The Design of Coalitional Model Predictive Control for Large-Scale Systems

Zaiful Ulum, Salmah*, Ari Suparwanto, and Solikhatun

Abstract—**This study provides a hierarchical control approach for large-scale systems composed of interconnected subsystems. A local controller or agent that can communicate via a network manages each subsystem. The upper layer is responsible of network topology selection to determine the optimal trade-off between the communication load and the control performance. The selected network topology splits the set of agents into cooperating groups of agents referred to as coalitions. At the bottom layer, each coalition computes the value of its control input variable by employing a decentralized model predictive control (MPC) scheme. Recursive feasibility and closed-loop stability analysis are carried out by considering fixed and varied network topologies. As a final point, a numerical simulation is carried out to verify the performance of the control scheme that is proposed.**

Index Terms—**coalitional control, model predictive control, hierarchical control, switched systems**

I. INTRODUCTION

OVER the last years, MPC-based non-centralized control schemes such as decentralized and distributed schemes are becoming more and more popular than MPC-based centralized schemes for controlling large-scale systems like traffic, irrigation, and power systems [1], [2], [3], [4], [5]. This is because the non-centralized scheme offers less computational demands, better fault tolerance, and more flexibility to apply to the system compared to the centralized one [6]. The idea of the non-centralized scheme is to assign a different local controller or agent for each subsystem to compute the local control input for its subsystem.

In the decentralized scheme, there is no communication among the agents in deciding their corresponding control actions. Therefore, this scheme may degrade the system's control performance and stability when the interactions between different subsystems are strong [7]. On the contrary,some information is shared among the agents to calculate their control inputs in the distributed scheme. This information plays an essential role in achieving performance close to those provided by the centralized approach. Furthermore, the basic assumption employed in the distributed control strategy is that the topology of the communication network is fixed, regardless of the alteration in the interaction among different subsystems.

Recently, a control strategy that adjusts the network topology to the coupling conditions between the agents has emerged, referred to as a coalitional control scheme [8]. This scheme aims to ensure that communication is carried out among only strongly coupled agents so that a control performance close to the centralized strategy is obtained while simultaneously reducing computational and communicational burdens. Furthermore, due to network topology adjustment, the agents can dynamically establish coalitions, i.e., groups of agents who cooperate in deciding their control inputs or work in a decentralized manner.

The coalitional control scheme initially appeared to control a set of unconstrained dynamically coupled linear systems [9]. In particular, a supervisor decides the network topology among the agents by solving a network topology optimization problem. Then, based on the selected network topology, linear matrix inequalities (LMIs) are solved subject to communication constraints to obtain the coalition's feedback matrix. Furthermore, the Shapley value, which is one of the tools to solve a cooperative game, is employed to investigate the relation of the distinct links and agents. This coalitional scheme is then developed by employing other tools from cooperative game theory, such as Harsanyi solutions [10] and the Banzhaf value [11]. In addition, a coalitional MPC scheme is proposed to deal with constrained systems in [12], where the network topology is determined through a pairwise bargaining procedure between MPC controllers/agents. Finally, input-to-state stability (ISS) is established under the assumption that the coupling between agents is weak and the system is recursively feasible.

Recursive feasibility and closed-loop stability issues in coalitional MPC are challenging to address, because the time-varying network topology leads the controlled system to the switched system. It is commonly understood that switching between multiple feasible and stable systems might result in an infeasible and unstable system. Since the coalitional MPC scheme in [12] doesn't address the feasibility issue, we propose a coalitional MPC scheme for constrained linear systems built by a number of dynamically interconnected subsystems that tackles the feasibility and stability issues. The recursive feasibility issue is addressed by modifying the decentralized MPC scheme for non-switched systems proposed in [13] to switched systems. Specifically, the disturbance-and-switch-robust control invariant (DS-RCI) sets proposed in [14] will be adapted to state the coalitional MPC optimization problem. Then, the result in [15] will be employed to address the stability of the closed-loop system. Furthermore, to validate the performance of the proposed control scheme, a numerical simulation is performed and compared to the performance of the centralized MPC

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Z. Ulum is a PhD candidate at Mathematics Department, Universitas Gadjah Mada, Yogyakarta 55281, Indonesia (email: zaifululum@mail.ugm.ac.id)

*Salmah (corresponding author) is a Professor at Mathematics Department, Universitas Gadjah Mada, Yogyakarta 55281, Indonesia (email: syalmah@yahoo.com)

A. Suparwanto is an Associate Professor at Mathematics Department, Universitas Gadjah Mada, Yogyakarta 55281, Indonesia (email: ari_suparwanto@mail.ugm.ac.id)

Solikhatun is an Associate Professor at Mathematics Department, Universitas Gadjah Mada, Yogyakarta 55281, Indonesia (email: solikhatun@mail.ugm.ac.id)

method.

The remaining of this work are structured as follows. First, in Section II, the problem to be solved is stated. Then, the control algorithm that includes the formulation of coalitional MPC optimization problems and the recursive feasibility and stability analysis is provided in Section III. Next, Section IV presents a simulation to assess the proposed control scheme's performance. Finally, conclusions are provided in Section V.

Notation 1. *A^T indicates the transpose of matrix A. $\mathbb{R}_+$ represents the set of all real numbers t such that $t \geq 0$, and $\mathbb{Z}_+$ represents the set of all integers k such that $k \geq 0$. $\mathbb{Z}_{\geq s}$ stands for the set of all $k \in \mathbb{Z}_+$ such that $k \geq s$ for some $s \in \mathbb{Z}_+$. A square matrix A is called Schur if its spectral radius is less than one. For two sets $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$, their Minkowski sum and Pontryagin difference are respectively defined as $\mathbb{X} \oplus \mathbb{Y} \triangleq \{x+y | x \in \mathbb{X}, y \in \mathbb{Y}\}$ and $\mathbb{X} \ominus \mathbb{Y} \triangleq \{x | x \oplus \mathbb{Y} \subseteq \mathbb{X}\}$. A strictly increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a $\mathcal{K}$-function if it is continuous and taking values zero at zero. $\mathbb{B}_\rho(y) \subset \mathbb{R}^n$ denotes the ball with center y and radius $\rho > 0$, i.e. the set of all $x \in \mathbb{R}^n$ such that $||x-y|| \leq \rho$, where $||.||$ denotes the Euclidian norm in $\mathbb{R}^n$. A set $\mathbb{X}$ is said to be robust positively invariant (RPI) for the system $x(k+1) = f(x(k), w(k))$ if the state trajectory $\{x(k)\}_{k=0}^{\infty}$ starting at $\mathbb{X}$ always stays at $\mathbb{X}$ for any $w(k) \in \mathbb{W}, k = 0, 1, 2, ...$*

II. Problem Formulation

A. System Description

We consider a system that can be split into a set $\mathcal{N} = \{1, 2, \dots, M\}$ of coupled subsystems. The dynamic and the constraints of each subsystem $i \in \mathcal{N}$ are given by

$$x_i(k+1) = A_{ii}x_i(k) + B_{ii}u_i(k) + d_i(k), \qquad (1a)$$

$$d_i(k) = \sum_{j \in \mathcal{M}_i} \left(A_{ij}x_j(k) + B_{ij}u_j(k)\right), \qquad (1b)$$

$$x_i \in \mathbb{X}_i \subset \mathbb{R}^{n_i}, \quad u_i \in \mathbb{U}_i \subset \mathbb{R}^{q_i}, \qquad (1c)$$

where $k \in \mathbb{Z}_+$ denotes the discrete-time index, and $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{q_i}$ are the state and input vectors of subsystem i. Matrices $A_{ii} \in \mathbb{R}^{n_i \times n_i}$ and $B_{ii} \in \mathbb{R}^{n_i \times q_i}$ are transition and input to state matrices of subsystem i, respectively. Vector $d_i \in \mathbb{R}^{n_i}$ gathers the effect of the state and input of all subsystems in the set $\mathcal{M}_i$ on the dynamic of subsystem i, where $\mathcal{M}_i$ is defined as $\mathcal{M}_i = \{j \in \mathcal{N} | A_{ij} \neq 0 \vee B_{ij} \neq 0\}$.

Assumption 1. *$\mathbb{X}_i$ and $\mathbb{U}_i$ are compact polyhedral sets having the origin within their interior for all $i \in \mathcal{N}$.*

B. Information Exchange

A different controller or agent manage each subsystem $i \in \mathcal{N}$ with access only to its state x_i. Therefore, the number of agents is equal to the number of subsystems. The symbol $\mathcal{N}$ is also used to denote the set of agents from now on. Each agent $i \in \mathcal{N}$ has a task to compute the value of its input variable u_i at every time step. For this purpose, agents can communicate through a communication network represented by the undirected graph $(\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of agents, and $\mathcal{L}$ is the set of links connecting them. Specifically, the set $\mathcal{L}$ is defined as

$$\mathcal{L} \subseteq \mathcal{L}^{\mathcal{N}} = \{\{i,j\} | \{i,j\} \subseteq \mathcal{N}, i,j \in \mathcal{N}, i \neq j\}. \qquad (2)$$

Each link $\{i, j\} \in \mathcal{L}$ may be enabled or disabled, with each enabled link having a specific stage cost $c > 0$. When the link $\{i, j\}$ is enabled, agents i and j can communicate so that the states x_i and x_j are respectively available for agents j and i. In addition, any two indirectly connected agents can communicate as long as there is a path of activated links between them. Hereafter, the symbol g is used to denote the set of all enabled links, referred to as *topology*, and the set of all possible topologies is represented by $G = \{g | g \subseteq \mathcal{L}\}$. Consider that if the cardinality of $\mathcal{L}$ is $|\mathcal{L}|$, then the cardinality of G is $2^{|\mathcal{L}|}$.

Any topology $g \in G$ splits the set of agents $\mathcal{N}$ into disjoint sets of cooperation groups of agents, referred to as *coalitions*. In addition, two agents are inside the same coalition if and only if they are either directly connected through an enabled link or indirectly through a path of enabled links. Furthermore, the set of all coalitions induced by the topology g is denoted by $\Pi(g) = \{\mathcal{C} | \mathcal{C} \subseteq \mathcal{N}\}$, where $\mathcal{C} \neq \emptyset$ for all $\mathcal{C} \in \Pi(g)$, $\mathcal{C} \cap \mathcal{C}' = \emptyset$ for any pair $\mathcal{C}, \mathcal{C}' \in \Pi(g)$ and $\bigcup_{\mathcal{C} \in \Pi(g)} \mathcal{C} = \mathcal{N}$. Hence, the set $\Pi(g)$ is a partition of the set $\mathcal{N}$. The cardinality of the set $\Pi(g)$, denoted by $|\Pi(g)|$, ranges from 1 to M.

