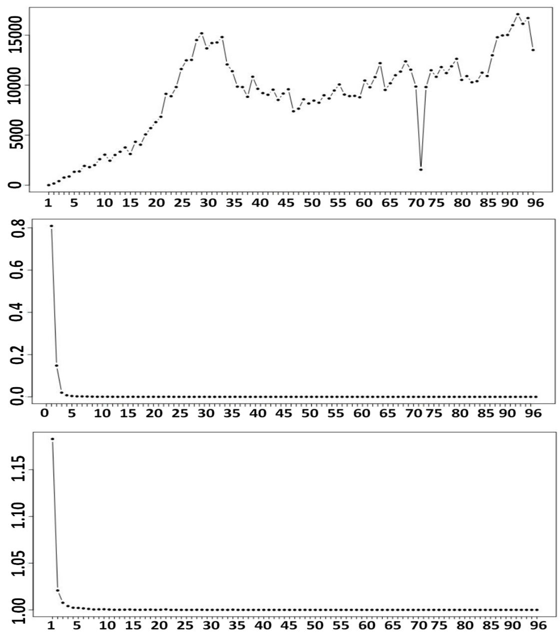

Fig. 4. The estimate of chain ladder parameters and the development factors in the monthly period.

In the CL method under Mack's conditions, the prediction of the number of IBNR claims is equal to 131. Thus, to compare with the real data, we conclude it has underestimated the number of claims. The results for the prediction of future payments are equal to 19814 million IRR. Thus, to compare with the real data, we conclude it has been underestimated in future payments. Moreover, in the CL method, it is not possible to split the IBNR and RBNS reserve and the reserve includes both of them.

Moreover, to study the accuracy of prediction methods, we approximate a 95% prediction interval by the monthly predictions ± 1.96 times the square of the conditional MSEP, i.e. (LCB, UCB), where

$$\begin{aligned} LCB &= \hat{X}_{ij}^{DCL} - 1.96\sqrt{MSEP_{\mathcal{F}_t}(\hat{X}_{ij}^{DCL})} \\ UCB &= \hat{X}_{ij}^{DCL} + 1.96\sqrt{MSEP_{\mathcal{F}_t}(\hat{X}_{ij}^{DCL})}). \end{aligned}$$

Figure 5 represents such point prediction along with its corresponding 95% predictions interval for the period of 20-March-2019 to 19-March-2020.

The length of the 95% predictions interval which is evaluated based on our model, for both the claim counts/payments, has been increased. However, we may observe a dramatic jump in the prediction's interval which is evaluated based on the CL model. These two observation may convince us to suggest our method rather than the CL one.

In Figure 6, we run 10000 bootstrap simulations, and the results are presented in the IBNR and RBNS.

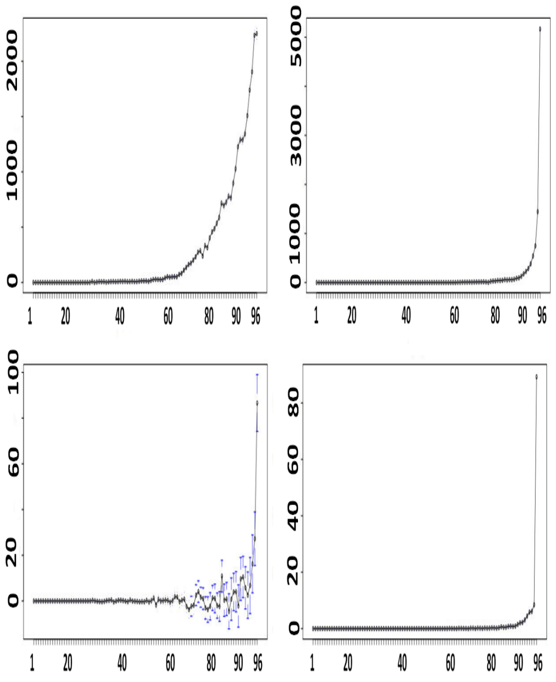

Fig. 5. A 95% prediction interval for the claim counts reserves (Panel, a evaluated under our model and Panel b evaluated under the CL model) and the claim payments reserves(Panel, c evaluated under our model and Panel d evaluated under the CL model).

IV. Conclusion and Suggestion

One of the most important questions in non-life insurance research is: "what is appropriate, in some sense, the method to predict claim counts/payments reserves for a given insurance contract"

This article employs a Zero-Inflated Gamma Mixture (ZIGM) framework to model the reporting delay probabilities. Moreover, it considers settlement delay probabilities in its model. We believe that employing the reporting/settlement delay probabilities allows one to provide more accurate predictions for claim counts/payments reserves. Given past observations, we study the prediction of future payments (in number and amount) and their prediction errors and derived reasonable expressions for them. The model advantage in the IBNR reserve is insurance companies can predict the number of future claims for each future date. This enables them to change the claim reporting process.

Our results for predicting the number of IBNR claims and the amount of IBNR and RBNS claims compared with real data were reasonable. Moreover, we compare our results with Mack's CL method. The results show that our method was better than the CL method. Also, in the CL method, it is not possible to split the IBNR and RBNS reserve and the reserve includes both of them.

The approach proposed in this article can be improved with additional information about claims, such as the seasons that claims occur, the zone of accident, online reporting claims, etc. These characteristics can be considered in the loss reserving model and make the prediction of amount more accurate.

Certainly, the same approach can be adopted with generalized classical cumulative shock models for claims arrival and reporting delay to insurance companies. For example, when we have a claim it causes a stream of payments from the insurer to the insured. We believe that this approach has the potential to create more accuracy in the practical approach to reserving.

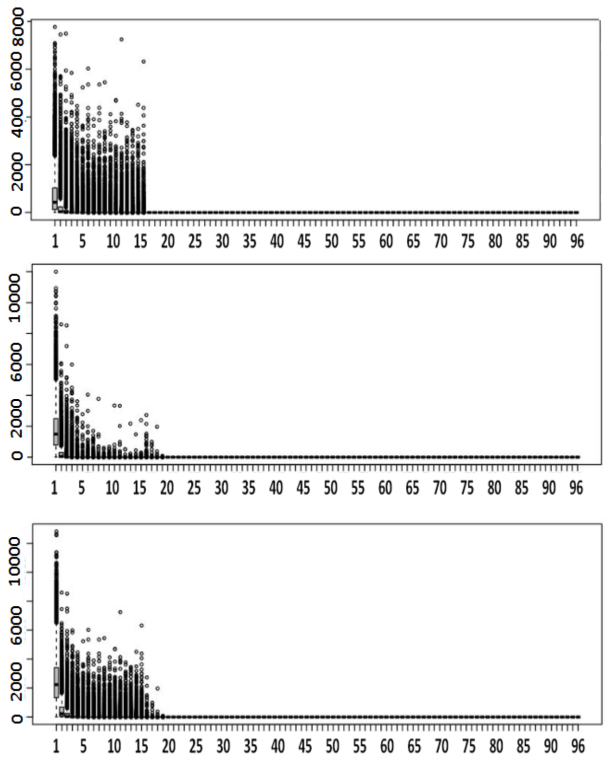

Fig. 6. Simulation results in splits of the IBNR and RBNS reserves.

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