All agents in the same coalition $\mathcal{C} \in \Pi(g)$ determine their control input cooperatively and act as a single system. Let $x_\mathcal{C} = (x_i)_{i \in \mathcal{C}} \in \mathbb{R}^{n_\mathcal{C}}$ and $u_\mathcal{C} = (u_i)_{i \in \mathcal{C}} \in \mathbb{R}^{q_\mathcal{C}}$ be the state and control input of coalition $\mathcal{C}$ obtained by stacking the state and input vectors of all subsystems in $\mathcal{C}$, the dynamic of the coalition $\mathcal{C}$ is specified by

$$x_\mathcal{C}(k+1) = A_\mathcal{C}x_\mathcal{C}(k) + B_\mathcal{C}u_\mathcal{C}(k) + d_\mathcal{C}(k), \qquad (3)$$

where $A_\mathcal{C} = [A_{ij}]_{i,j \in \mathcal{C}} \in \mathbb{R}^{n_\mathcal{C} \times n_\mathcal{C}}$ and $B_\mathcal{C} = [B_{ij}]_{i,j \in \mathcal{C}} \in \mathbb{R}^{n_\mathcal{C} \times q_\mathcal{C}}$ with $n_\mathcal{C} = \sum_{i \in \mathcal{C}} n_i$ and $q_\mathcal{C} = \sum_{i \in \mathcal{C}} q_i$. Furthermore, the vector $d_\mathcal{C} = (d_i^\mathcal{C})_{i \in \mathcal{C}} \in \mathbb{R}^{n_\mathcal{C}}$ represents the influence of state and input of subsystems outside $\mathcal{C}$ on the dynamic of coalition $\mathcal{C}$. For each $i \in \mathcal{C}$, $d_i^\mathcal{C}$ is given by

$$d_i^\mathcal{C}(k) = \sum_{j \in \mathcal{M}_i \backslash \mathcal{C}} A_{ij}x_j(k) + B_{ij}u_j(k). \qquad (4)$$

From (1c), we have

$$x_\mathcal{C} \in \mathbb{X}_\mathcal{C}^g = \prod_{i \in \mathcal{C}} \mathbb{X}_i, \qquad u_\mathcal{C} \in \mathbb{U}_\mathcal{C}^g = \prod_{i \in \mathcal{C}} \mathbb{U}_i. \qquad (5)$$

Note that $\mathbb{X}_\mathcal{C}^g$ and $\mathbb{U}_\mathcal{C}^g$ are compact polyhedral sets including the origin inside their interior based on Assumption 1. Moreover, from (4), we have

$$d_i^\mathcal{C} \in \mathbb{D}_i^\mathcal{C} = \bigoplus_{j \in \mathcal{M}_i \backslash \mathcal{C}} A_{ij}\mathbb{X}_j \oplus B_{ij}\mathbb{U}_j. \qquad (6)$$

Therefore, the constraint of $d_\mathcal{C}$ is as follows.

$$d_\mathcal{C} \in \mathbb{D}_\mathcal{C}^g = \prod_{i \in \mathcal{C}} \mathbb{D}_i^\mathcal{C}. \qquad (7)$$

Since $\mathbb{X}_j$ and $\mathbb{U}_j$ are bounded and include the origin inside their interior for all $j \in \mathcal{M}_i \backslash \mathcal{C}$ based on Assumption 1, then $\mathbb{D}_i^\mathcal{C}$ is also bounded and includes the origin within its interior. Consequently, $\mathbb{D}_\mathcal{C}^g$ is bounded that contains the origin within its interior.

Denoting the aggregate of state and input vectors of all coalitions $\mathcal{C} \in \Pi(g)$ as $x_\mathcal{N} = (x_\mathcal{C})_{\mathcal{C} \in \Pi(g)}$ and $u_\mathcal{N} =$

$(u_{\mathcal{C}})_{\mathcal{C}\in\Pi(g)}$, the dynamic of the overall system corresponding to the topology g can be modeled as

$$x_{\mathcal{N}}(k+1) = A_{\mathcal{N}}x_{\mathcal{N}}(k) + B_{\mathcal{N}}u_{\mathcal{N}}(k), \quad (8)$$

where $A_{\mathcal{N}} = [A_{\mathcal{C}}]_{\mathcal{C}\in\Pi(g)}$ and $B_{\mathcal{N}} = [B_{\mathcal{C}}]_{\mathcal{C}\in\Pi(g)}$. In addition, the constraints of the overall system are

$$x_{\mathcal{N}} \in \mathbb{X}^g_{\mathcal{N}} = \prod_{\mathcal{C}\in\Pi(g)} \mathbb{X}^g_{\mathcal{C}}, \quad u_{\mathcal{N}} \in \mathbb{U}^g_{\mathcal{N}} = \prod_{\mathcal{C}\in\Pi(g)} \mathbb{U}^g_{\mathcal{C}}. \quad (9)$$

Since $\mathbb{X}^g_{\mathcal{C}}$ and $\mathbb{U}^g_{\mathcal{C}}$ are compact polyhedral sets containing the origin in their interior for all $g \in \Pi(g)$, then $\mathbb{X}^g_{\mathcal{N}}$ and $\mathbb{U}^g_{\mathcal{N}}$ are also compact polyhedral sets containing the origin in their interior.

Regarding systems (3) and (8), the following decentralized stabilization assumption is introduced.

Assumption 2. *For each topology $g \in G$, a block-diagonal matrix $K^g_{\mathcal{N}} = diag(K^g_{\mathcal{C}})_{\mathcal{C}\in\Pi(g)}$ exists, with $K^g_{\mathcal{C}} \in \mathbb{R}^{q_{\mathcal{C}}\times n_{\mathcal{C}}}$, such that (i) $F^g_{\mathcal{N}} \triangleq A_{\mathcal{N}} + B_{\mathcal{N}}K^g_{\mathcal{N}}$ is Schur, (ii) $F^g_{\mathcal{C}} \triangleq A_{\mathcal{C}} + B_{\mathcal{C}}K^g_{\mathcal{C}}$ is Schur for all $\mathcal{C} \in \Pi(g)$.*

The algorithm to compute the matrix $K^g_{\mathcal{N}}$ satisfying Assumption 2 is provided in [9].

C. Coalitional Control Problem

Coalitional control schemes aim to seek the best trade-off between the control performance and the communicational cost by periodically altering the network topology. To this end, the objective function below is minimized at every time step k.

$$\sum_{t=0}^{\infty} \sum_{\mathcal{C}\in\Pi(g(k))} \ell^{g(k)}_{\mathcal{C}}(x_{\mathcal{C}}(t|k), u_{\mathcal{C}}(t|k)) + c|g(k)| \quad (10a)$$

subject to

$$x_{\mathcal{C}}(t+1|k) = A_{\mathcal{C}}x_{\mathcal{C}}(t|k) + B_{\mathcal{C}}u_{\mathcal{C}} + d_{\mathcal{C}}(t|k), \ \forall \mathcal{C} \in \Pi(g(k)) \quad (10b)$$

$$x_{\mathcal{C}}(t|k) \in \mathbb{X}^{g(k)}_{\mathcal{C}}, u_{\mathcal{C}}(t|k) \in \mathbb{U}^{g(k)}_{\mathcal{C}}, \ \forall \mathcal{C} \in \Pi(g(k)) \quad (10c)$$

$$g(k) \in G, \quad (10d)$$

where $g(k)$ denotes the network topology at time step k. $x_{\mathcal{C}}(t|k)$ and $u_{\mathcal{C}}(t|k)$ respectively denote the prediction of $x_{\mathcal{C}}$ and $u_{\mathcal{C}}$ at time step $k+t$ based on the measurement at time step k, with $x_{\mathcal{C}}(0|k) = x_{\mathcal{C}}(k)$ for all $\mathcal{C} \in \Pi(g(k))$. The first and second terms of (10a) respectively represent the cost-to-go and the communicational cost. In order to regulate the state of the overall system towards the origin, the stage cost $\ell^{g(k)}_{\mathcal{C}}$ in (10a) is defined as

$$\ell^{g(k)}_{\mathcal{C}}(x_{\mathcal{C}}(k), u_{\mathcal{C}}(k)) = x^T_{\mathcal{C}}(k)Q^{g(k)}_{\mathcal{C}}x_{\mathcal{C}}(k) + u^T_{\mathcal{C}}(k)R^{g(k)}_{\mathcal{C}}u_{\mathcal{C}}(k), \quad (11)$$

where $Q^{g(k)}_{\mathcal{C}} = \text{diag}(Q_i)_{i\in\mathcal{C}}$ and $R^{g(k)}_{\mathcal{C}} = \text{diag}(R_i)_{i\in\mathcal{C}} \in \mathbb{R}^{q_{\mathcal{C}}\times q_{\mathcal{C}}}$.

Assumption 3. *Matrices $Q_i \in \mathbb{R}^{n_i\times n_i}$ and $R_i \in \mathbb{R}^{q_i\times q_i}$ are positive definite for all $i \in \mathcal{N}$.*

The decision variables of the optimization problem (10) consist of continuous and integer variables. More specifically, the input control variables are continuous, and the network topology selection leads to integer variables. Hence, the problem (10) is a mixed-integer optimization problem, which is generally impractical to solve. In addition, the cardinality of the set of all network topologies generally increases exponentially as the number of links increases. Therefore, a hierarchical multi-agent control algorithm will be proposed to address these issues to get a sub-optimal solution for the problem. (10).

III. Control Algorithm

The hierarchical multi-agent control algorithm which yields an approximate solution to the problem (10) is presented in this section. The top layer is in charge of selecting the network topology, and the bottom layer is responsible for computing the control input. For this purpose, we provide the following assumption.

Assumption 4. *For each $g \in G$, there exists a positive definite matrix $P^g_{\mathcal{N}} = diag(P^g_{\mathcal{C}})_{\mathcal{C}\in\Pi(g)}$ that, $\forall x_{\mathcal{N}}(k)$, satisfies*

$$\begin{aligned} x_{\mathcal{N}}(k)^T P^g_{\mathcal{N}} x_{\mathcal{N}}(k) &= \sum_{\mathcal{C}\in\Pi(g)} x^T_{\mathcal{C}}(k)P^g_{\mathcal{C}}x_{\mathcal{C}}(k) \\ &\geq \sum_{t=0}^{\infty}\sum_{\mathcal{C}\in\Pi(g)} \ell^g_{\mathcal{C}}(x_{\mathcal{C}}(t|k), u_{\mathcal{C}}(t|k)) \end{aligned}$$

when the control feedback law $u_{\mathcal{N}}(k) = K^g_{\mathcal{N}}x_{\mathcal{N}}(k)$ is applied to the system (8).

The matrices $K^g_{\mathcal{N}}$ and $P^g_{\mathcal{N}}$ that meet Assumption 2 and Assumption 4, respectively, could be computed by using a method in [9]. Note that the matrix $P^g_{\mathcal{N}}$ provides an upper bound on the cost-to-go. Therefore, the upper bound on the cost function (10a) is given by

$$r(g(k), x_{\mathcal{N}}(k)) = x_{\mathcal{N}}(k)^T P^{g(k)}_{\mathcal{N}} x_{\mathcal{N}}(k) + c|g(k)|. \quad (12)$$

Given the value of the state $x_{\mathcal{N}}(k)$, the following optimization should be solved at the top layer to determine the network topology.

$$\min_{g(k)} r(g(k), x_{\mathcal{N}}(k)) \quad (13a)$$

subject to

$$g(k) \in G. \quad (13b)$$

Note the cardinality of G grows exponentially to the cardinality of $\mathcal{L}$. Therefore, the problem (13) will be solved at some given time step to reduce the computational burden.

At the bottom layer, each coalition $\mathcal{C} \in \Pi(g(k))$ computes the value of its corresponding input variable in a decentralized manner. The term decentralized here refers to the absence of communication among coalitions when they calculate their control input. Furthermore, the decentralized MPC scheme in [13] will be employed to compute the input of the overall system to address constraint satisfaction and stability of closed-loop system issues. In addition, since the coalitional control scheme leads the system under consideration to the switched system, the decentralized MPC scheme will be modified for the switched system.

Remark 1. *The dynamic of each subsystem considered in this research differs from [13], since the control input coupling $B_{ij}u_j(k)$ in (1b) is neglected in [13], i.e. $B_{ij}u_j(k) = 0$ for all $j \in \mathcal{M}_i$.*

A. MPC for Coalitions

The decentralized MPC scheme in [13] is developed based on the tube-based robust MPC technique in [16]. The tube-based robust MPC approach works on the nominal system corresponding to the actual system and the tightened constraints to obtain the trajectory of the nominal state and control input over a finite horizon. The trajectory of the nominal state is then used as a reference for the state of the actual system. To ensure the actual state near to the nominal state,a feedback control is applied to the actual system.

Treating the coupling term $d_{\mathcal{C}}$ as a disturbance, the nominal system that corresponds to (3) is given by

$$\hat{x}_{\mathcal{C}}(k+1) = A_{\mathcal{C}}\hat{x}_{\mathcal{C}}(k) + B_{\mathcal{C}}\hat{u}_{\mathcal{C}}(k), \tag{14}$$

in which $\hat{x}_{\mathcal{C}}$ and $\hat{u}_{\mathcal{C}}$ are the nominal state and input of coalition $\mathcal{C}$. Following [16], the control input for (3) at time step k is given by

$$u_{\mathcal{C}}(k) = \hat{u}_{\mathcal{C}}(k) + K^g_{\mathcal{C}}(x_{\mathcal{C}}(k) - \hat{x}_{\mathcal{C}}(k)). \tag{15}$$

Letting $z_{\mathcal{C}} = x_{\mathcal{C}} - \hat{x}_{\mathcal{C}}$, from (3), (14), and (15), we obtain

$$z_{\mathcal{C}}(k+1) = F^g_{\mathcal{C}} z_{\mathcal{C}}(k) + d_{\mathcal{C}}(k), \tag{16}$$

where $d_{\mathcal{C}} \in \mathbb{D}^g_{\mathcal{C}}$. Furthermore, since $F^g_{\mathcal{C}}$ is Schur and $\mathbb{D}^g_{\mathcal{C}}$ is bounded, an RPI set $\mathcal{Z}^g_{\mathcal{C}}$ exists for each $\mathcal{C} \in \Pi(g)$. To reduce conservativeness, the RPI set $\mathcal{Z}^g_{\mathcal{C}}$ should be as small as possible. Since $F^g_{\mathcal{C}}$ is Schur, the minimal RPI set is $\bigoplus_{j=0}^{\infty}(F^g_{\mathcal{C}})^j\mathbb{D}^g_{\mathcal{C}}$, according to [17]. However, to reduce the complexity of computation, the RPI set $\mathcal{Z}^g_{\mathcal{C}}$ in this research is computed by the following algorithm. This algorithm is the modification of Algorithm 1 in [18].

Algorithm 1: The computation of the RPI $\mathcal{Z}^g_{\mathcal{C}}$

1) Initialize $\mathcal{Z}^g_{\mathcal{C}} = \text{box}(\mathbb{D}^g_{\mathcal{C}})$ for all $\mathcal{C} \in \Pi(g)$.
2) Compute $\mathcal{Z}^{g^+}_{\mathcal{C}} = \text{box}(F^g_{\mathcal{C}}\mathcal{Z}^g_{\mathcal{C}}) \oplus \text{box}(\mathbb{D}^g_{\mathcal{C}})$ for all $\mathcal{C} \in \Pi(g)$.
3) If $\mathcal{Z}^{g^+}_{\mathcal{C}} \subseteq \mathcal{Z}^g_{\mathcal{C}}$ for all $\mathcal{C} \in \Pi(g)$, then stop. The desired RPI set is $\mathcal{Z}^g_{\mathcal{C}}$. Otherwise, set $\mathcal{Z}^g_{\mathcal{C}} = \mathcal{Z}^{g^+}_{\mathcal{C}}$ and repeat step 2.

The notation $\text{box}(X)$ indicates the smallest hyperrectangle that contains X with faces perpendicular to the cartesian axis. In particular, the *outerApprox* method from [19] can be used to compute the operator box. Algorithm 1 generates a compact set $\mathcal{Z}^g_{\mathcal{C}}$ since the operator box yields a compact set, and the Minkowski sum operation preserves the compactness property. In addition, the set $\mathcal{Z}^g_{\mathcal{C}}$ contains the origin in its interior since $\mathbb{D}^g_{\mathcal{C}}$ contains the origin in its interior.

The following assumption is needed to guarantee the feasibility and stability of switched systems.

Assumption 5. *(Dwell time): Let* $\mathcal{T} \triangleq \{k_0, k_1, \ldots, k_l, \ldots\} \subseteq \mathbb{Z}_+$, *with* $k_0 = 0$, *denote the set of switching instants. Then, for each topology* $g \in G$, *there exist a dwell time* τ_g *such that* $\tau_g \geq k_l - k_{l-1}$ *when* $g(k) = g$ *for all* $k \in [k_{l-1}, k_l)$, $l \in \mathbb{Z}_{\geq 1}$.

Let t_e denote the amount of time that has passed from the most current switching instant, with $t_e(0) = 0$. Let the network topology at time step k is $g(k) = g$ with dwell time τ_g. Then, the following optimization problem is solved by each coalition $\mathcal{C} \in \Pi(g)$ to compute its corresponding input $u_{\mathcal{C}}(k)$ when its state is $x_{\mathcal{C}}(k)$.

$$J^{g,N_p^*}_{\mathcal{C}}(x_{\mathcal{C}}(k)) = \min_{\substack{(\hat{x}_{\mathcal{C}}(0|k),\\ \{\hat{u}_{\mathcal{C}}(t|k)\}_{t=0}^{N_p-1})}} J^{g,N_p}_{\mathcal{C}}(\hat{x}_{\mathcal{C}}(0|k), \{\hat{u}_{\mathcal{C}}(t|k)\}_{t=0}^{N_p-1}), \tag{17a}$$

subject to

$$x_{\mathcal{C}}(k) - \hat{x}_{\mathcal{C}}(0|k) \in \mathcal{Z}^g_{\mathcal{C}}, \tag{17b}$$

$$\hat{x}_{\mathcal{C}}(t+1|k) = A_{\mathcal{C}}\hat{x}_{\mathcal{C}}(t|k) + B_{\mathcal{C}}\hat{u}_{\mathcal{C}}(t|k), \tag{17c}$$

$$\hat{x}_{\mathcal{C}}(t|k) \in \begin{cases} \hat{\mathbb{X}}^g_{\mathcal{C}}, & t < r(k) \\ \hat{\mathbb{T}}^g_{\mathcal{C}}, & t \geq r(k) \end{cases} \tag{17d}$$

$$\hat{u}_{\mathcal{C}}(t|k) \in \hat{\mathbb{U}}^g_{\mathcal{C}}, \tag{17e}$$

$$\hat{x}_{\mathcal{C}}(N_p|k) \in \hat{\mathbb{T}}^g_{\mathcal{C}}, \tag{17f}$$

where $\hat{\mathbb{X}}^g_{\mathcal{C}} = \mathbb{X}^g_{\mathcal{C}} \ominus \mathcal{Z}^g_{\mathcal{C}}$, $\hat{\mathbb{U}}^g_{\mathcal{C}} = \mathbb{U}^g_{\mathcal{C}} \ominus K^g_{\mathcal{C}}\mathcal{Z}^g_{\mathcal{C}}$, $N_p \geq \tau_g$ is the length of horizon prediction, and $r(k) = \tau_g - t_e(k)$. The set $\hat{\mathbb{T}}^g_{\mathcal{C}}$ is the terminal constraint of the nominal system (14) whose properties will be specified later. The objective function $J^{g,N_p}_{\mathcal{C}}$ is defined by

$$J^{g,N_p}_{\mathcal{C}}(\hat{x}_{\mathcal{C}}(0|k), \{\hat{u}_{\mathcal{C}}(t|k)\}_{t=0}^{N_p-1}) = \sum_{t=0}^{N_p-1} \ell^g_{\mathcal{C}}(\hat{x}_{\mathcal{C}}(t|k), \hat{u}_{\mathcal{C}}(t|k)) + V^g_{\mathcal{C}}(\hat{x}_{\mathcal{C}}(N_p|k)), \tag{18}$$

where the stage cost $\ell^g_{\mathcal{C}}$ is given by (11), and the terminal cost $V^g_{\mathcal{C}}$ is defined as

$$V^g_{\mathcal{C}}(\hat{x}_{\mathcal{C}}(N_p|k)) = \hat{x}^T_{\mathcal{C}}(N_p|k)P^g_{\mathcal{C}}\hat{x}_{\mathcal{C}}(N_p|k). \tag{19}$$

Let $(\hat{x}^*_{\mathcal{C}}(0|k), \{\hat{u}^*_{\mathcal{C}}(t|k)\}_{t=0}^{N_p-1})$ denote the optimal solution to the problem (17). Then the control input of coalitions $\mathcal{C}$ at time step k is given by

$$u^*_{\mathcal{C}}(k) = \hat{u}^*_{\mathcal{C}}(0|k) + K^g_{\mathcal{C}}(x_{\mathcal{C}}(k) - \hat{x}^*_{\mathcal{C}}(0|k)). \tag{20}$$

To guarantee the existence of the sets $\hat{\mathbb{X}}^g_{\mathcal{C}}$ and $\hat{\mathbb{U}}^g_{\mathcal{C}}$, the following assumption is assumed to hold.

Assumption 6. *Given a topology* $g \in G$. *For any coalition* $\mathcal{C} \in \Pi(g)$, *there exist* $\rho_{\mathcal{C},1} > 0$ *and* $\rho_{\mathcal{C},2} > 0$ *such that* $\mathcal{Z}^g_{\mathcal{C}} \oplus \mathbb{B}_{\rho_{\mathcal{C},1}}(0) \subseteq \mathbb{X}^g_{\mathcal{C}}$ *and* $K^g_{\mathcal{C}}\mathcal{Z}^g_{\mathcal{C}} \oplus \mathbb{B}_{\rho_{\mathcal{C},2}}(0) \subseteq \mathbb{U}^g_{\mathcal{C}}$ *where* $\mathbb{B}_{\rho_{\mathcal{C},1}}(0) \subset \mathbb{R}^{n_{\mathcal{C}}}$ *and* $\mathbb{B}_{\rho_{\mathcal{C},2}}(0) \subset \mathbb{R}^{q_{\mathcal{C}}}$.

The following algorithm summarizes the hierarchical control scheme proposed in this research.

Algorithm 2: The hierarchical control scheme

1) a) If $k \in \mathcal{T}$, all agents announce their state vector so that the problem (13) can be solved. The optimal solution of (13) is then selected as the next topology.
   b) If $k \notin \mathcal{T}$, each agent only shares their state vector with all agents inside the same coalition.
2) Each coalition $\mathcal{C} \in \Pi(g(k))$ solves the problem (17) to obtain the coalitional control input (20).

B. Feasibility and Stability Analysis

To analyze the feasibility and the stability of the system, we initiate by formulating the collective MPC optimization problem for all coalitions $\mathcal{C} \in \Pi(g)$. Letting $\tilde{A}_{\mathcal{N}} = \text{diag}(A_{\mathcal{C}})_{\mathcal{C}\in\Pi(g)}$, $\tilde{B}_{\mathcal{N}} = \text{diag}(B_{\mathcal{C}})_{\mathcal{C}\in\Pi(g)}$, $A'_{\mathcal{N}} = A_{\mathcal{N}} - \tilde{A}_{\mathcal{N}}$, and $B'_{\mathcal{N}} = B_{\mathcal{N}} - \tilde{B}_{\mathcal{N}}$, the overall system (8) can be rewritten as follows

$$x_{\mathcal{N}}(k+1) = \tilde{A}_{\mathcal{N}} x_{\mathcal{N}}(k) + \tilde{B}_{\mathcal{N}} u_{\mathcal{N}}(k) + d_{\mathcal{N}}(k), \quad (21)$$

where $d_{\mathcal{N}}(k) = (d_{\mathcal{C}})_{\mathcal{C}\in\Pi(g)} = A'_{\mathcal{N}} x_{\mathcal{N}}(k) + B'_{\mathcal{N}} u_{\mathcal{N}}(k)$. Furthermore, viewing $d_{\mathcal{N}}$ as a disturbance vector, the nominal system corresponding to (21) and its constraints are given by

$$\hat{x}_{\mathcal{N}}(k+1) = \tilde{A}_{\mathcal{N}} \hat{x}_{\mathcal{N}}(k) + \tilde{B}_{\mathcal{N}} \hat{u}_{\mathcal{N}}(k), \quad (22a)$$

$$\hat{x}_{\mathcal{N}} \in \hat{\mathbb{X}}^g_{\mathcal{N}} = \prod_{\mathcal{C}\in\Pi(g)} \hat{\mathbb{X}}^g_{\mathcal{C}},\ \hat{u}_{\mathcal{N}} \in \hat{\mathbb{U}}^g_{\mathcal{N}} = \prod_{\mathcal{C}\in\Pi(g)} \hat{\mathbb{U}}^g_{\mathcal{C}}, \quad (22b)$$

where $\hat{x}_{\mathcal{N}} = (\hat{x}_{\mathcal{C}})_{\mathcal{C}\in\Pi(g)}$ and $\hat{u}_{\mathcal{N}} = (\hat{u}_{\mathcal{C}})_{\mathcal{C}\in\Pi(g)}$. In view of (15), the control input of the system (21) is

$$u_{\mathcal{N}}(k) = \hat{u}_{\mathcal{N}}(k) + K^g_{\mathcal{N}}(x_{\mathcal{N}}(k) - \hat{x}_{\mathcal{N}}(k)). \quad (23)$$

Moreover, from (16), we have the error system between (21) and (22) as follows

$$z_{\mathcal{N}}(k+1) = F^g_{\mathcal{N}} z_{\mathcal{N}}(k) + d_{\mathcal{N}}(k), \quad (24)$$

in which $z_{\mathcal{N}} = (z_{\mathcal{C}})_{\mathcal{C}\in\Pi(g)}$.

Solving (17) for all $\mathcal{C} \in \Pi(g)$ is equivalent to solve the following problem.

$$J^{g,N_p^*}_{\mathcal{N}}(x_{\mathcal{N}}(k)) = \min_{\substack{(\hat{x}_{\mathcal{N}}(0|k),\\ \{\hat{u}_{\mathcal{N}}(t|k)\}_{t=0}^{N_p-1})}} J^{g,N_p}_{\mathcal{N}}(\hat{x}_{\mathcal{N}}(0|k), \{\hat{u}_{\mathcal{N}}(t|k)\}_{t=0}^{N_p-1}), \quad (25a)$$

subject to

$$x_{\mathcal{N}}(k) - \hat{x}_{\mathcal{N}}(0|k) \in \mathcal{Z}^g_{\mathcal{N}}, \quad (25b)$$

$$\hat{x}_{\mathcal{N}}(t+1|k) = \tilde{A}_{\mathcal{N}} \hat{x}_{\mathcal{N}}(t|k) + \tilde{B}_{\mathcal{N}} \hat{u}_{\mathcal{N}}(t|k), \quad (25c)$$

$$\hat{x}_{\mathcal{N}}(t|k) \in \begin{cases} \hat{\mathbb{X}}^g_{\mathcal{N}}, & t < r(k) \\ \hat{\mathbb{T}}^g_{\mathcal{N}}, & t \geq r(k) \end{cases} \quad (25d)$$

$$\hat{u}_{\mathcal{N}}(t|k) \in \hat{\mathbb{U}}^g_{\mathcal{N}}, \quad (25e)$$

$$\hat{x}_{\mathcal{N}}(N_p|k) \in \hat{\mathbb{T}}^g_{\mathcal{N}}, \quad (25f)$$

where $\mathcal{Z}^g_{\mathcal{N}} = \prod_{\mathcal{C}\in\Pi(g)} \mathcal{Z}^g_{\mathcal{C}}$, $\hat{\mathbb{X}}^g_{\mathcal{N}} = \prod_{\mathcal{C}\in\Pi(g)} \hat{\mathbb{X}}^g_{\mathcal{C}}$, $\hat{\mathbb{U}}^g_{\mathcal{N}} = \prod_{\mathcal{C}\in\Pi(g)} \hat{\mathbb{U}}^g_{\mathcal{C}}$, $\hat{\mathbb{T}}^g_{\mathcal{N}} = \prod_{\mathcal{C}\in\Pi(g)} \hat{\mathbb{T}}^g_{\mathcal{C}}$, and

$$J^{g,N_p}_{\mathcal{N}}(\hat{x}_{\mathcal{N}}(0|k), \{\hat{u}_{\mathcal{N}}(t|k)\}_{t=0}^{N_p-1}) = \sum_{t=0}^{N_p-1} \ell^g_{\mathcal{N}}(\hat{x}_{\mathcal{N}}(t|k), \hat{u}_{\mathcal{N}}(t|k)) + V^g_{\mathcal{N}}(\hat{x}_{\mathcal{N}}(N_p|k)), \quad (26)$$

with

$$\ell^g_{\mathcal{N}}(\hat{x}_{\mathcal{N}}(k), \hat{u}_{\mathcal{N}}(k)) = \sum_{\mathcal{C}\in\Pi(g)} \ell^g_{\mathcal{C}}(\hat{x}_{\mathcal{C}}(k), \hat{u}_{\mathcal{C}}(k)),$$

and

$$V^g_{\mathcal{N}}(\hat{x}_{\mathcal{N}}(k), \hat{u}_{\mathcal{N}}(k)) = \sum_{\mathcal{C}\in\Pi(g)} V^g_{\mathcal{C}}(\hat{x}_{\mathcal{C}}(k), \hat{u}_{\mathcal{C}}(k)).$$

Expanding the right-hand side of the last two equations, we have

$$\ell^g_{\mathcal{N}}(\hat{x}_{\mathcal{N}}(k), \hat{u}_{\mathcal{N}}(k)) = \hat{x}^T_{\mathcal{N}}(k) Q^g_{\mathcal{N}} \hat{x}_{\mathcal{N}}(k) + \hat{u}^T_{\mathcal{N}}(k) R^g_{\mathcal{N}} \hat{u}_{\mathcal{N}}(k) \quad (27)$$

and

$$V^g_{\mathcal{N}}(\hat{x}_{\mathcal{N}}(k)) = \hat{x}^T_{\mathcal{N}}(k) P^g_{\mathcal{N}} \hat{x}_{\mathcal{N}}(k), \quad (28)$$

where $Q^g_{\mathcal{N}} = \text{diag}(Q^g_{\mathcal{C}})_{\mathcal{C}\in\Pi(g)}$ and $R^g_{\mathcal{N}} = \text{diag}(R^g_{\mathcal{C}})_{\mathcal{C}\in\Pi(g)}$ are positive definite. In view of (20), the control input to (8) at time step k is given by

$$u^*_{\mathcal{N}}(k) = \hat{u}^*_{\mathcal{N}}(0|k) + K^g_{\mathcal{N}}(x_{\mathcal{N}}(k) - \hat{x}^*_{\mathcal{N}}(0|k)), \quad (29)$$

where $\hat{u}^*_{\mathcal{N}}(0|k) = (\hat{u}^*_{\mathcal{C}}(0|k))_{\mathcal{C}\in\Pi(g)}$ and $\hat{x}^*_{\mathcal{N}}(0|k) = (\hat{x}^*_{\mathcal{C}}(0|k))_{\mathcal{C}\in\Pi(g)}$.

Remark 2. *For the case fixed network topology, i.e. $g(k) = g$ for all $k \geq 0$, for any $g \in G$, the constraint (25d) becomes $\hat{x}_{\mathcal{N}}(t|k) \in \hat{\mathbb{X}}^g_{\mathcal{N}}$ since $\tau_g \to \infty$, so that $r(k) \to \infty$.*

The definition below provides the feasibility region of the problem (25).

Definition 7. *The feasibility region of the problem (25) is defined as*

$$\begin{aligned}\mathbb{X}^{g,N_p}_{\mathcal{N}} = \{ & x_{\mathcal{N}} \in \mathbb{X}^g_{\mathcal{N}} | \text{ if } x_{\mathcal{N}}(k) = x_{\mathcal{N}}, \text{ then there exists} \\ & (\hat{x}_{\mathcal{N}}(0|k), \{\hat{u}_{\mathcal{N}}(t|k)\}_{t=0}^{N_p-1}) \text{ such that} \\ & (25b) - (25f) \text{ are satisfied}\}.\end{aligned}$$

Note that the set $\mathbb{X}^{g,N_p}_{\mathcal{N}}$ is the domain of $J^{g,N_p^*}_{\mathcal{N}}(.)$.

To ensure the feasibility and stability of the proposed control scheme, the terminal cost $V^g_{\mathcal{N}}(.)$ and the terminal set $\hat{\mathbb{T}}^g_{\mathcal{N}}$ should satisfy the following assumption. This assumption is commonly used in the MPC design to stabilize the origin of the closed-loop system.

Assumption 8. *For each topology $g \in G$, it holds that*

1) *$\hat{\mathbb{T}}^g_{\mathcal{N}} \subseteq \hat{\mathbb{X}}^g_{\mathcal{N}}$ is an invariant set for $\hat{x}_{\mathcal{N}}(k+1) = (\tilde{A}_{\mathcal{N}} + \tilde{B}_{\mathcal{N}} K^g_{\mathcal{N}})\hat{x}_{\mathcal{N}}(k)$;*
2) *$\hat{u}_{\mathcal{N}}(k) = K^g_{\mathcal{N}} \hat{x}_{\mathcal{N}} \in \hat{\mathbb{U}}^g_{\mathcal{N}}$ for any $x(k) \in \hat{\mathbb{T}}^g_{\mathcal{N}}$;*
3) *$V^g_{\mathcal{N}}(\hat{x}_{\mathcal{N}}(k+1)) - V^g_{\mathcal{N}}(\hat{x}_{\mathcal{N}}(k)) \leq -\ell^g_{\mathcal{N}}(\hat{x}_{\mathcal{N}}(k), \hat{u}_{\mathcal{N}}(k))$, $\forall \hat{x}_{\mathcal{N}}(k) \in \hat{\mathbb{T}}^g_{\mathcal{N}}$.*

To maintain the feasibility of the overall system when the switching of the network topology appears, we introduce the following definition and assumption.

Definition 9. *Let Γ_g denote the global nominal system (22). Then the set of all $\hat{x}_{\mathcal{N}} \in \hat{\mathbb{X}}^g_{\mathcal{N}}$ that can be feasibly driven into $\hat{\mathbb{T}}^g_{\mathcal{N}}$ in h steps relative to Γ_g is defined as*

$$\begin{aligned}\text{Pre}^h_{\Gamma_g}(\hat{\mathbb{T}}^g_{\mathcal{N}}) = \{ & \hat{x}_{\mathcal{N}} \in \hat{\mathbb{X}}^g_{\mathcal{N}} | \text{ if } \hat{x}_{\mathcal{N}}(0) = \hat{x}_{\mathcal{N}}, \exists \{\hat{u}_{\mathcal{N}}(k)\}_{k=0}^{h-1}, \\ & \hat{u}_{\mathcal{N}}(k) \in \hat{\mathbb{U}}^g_{\mathcal{N}}, \forall k = 0, \ldots, h-1, \text{ such that} \\ & \hat{x}_{\mathcal{N}}(k) \in \hat{\mathbb{X}}^g_{\mathcal{N}}, \forall k = 0, \ldots, h-1, \\ & \hat{x}_{\mathcal{N}}(h) \in \hat{\mathbb{T}}^g_{\mathcal{N}}\},\end{aligned}$$

with $\text{Pre}^0_{\Gamma_g}(\hat{\mathbb{T}}^g_{\mathcal{N}}) = \hat{\mathbb{T}}^g_{\mathcal{N}}$.

The *reachableSet* method provided by [19] can be used to compute the set $\text{Pre}^h_{\Gamma_g}(\hat{\mathbb{T}}^g_{\mathcal{N}})$ by inserting the appropriate arguments.

Assumption 10. *For any $g \in G$, it holds*

1) *$\hat{\mathbb{T}}^g_{\mathcal{N}} \subseteq \text{Pre}^{\tau_{g'}}_{\Gamma_{g'}}(\hat{\mathbb{T}}^{g'}_{\mathcal{N}})$, $\forall g' \in G$;*

2) $\hat{\mathbb{T}}^g_{\mathcal{N}} \oplus \mathcal{Z}^g_{\mathcal{N}} \subseteq \text{Pre}^{\tau_{g'}}_{\Gamma_{g'}}(\hat{\mathbb{T}}^{g'}_{\mathcal{N}}) \oplus \mathcal{Z}^{g'}_{\mathcal{N}}, \forall g' \in G$.

The collection of sets $\hat{\mathbb{T}}^g_{\mathcal{N}}$ for all $g \in G$ that satisfies Assumption 8.1 and 8.2 and Assumption 10.1 is called a switch-robust control invariant (switch-RCI) set [20]. In addition, the switch-RCI set $\{\hat{\mathbb{T}}^g_{\mathcal{N}}\}g \in G$ that meets Assumption 10.2 is called a disturbance-and-switch robust control-invariant (DS-RCI) set [14]. In this research, the set $\hat{\mathbb{T}}^g_{\mathcal{N}}$ is computed using the following algorithm, which is adapted from Algorithm 1 in [20]. This algorithm provides the computation of maximal switch-RCI sets.

Algorithm 3: The computation of the set $\hat{\mathbb{T}}^g_{\mathcal{N}}$

1) Initialize $\Omega^g_{\mathcal{N}}(0) = \hat{\mathbb{X}}^g_{\mathcal{N}}$ for all $g \in G$.
2) For all $g \in G$, compute
$$\Omega^g_{\mathcal{N}}(k+1) = \Omega^g_{\mathcal{N}}(k) \cap \text{Pre}^{\tau_g}_{\Gamma_g}(\Omega^g_{\mathcal{N}}(k)) \cap \left(\cap_{g' \in G}\text{Pre}^{\tau_{g'}}_{\Gamma_{g'}}(\Omega^{g'}_{\mathcal{N}}(k))\right).$$
3) If $\Omega^g_{\mathcal{N}}(k+1) = \Omega^g_{\mathcal{N}}(k)$ for all $g \in G$, then stop and set $\hat{\mathbb{T}}^g_{\mathcal{N}} = \Omega^g_{\mathcal{N}}(k)$. Otherwise, set $\Omega^g_{\mathcal{N}}(k) = \Omega^g_{\mathcal{N}}(k+1)$ for all $g \in G$ and repeat step 2.

Since the system Γ_g is linear and the constraints $\hat{\mathbb{X}}^g_{\mathcal{N}}$ and $\hat{\mathbb{U}}^g_{\mathcal{N}}$ are compact for all $g \in G$, then the sets $\hat{\mathbb{T}}^g_{\mathcal{N}}$ produced by Algorithm 3 are compact based on Lemma 3 in [20]. Having computed the set $\hat{\mathbb{T}}^g_{\mathcal{N}}$, we can calculate the set $\hat{\mathbb{T}}^g_{\mathcal{C}}$ for all $\mathcal{C} \in \Pi(g)$ by projecting $\hat{\mathbb{T}}^g_{\mathcal{N}}$ to the state space of coalition $\mathcal{C}$.

In order to compute the feasibility region $\mathbb{X}^{g,N_p}_{\mathcal{N}}$, we provide the following algorithm.

Algorithm 4: The computation of the set $\mathbb{X}^{g,N_p}_{\mathcal{N}}$

input: the global nominal system Γ_g given by (22), the set $\mathcal{Z}^g_{\mathcal{N}}$, the set $\hat{\mathbb{T}}^g_{\mathcal{N}}$, and the length of prediction horizon N_p.

output: the set $\mathbb{X}^{g,N_p}_{\mathcal{N}}$.

Initialize $\Psi^g_{\mathcal{N}}(0) = \hat{\mathbb{T}}^g_{\mathcal{N}}$.
for $j = 1$ **to** N_p **do**
1. compute $\Upsilon^g_{\mathcal{N}}(j) = \text{Pre}^1_{\Gamma_g}(\Psi^g_{\mathcal{N}}(j-1)) \cap \hat{\mathbb{X}}^g_{\mathcal{N}}$,
2. update $\Psi^g_{\mathcal{N}}(j-1) = \Upsilon^g_{\mathcal{N}}(j)$,
end for
Compute $\mathbb{X}^{g,N_p}_{\mathcal{N}} = \Upsilon^g_{\mathcal{N}}(N_p) \oplus \mathcal{Z}^g_{\mathcal{N}}$.

The proposition below provides the feasibility property of the proposed coalitional control scheme.

Proposition 11. *Let Assumptions 1, 2, 4, 5, 8 and 10 hold. Then, if the problem (25) is feasible at time step k, it is also feasible at time step $k+1$.*

Proof: Let g and the pair $(\hat{x}^*_{\mathcal{N}}(0|k), \{\hat{u}^*_{\mathcal{N}}(t|k)\}^{N_p-1}_{t=0})$ be the topology and the optimal solution of (25) at time step k. Then, denote the trajectory of the nominal state corresponding to the optimal solution as $\{\hat{x}^*_{\mathcal{N}}(t|k)\}^{N_p}_{t=0}$. The proof will consider two cases, i.e, (i) $g(k+1) = g(k)$ and (ii) $g(k+1) \neq g(k)$.

Case 1. ($g(k+1) = g(k) = g$). Define the pair $(\hat{x}^0_{\mathcal{N}}(0|k+1), \{\hat{u}^0_{\mathcal{N}}(t|k+1)\}^{N_p-1}_{t=0})$ as a solution to the problem (25) at time step $k+1$, with $\hat{x}^0_{\mathcal{N}}(0|k+1) = \hat{x}^*_{\mathcal{N}}(1|k)$, $\hat{u}^0_{\mathcal{N}}(t|k+1) = \hat{u}^*_{\mathcal{N}}(t+1|k)$ for $t = 0, \ldots, N_p-2$, and $\hat{u}^0_{\mathcal{N}}(N_p-1|k+1) = K^g_{\mathcal{N}}\hat{x}^*_{\mathcal{N}}(N_p|k)$. Then, the trajectory of the nominal state corresponding to this solution is $\{\hat{x}^0_{\mathcal{N}}(t|k+1)\}^{N_p}_{t=0}$ with $\hat{x}^0_{\mathcal{N}}(t|k+1) = \hat{x}^*_{\mathcal{N}}(t+1|k)$ for $t = 0, \ldots, N_p-1$, and $\hat{x}^0_{\mathcal{N}}(N_p|k+1) = (\tilde{A}_{\mathcal{N}} + \tilde{B}_{\mathcal{N}}K^g_{\mathcal{N}})\hat{x}^*(N_p|k)$. Furthermore, based on the feasibility at time step k, we have $\hat{u}^0_{\mathcal{N}}(t|k+1) \in \hat{\mathbb{U}}^g_{\mathcal{N}}$ for all $t = 0, \ldots, N_p-2$, and $\hat{x}^0_{\mathcal{N}}(t|k) \in \hat{\mathbb{X}}^g_{\mathcal{N}}$ if $t < r(k+1) = r(k)-1$, $\hat{x}^0_{\mathcal{N}}(t|k) \in \hat{\mathbb{T}}^g_{\mathcal{N}}$ if $t \geq r(k+1)$, for $t = 0, \ldots, N_p-1$. In addition, since $\hat{x}^*_{\mathcal{N}}(N_p|k) \in \hat{\mathbb{T}}^g_{\mathcal{N}}$, then $\hat{u}^0_{\mathcal{N}}(N_p-1|k+1) \in \hat{\mathbb{U}}^g_{\mathcal{N}}$ and $\hat{x}^0_{\mathcal{N}}(N_p|k+1) \in \hat{\mathbb{T}}^g_{\mathcal{N}}$ according to Assumptions 8.1 and 8.2. Finally, since $x_{\mathcal{N}}(k) - \hat{x}^*_{\mathcal{N}}(0|k) \in \mathcal{Z}^g_{\mathcal{N}}$, and $u_{\mathcal{N}}(k)$ is given by (29), then, based on Proposition 1 in [16], $x_{\mathcal{N}}(k+1) - \hat{x}^*(1|k) = x_{\mathcal{N}}(k+1) - \hat{x}^0(0|k+1) \in \mathcal{Z}^g_{\mathcal{N}}$. Thus, we conclude that the solution $(\hat{x}^0_{\mathcal{N}}(0|k+1), \{\hat{u}^0_{\mathcal{N}}(t|k+1)\}^{N_p-1}_{t=0})$ is feasible.

Case 2.($g(k+1) \neq g(k)$). In this case, $r(k) = 0$ and $r(k+1) = \tau_{g(k+1)}$, so $\hat{x}^0_{\mathcal{N}}(0|k+1) = \hat{x}^*_{\mathcal{N}}(1|k) \in \hat{\mathbb{T}}^{g(k)}_{\mathcal{N}} \subseteq \text{Pre}^{\tau_{g(k+1)}}_{\Gamma_{g(k+1)}}(\hat{\mathbb{T}}^{g(k+1)}_{\mathcal{N}})$. From case 1, we have $x_{\mathcal{N}}(k+1) - \hat{x}^0_{\mathcal{N}}(0|k+1) \in \mathcal{Z}^{g(k)}_{\mathcal{N}}$. Therefore, $x_{\mathcal{N}}(k+1) \in \hat{\mathbb{T}}^{g(k)}_{\mathcal{N}} \oplus \mathcal{Z}^{g(k)}_{\mathcal{N}} \subseteq \text{Pre}^{\tau_{g(k+1)}}_{\Gamma_{g(k+1)}}(\hat{\mathbb{T}}^{g(k+1)}_{\mathcal{N}}) \oplus \mathcal{Z}^{g(k+1)}_{\mathcal{N}}$, based on Assumption 10. Hence, constraint (25b) holds at time step $k+1$. Since $\hat{x}^0_{\mathcal{N}}(0|k+1) \in \text{Pre}^{\tau_{g(k+1)}}_{\Gamma_{g(k+1)}}(\hat{\mathbb{T}}^{g(k+1)}_{\mathcal{N}})$, then there exists a sequence $\{\hat{u}^0_{\mathcal{N}}(t|k+1) \in \hat{\mathbb{U}}^{g(k+1)}_{\mathcal{N}}\}^{\tau_{g(k+1)}-1}_{t=0}$ such that $\hat{x}^{g(k+1)}_{\mathcal{N}} \in \hat{\mathbb{X}}^{g(k+1)}_{\mathcal{N}}$ for $t = 0, \ldots, \tau_{g(k+1)}-1$, and $\hat{x}^0_{\mathcal{N}}(\tau_{g(k+1)}|k+1) \in \hat{\mathbb{T}}^{g(k+1)}_{\mathcal{N}}$. Based on Assumptions 8.1 and 8.2, we have $\hat{u}^0_{\mathcal{N}}(t|k+1) = K^{g(k+1)}_{\mathcal{N}}\hat{x}^0_{\mathcal{N}}(t|k+1) \in \hat{\mathbb{U}}^{g(k+1)}_{\mathcal{N}}$ and $\hat{x}^0_{\mathcal{N}}(t+1|k+1) \in \hat{\mathbb{T}}^{g(k+1)}_{\mathcal{N}}$ for $t = \tau_{g(k+1)}, \ldots, N_p-1$. Hence, we conclude that the pair $(\hat{x}^0_{\mathcal{N}}(0|k+1), \{\hat{u}^0_{\mathcal{N}}(t|k+1)\}^{N_p-1}_{t=0})$ is the feasible solution of (25) at time step $k+1$. ■

The following proposition provides the result on the stability of the closed-loop system (8) when the network topology is fixed. This proposition will be employed later to state the result on the stability of the closed-loop system (8) for switched topology cases.

Proposition 12. *If $g(k) = g$ for all $k \geq 0$, then the origin of the closed-loop system $x_{\mathcal{N}}(k+1) = A_{\mathcal{N}}x_{\mathcal{N}}(k) + B_{\mathcal{N}}u^*_{\mathcal{N}}(k)$ where $u^*_{\mathcal{N}}(k)$ is given by (29) is asymptotically stable with the region of attraction $\mathbb{X}^{g,N_p}_{\mathcal{N}}$.*

Proof: The proof follows the proof of theorem 1 in [13]. ■

The following result is provided by [15] and also will be employed to state the result on the stability of the closed-loop system (8).

Lemma 13. *Consider the switched system $x(k+1) = f_{\sigma(k)}(x(k))$, where $f_\sigma(.)$ is globally Lipschitz continuous having values 0 at 0, and the switching signal σ is a function from $\mathbb{Z}_+$ to a finite set $\mathcal{I}$. The origin of the system is locally stable if there exist a scalar $\gamma > 0$ and a set of continuous positive-definite functions $\mathcal{V}_m : \mathbb{R}^n \to \mathbb{R}_+$, $\sigma(k) = m \in \mathcal{I}$ satisfying*

$$\beta_1(\|x(k)\|) \leq \mathcal{V}_m(x(k)) \leq \beta_2(\|x(k)\|), \tag{30a}$$

$$\mathcal{V}_m(x(k)) \leq \gamma\mathcal{V}_{\sigma(0)}(x(0)), \tag{30b}$$

where $\beta_1, \beta_2 \in \mathcal{K}$.

In this research, the switching signal is defined by $g : \mathbb{Z}_+ \to G$, and the family of continuous positive-definite functions is given by $J^{g,N^*_p}_{\mathcal{N}} : \mathbb{X}^{g,N_p}_{\mathcal{N}} \to \mathbb{R}_+$ for all $g \in G$. Thus, to establish the stability of the closed-loop system (8) and (29), we have to show that the functions $J^{g,N^*_p}_{\mathcal{N}}(.)$ satisfy (30a) and (30b). To this end, we provide the following results.

Proposition 14. *Letting $\mathbb{X}_{\mathcal{N}}^{g,0} \triangleq \hat{\mathbb{T}}_{\mathcal{N}}^{g} \oplus \mathcal{Z}_{\mathcal{N}}^{g}$ for all $g \in G$. Then, the set $\mathbb{X}_{\mathcal{N}}^{g,0}$ is compact and control invariant for (8) and (9).*

Proof: The compactness of $\mathbb{X}_{\mathcal{N}}^{g,0}$ is immediately implied by the compactness of $\hat{\mathbb{T}}_{\mathcal{N}}^{g}$ and $\mathcal{Z}_{\mathcal{N}}^{g}$. Next, for any $x_{\mathcal{N}}(k) \in \mathbb{X}_{\mathcal{N}}^{g,0}$, we have $x_{\mathcal{N}}(k) - \hat{x}_{\mathcal{N}}(k) \in \mathcal{Z}_{\mathcal{N}}^{g}$, where $\hat{x}_{\mathcal{N}}(k) \in \hat{\mathbb{T}}_{\mathcal{N}}^{g}$. Therefore, $\hat{u}_{\mathcal{N}}(k) = K_{\mathcal{N}}^{g}\hat{x}_{\mathcal{N}}(k) \in \hat{\mathbb{U}}_{\mathcal{N}}^{g}$ and $\hat{x}_{\mathcal{N}}(k+1) \in \hat{\mathbb{T}}_{\mathcal{N}}^{g}$, based on Assumptions 8.1 and 8.2. Consequently, $u_{\mathcal{N}}(k) = \hat{u}_{\mathcal{N}}(k) + K_{\mathcal{N}}^{g}(x_{\mathcal{N}}(k) - \hat{x}_{\mathcal{N}}(k)) \in \hat{\mathbb{U}}_{\mathcal{N}}^{g} \oplus K_{\mathcal{N}}^{g}\mathcal{Z}_{\mathcal{N}}^{g} = \mathbb{U}_{\mathcal{N}}^{g}$. Moreover, since $x_{\mathcal{N}}(k) - \hat{x}_{\mathcal{N}}(k) \in \mathcal{Z}_{\mathcal{N}}^{g}$ and $u_{\mathcal{N}}(k) = \hat{u}_{\mathcal{N}}(k) + K_{\mathcal{N}}^{g}(x_{\mathcal{N}}(k) - \hat{x}_{\mathcal{N}}(k))$, then, based on Proposition 1 in [16] , $x_{\mathcal{N}}(k+1) - \hat{x}_{\mathcal{N}}(k+1) \in \mathcal{Z}_{\mathcal{N}}^{g}$. In addition, since $\hat{x}_{\mathcal{N}}(k+1) \in \hat{\mathbb{T}}_{\mathcal{N}}^{g}$, then $x_{\mathcal{N}}(k+1) \in \hat{\mathbb{T}}_{\mathcal{N}}^{g} \oplus \mathcal{Z}_{\mathcal{N}}^{g} = \mathbb{X}_{\mathcal{N}}^{g,0}$. Thus the set $\mathbb{X}_{\mathcal{N}}^{g,0}$ is control invariant for (8) and (9). ■

Since the dynamic of the overall system (8) is linear where its constraints (9) are compact polyhedral sets containing the origin in their interior, and the set $\mathbb{X}_{\mathcal{N}}^{g,0}$ is compact and control invariant for (8) and (9), then, according to Proposition 2.10 in [21], the sets $\mathbb{X}_{\mathcal{N}}^{g,j}$, for all $j \in \{1,\ldots,N_p\}$, are compact and control invariant for (8) and (9), and satisfy $\mathbb{X}_{\mathcal{N}}^{g,N_p} \supseteq \mathbb{X}_{\mathcal{N}}^{g,N_p-1} \supseteq \ldots \supseteq \mathbb{X}_{\mathcal{N}}^{g,0}$. Furthermore, if $\mathbb{X}_{\mathcal{N}}^{g,0}$ contains the origin, $\mathbb{X}_{\mathcal{N}}^{g,j}$ also contains the origin for all $j \in \{0,1,\ldots,N_p\}$.

The result below provides the lower and upper bound on the value function $J_{\mathcal{N}}^{g,N_p^*}(.)$ and will be employed to state that $J_{\mathcal{N}}^{g,N_p^*}(.)$ satisfies (30a).

Proposition 15. *Let $(\hat{x}_{\mathcal{N}}^*(0|k), \{\hat{u}_{\mathcal{N}}^*(t|k)\}_{t=0}^{N_p-1})$ be the optimal solution for (25) when the state of the overall system and the network topology and at time step k are $x_{\mathcal{N}}(k) = x_{\mathcal{N}}$ and $g(k) = g$, respectively. Then there exist positive scalars $b_g > a_g > 0$ such that*

$$J_{\mathcal{N}}^{g,{N_p}^*}(x_{\mathcal{N}}) \geq a_g\|\hat{x}_{\mathcal{N}}^*(0|k)\|^2, \quad \forall x_{\mathcal{N}} \in \mathbb{X}_{\mathcal{N}}^{g,N_p}, \tag{31}$$

and

$$J_{\mathcal{N}}^{g,{N_p}^*}(x_{\mathcal{N}}) \leq b_g\|\hat{x}_{\mathcal{N}}^*(0|k)\|^2, \quad \forall x_{\mathcal{N}} \in \mathbb{X}_{\mathcal{N}}^{g,0}. \tag{32}$$

Proof:

1) Since $(\hat{x}_{\mathcal{N}}^*(0|k), \{\hat{u}_{\mathcal{N}}^*(t|k)\}_{t=0}^{N_p-1})$ is the optimal solution for (25) corresponding to $g(k) = g$ and $x_{\mathcal{N}}(k) = x_{\mathcal{N}}$, then $J_{\mathcal{N}}^{g,N_p^*}(x_{\mathcal{N}}) \geq \ell_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}^*(0|k), \hat{u}_{\mathcal{N}}^*(0|k))$, where $\ell_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}, \hat{u}_{\mathcal{N}}) = \hat{x}_{\mathcal{N}}^T Q_{\mathcal{N}}^{g}\hat{x}_{\mathcal{N}} + \hat{u}_{\mathcal{N}}^T R_{\mathcal{N}}^{g}\hat{u}_{\mathcal{N}}$. Furthermore, based on Assumption 3, matrices $Q_{\mathcal{N}}^{g}$ and $R_{\mathcal{N}}^{g}$ are positive definite. Therefore, there exists a scalar $a_g > 0$ such that $\ell_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}^*(0|k), \hat{u}_{\mathcal{N}}^*(0|k)) \geq a_g||\hat{x}_{\mathcal{N}}^*(0|k)||^2$. Hence,
$$J_{\mathcal{N}}^{g,N_p^*}(x_{\mathcal{N}}) \geq a_g||\hat{x}_{\mathcal{N}}^*(0|k)||^2,\ \forall x_{\mathcal{N}} \in \mathbb{X}_{\mathcal{N}}^{g,N_p}.$$
2) Since $x_{\mathcal{N}} \in \mathbb{X}_{\mathcal{N}}^{g,0}$, then there exists $\hat{x}_{\mathcal{N}} \in \hat{\mathbb{T}}_{\mathcal{N}}^{g}$ that satisfies $x_{\mathcal{N}} - \hat{x}_{\mathcal{N}} \in \mathcal{Z}_{\mathcal{N}}^{g}$. Letting $\hat{x}_{\mathcal{N}}^*(0|k) \triangleq \hat{x}_{\mathcal{N}}$, we have $\hat{u}_{\mathcal{N}}^*(0|k) \triangleq K_{\mathcal{N}}^{g}\hat{x}_{\mathcal{N}}^*(0|k) \in \hat{\mathbb{U}}_{\mathcal{N}}^{g}$ and $\hat{x}_{\mathcal{N}}^*(1|k) = (\tilde{A}_{\mathcal{N}}^{g} + \tilde{B}_{\mathcal{N}}^{g}K_{\mathcal{N}}^{g})\hat{x}_{\mathcal{N}}^*(0|k) \in \hat{\mathbb{T}}_{\mathcal{N}}^{g}$, based on Assumptions 8.1 and 8.2. Repeating this, we can form trajectories of nominal input $\{\hat{u}_{\mathcal{N}}^*(t|k)\}_{t=0}^{N_p-1}$ and nominal state $\{\hat{x}_{\mathcal{N}}^*(t|k)\}_{t=0}^{N_p}$, where $\hat{u}_{\mathcal{N}}^*(t|k) = K_{\mathcal{N}}^{g}\hat{x}_{\mathcal{N}}^*(t|k) \in \hat{\mathbb{U}}_{\mathcal{N}}^{g}$ and $\hat{x}_{\mathcal{N}}^*(t+1|k) = (\tilde{A}_{\mathcal{N}}^{g} + \tilde{B}_{\mathcal{N}}^{g}K_{\mathcal{N}}^{g})\hat{x}_{\mathcal{N}}^*(t|k) \in \hat{\mathbb{T}}_{\mathcal{N}}^{g}$ for all $t = 0,\ldots,N_p-1$. Consequently, from Assumption 8.3, we have
$$\begin{aligned} V_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}^*(0|k)) &\geq V_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}^*(1|k)) + \\ &\quad \ell_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}^*(0|k), \hat{u}_{\mathcal{N}}^*(0|k)) \\ &\geq V_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}^*(2|k)) + \\ &\quad \sum_{t=0}^{1} \ell_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}^*(t|k), \hat{u}_{\mathcal{N}}^*(t|k)) \\ &\ \vdots \\ &\geq V_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}^*(N_p|k)) + \\ &\quad \sum_{t=0}^{N_p-1} \ell_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}^*(t|k), \hat{u}_{\mathcal{N}}^*(t|k)) \\ &= J_{\mathcal{N}}^{g,N_p^*}(x_{\mathcal{N}}(k)), \end{aligned}$$
where $V_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}) = \hat{x}_{\mathcal{N}}^T P_{\mathcal{N}}^{g}\hat{x}_{\mathcal{N}}$. Since $P_{\mathcal{N}}^{g}$ is positive definite, there exists a scalar $b_g > a_g > 0$ such that $V_{\mathcal{N}}^{g}(\hat{x}_{\mathcal{N}}^*(0|k)) \leq b_g||\hat{x}_{\mathcal{N}}^*(0|k)||^2$. Consequently,
$$J_{\mathcal{N}}^{g,N_p^*}(x_{\mathcal{N}}) \leq b_g||\hat{x}_{\mathcal{N}}^*(0|k)||^2, \forall x_{\mathcal{N}} \in \mathbb{X}_{\mathcal{N}}^{g,0}.$$
■

Note that, if $\mathbb{X}_{\mathcal{N}}^{g,0}$ contains the origin in its interior, we can extend the upper bound on $J_{\mathcal{N}}^{g,N_p}(.)$ to $\mathbb{X}_{\mathcal{N}}^{g,N_p}$ according to Proposition 2.16 in [21]. That is,

$$J_{\mathcal{N}}^{g,N_p^*}(x_{\mathcal{N}}) \leq b_g||\hat{x}_{\mathcal{N}}^*(0|k)||^2,\ \forall x_{\mathcal{N}} \in \mathbb{X}_{\mathcal{N}}^{g,N_p}. \tag{33}$$

The result below provides the closed-loop stability of the overall system.

Theorem 16. *Let Assumptions 1, 2, 4, 5, 8, 10 hold and $\mathbb{X}_{\mathcal{N}}^{g,0}$ contains the origin in its interior. Then*

1) *the origin of the closed-loop system $x_{\mathcal{N}}(k+1) = f_{g(k)}(x(k)) = A_{\mathcal{N}}x_{\mathcal{N}}(k) + B_{\mathcal{N}}u_{\mathcal{N}}^*(k)$ is locally stable;*
2) *If there exists a finite time $k_f > 0$ such that $g(k) = g \in G$ for all $k \geq k_f$, then the closed-loop system $x_{\mathcal{N}}(k+1) = f_{g(k)}(x(k)) = A_{\mathcal{N}}x_{\mathcal{N}}(k) + B_{\mathcal{N}}u_{\mathcal{N}}^*(k)$ is asymptotically stable with the region of attraction $\mathbb{X}_{\mathcal{N}}^{g(0),N_p}$.*

Proof:

1) Since the overall system $f_g(x_{\mathcal{N}}(k), u_{\mathcal{N}}(k)) = A_{\mathcal{N}}x_{\mathcal{N}}(k) + B_{\mathcal{N}}u_{\mathcal{N}}(k)$ is linear and the constraints $\mathbb{X}_{\mathcal{N}}^{g}$ and $\mathbb{U}_{\mathcal{N}}^{g}$ are polyhedral for each $g \in G$, the value function $J_{\mathcal{N}}^{g,N_p^*} : \mathbb{X}_{\mathcal{N}}^{g,N_p} \to \mathbb{R}_{\geq 0}$ is continuous for each $g \in G$ according to Theorem 2.7(b) in [21]. Therefore, the set
$$\{\theta_g \triangleq \sup_{x_{\mathcal{N}} \in \mathbb{X}_{\mathcal{N}}^{g,N_p}} J_{\mathcal{N}}^{g,{N_p}^*}(x_{\mathcal{N}})\}_{g \in G}$$
is bounded and finite by the continuity of $J_{\mathcal{N}}^{g,N_p^*}(.)$ and the compactness of $\mathbb{X}_{\mathcal{N}}^{g,N_p^*}$. Since $J_{\mathcal{N}}^{g(0),N_p^*}(x_{\mathcal{N}}(0))$ is finite, set for all $g \in G$
$$\gamma_g \triangleq (J_{\mathcal{N}}^{g(0),N_p^*}(x_{\mathcal{N}}(0)))^{-1}(\theta_g + \varepsilon_g), \quad \forall \varepsilon_g \geq 0.$$
Then for any $x_{\mathcal{N}} \in \mathbb{X}_{\mathcal{N}}^{g,N_p}$, $\forall g \in G$, we have
$$J_{\mathcal{N}}^{g,N_p^*}(x_{\mathcal{N}}) \leq \theta_g \leq \theta_g + \varepsilon_g \leq \gamma J_{\mathcal{N}}^{g(0),N_p^*}(x_{\mathcal{N}}(0)), \tag{34}$$

where $\gamma \triangleq \max_{g\in G}\{\gamma_g\}$. Hence, (30b) is satisfied within topology $g \in G$. Let k_s be the switching instant with $g(k_s-1)=g$ and $g(k_s)=h$. Then, if $x_{\mathcal{N}}(k_s-1) \in \mathbb{X}_{\mathcal{N}}^{g,N_p}$, $x_{\mathcal{N}}(k_s) \in \mathbb{X}_{\mathcal{N}}^{h,N_p}$, $\forall h \in G$, according to Proposition 11. As a result, $J_{\mathcal{N}}^{h,N_p{}^*}(x_{\mathcal{N}}(k_s)) \leq \gamma J_{\sigma(0)}^{N_p{}^*}(x(0))$ according to (34). Thus, (30b) is also satisfied when the switching of topology occurs. Setting $a \triangleq \min\{a_g\}_{g\in G}$ and $b \triangleq \max\{b_g\}_{g\in G}$, by (31) and (33), we have, for all $x_{\mathcal{N}}(k) \in \mathbb{X}_{\mathcal{N}}^{g,N_p}$,

$$\beta_1(\|x_{\mathcal{N}}(k)\|) \leq J_{\mathcal{N}}^{g,N_p{}^*}(x_{\mathcal{N}}(k)) \leq \beta_1(\|x_{\mathcal{N}}(k)\|),$$

where $\beta_1(\|x_{\mathcal{N}}(k)\|) = a\|\hat{x}^*_{\mathcal{N}}(0|k)\|^2$ and $\beta_2(\|x_{\mathcal{N}}(k)\|) = b\|\hat{x}^*_{\mathcal{N}}(0|k)\|^2$. Thus, (30a) is satisfied. Therefore, by Lemma 13, the closed-loop system $x_{\mathcal{N}}(k+1) = f_{g(k)}(x(k)) = A_{\mathcal{N}}x_{\mathcal{N}}(k) + B_{\mathcal{N}}u^*_{\mathcal{N}}(k)$ is locally stable.

2) The asymptotically stability of the closed-loop system $x_{\mathcal{N}}(k+1) = f_{g(k)}(x(k)) = A_{\mathcal{N}}x_{\mathcal{N}}(k) + B_{\mathcal{N}}u^*_{\mathcal{N}}(k)$ is immediately implied by Proposition 12.

■

Based on the proof in part 1, we can guarantee the stability of the closed-loop system as long as the recursive feasibility of the overall system is assured.

IV. Simulation

To assess the performance of the proposed control scheme, we use a system that consists of three interconnected subsystems as an academic example. The dynamic and the constraints of subsystem $i \in \mathcal{N} = \{1,2,3\}$ are respectively given by

$$\begin{aligned} x_i(k+1) = & A_{ii}x_i(k) + B_{ii}u_i(k) + \\ & \sum_{j\neq i}(A_{ij}x_j(k) + B_{ij}u_j(k)), \end{aligned} \tag{35}$$

and

$$\mathbb{X}_i = \{x_i \in \mathbb{R}^2 : (-2,-2) \leq x_i \leq (2,2)\}, \tag{36}$$

$$\mathbb{U}_i = \{u_i \in \mathbb{R} : -0.5 \leq u_i \leq 0.5\}. \tag{37}$$

For $i = 1$, the matrices that describe (35) are provided by

$$A_{11} = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.4 \end{pmatrix}, A_{12} = A_{13} = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.05 \end{pmatrix},$$

$$B_{11} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}, B_{12} = B_{13} = \begin{pmatrix} 0 \\ 0.15 \end{pmatrix}.$$

Then, for $i = 2$, the matrices describing (35) are

$$A_{22} = \begin{pmatrix} 0.6 & 0.3 \\ 0 & -0.5 \end{pmatrix}, A_{21} = A_{23} = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix},$$

$$B_{22} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, B_{21} = B_{23} = \begin{pmatrix} 0 \\ 0.15 \end{pmatrix}.$$

Finally, the matrices that describe (35) for $i = 3$ are

$$A_{33} = \begin{pmatrix} 0.6 & -0.2 \\ 0 & 0.3 \end{pmatrix}, A_{31} = A_{32} = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.05 \end{pmatrix},$$

$$B_{33} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, B_{31} = B_{32} = \begin{pmatrix} 0 \\ 0.15 \end{pmatrix}.$$

Fig. 1. Communication network of agents

A set $\{A1, A2, A3\}$ of agents connected via a communication network consisting of two links $\{I, II\}$ rules the subsystems independently, as depicted in Fig.1. Furthermore, both links can be enabled and disabled at every time step. Therefore, there are four possible topologies that might be formed. All possible topologies and their corresponding partition of agents are displayed in Table I.

TABLE I
THE LIST OF ALL TOPOLOGIES

No	Network Topology	Partition of Agents
1.	$g_1 = \emptyset$	$\Pi(g_1) = \{\{A1\},\{A2\},\{A3\}\}$
2.	$g_2 = \{I\}$	$\Pi(g_2) = \{\{A1, A2\},\{A3\}\}$
3.	$g_3 = \{II\}$	$\Pi(g_3) = \{\{A1\},\{A2, A3\}\}$
4.	$g_4 = \{I, II\}$	$\Pi(g_4) = \{A1, A2, A3\}$

In the topology g_1, all links are disabled. As a result, each agent forms a coalition of itself. The link I is enabled in the topology g_2, accordingly agents A_1 and A_2 form a coalitions, and agent A_3 forms its own coalition. On the other hand, the link II is enabled in the topology g_3. Consequently, agents A_2 and A_3 form a coalition in the topology g_3. Finally, all links are enabled in the topology g_4, so all agents form a grand coalition.

Given the system and the network topology, we then set and compute some parameters. First, the weighting matrices Q_i and R_i are set to

$$Q_i = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}, \; R_i = 10, \; \forall i \in \mathcal{N}.$$

Then, we set the dwell time of all topology $g \in G$ as $\tau_g = 3$ and the length of horizon prediction as $N_p = \tau_g$. After that, for all $g \in G = \{g_1, g_2, g_3, g_4\}$, we compute matrices $K_{\mathcal{N}}^g$ and $P_{\mathcal{N}}^g$ using the method in [9], and the set $\mathcal{Z}_{\mathcal{C}}^g$ using Algorithm 1. Based on $K_{\mathcal{C}}^g$ and $\mathcal{Z}_{\mathcal{C}}^g$, we then calculate the sets $\hat{\mathbb{X}}_{\mathcal{C}}^g$ and $\hat{\mathbb{U}}_{\mathcal{C}}^g$ as $\hat{\mathbb{X}}_{\mathcal{C}}^g = \mathbb{X}_{\mathcal{C}}^g \ominus \mathcal{Z}_{\mathcal{C}}^g$ and $\hat{\mathbb{U}}_{\mathcal{C}}^g = \mathbb{U}_{\mathcal{C}}^g \ominus K_{\mathcal{C}}^g \mathcal{Z}_{\mathcal{C}}^g$ for all $\mathcal{C} \in \Pi(g)$. Then,we compute the set $\hat{\mathbb{T}}_{\mathcal{N}}^g$ for each g using Algorithm 3. Finally, the sets $\hat{\mathbb{T}}_{\mathcal{C}}^g$ for all $\mathcal{C} \in \Pi(g)$ are obtained by projecting $\hat{\mathbb{T}}_{\mathcal{N}}^g$ to the state space of coalition $\mathcal{C}$.

For example, for topology $g = g_1$, which corresponds to the case no agents are communicating, we have

$$K_{\mathcal{N}}^g = \mathrm{diag}(K_{\mathcal{C}_1}^g, K_{\mathcal{C}_2}^g, K_{\mathcal{C}_3}^g), P_{\mathcal{N}}^g = \mathrm{diag}(P_{\mathcal{C}_1}^g, P_{\mathcal{C}_2}^g, P_{\mathcal{C}_3}^g),$$

where

$$\mathcal{C}_1 = \{A1\}, \mathcal{C}_2 = \{A2\}, \mathcal{C}_3 = \{A3\},$$

$$K_{\mathcal{C}_1}^g = \begin{pmatrix} -0.1332 & -0.2651 \end{pmatrix}, K_{\mathcal{C}_2}^g = \begin{pmatrix} -0.0141 & 0.2131 \end{pmatrix},$$

$$K_{\mathcal{C}_3}^g = \begin{pmatrix} -0.2773 & -0.0158 \end{pmatrix}, P_{\mathcal{C}_1}^g = \begin{pmatrix} 14.8443 & 0.6639 \\ 0.6639 & 11.5167 \end{pmatrix},$$

$$P_{\mathcal{C}_2}^g = \begin{pmatrix} 18.3644 & 3.0565 \\ 3.0565 & 13.6719 \end{pmatrix}, P_{\mathcal{C}_3}^g = \begin{pmatrix} 16.8463 & -3.1652 \\ -3.1652 & 13.0495 \end{pmatrix}.$$

Then, the sets $\hat{\mathbb{U}}_{\mathcal{C}_1}^g$, $\hat{\mathbb{U}}_{\mathcal{C}_2}^g$, and $\hat{\mathbb{U}}_{\mathcal{C}_3}^g$ are given by

$$\hat{\mathbb{U}}_{\mathcal{C}_1}^g = \{\hat{u}_1 \in \mathbb{R} : -0.3206 \leq \hat{u}_1 \leq 0.3206\},$$

$$\hat{\mathbb{U}}^{g}_{\mathcal{C}_2} = \{\hat{u}_2 \in \mathbb{R} : -0.3906 \leq \hat{u}_1 \leq 0.3906\},$$

and

$$\hat{\mathbb{U}}^{g}_{\mathcal{C}_2} = \{\hat{u}_2 \in \mathbb{R} : -0.3209 \leq \hat{u}_1 \leq 0.3209\}.$$

Furthermore, the sets $\mathcal{Z}^{g_1}_{\mathcal{C}}$, $\hat{\mathbb{X}}^{g_1}_{\mathcal{C}}$, and $\hat{\mathbb{T}}^{g_1}_{\mathcal{C}}$ for all $\mathcal{C} \in \Pi(g_1)$ are depicted in Fig. 2, Fig. 3, and Fig. 4.

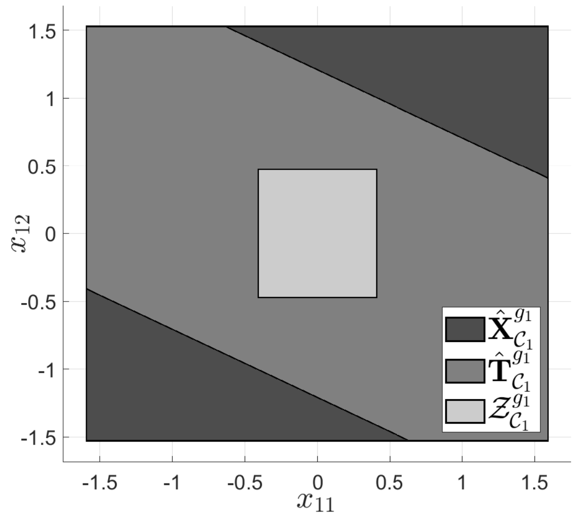

Fig. 2. The sets $\hat{\mathbb{X}}^{g_1}_{\mathcal{C}_1}$, $\hat{\mathbb{T}}^{g_1}_{\mathcal{C}_1}$, and $\mathcal{Z}^{g_1}_{\mathcal{C}_1}$.

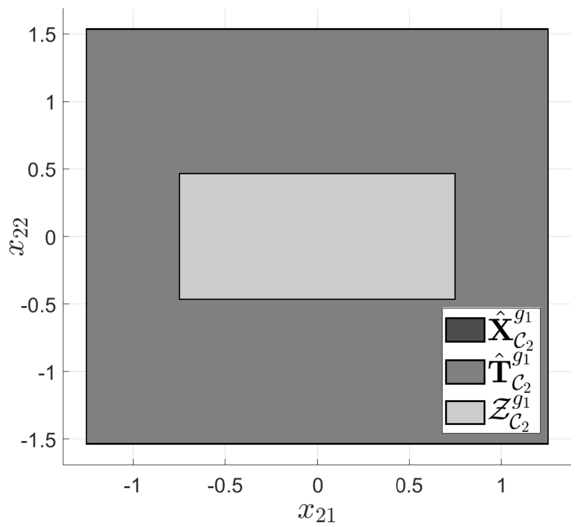

Fig. 3. The sets $\hat{\mathbb{X}}^{g_1}_{\mathcal{C}_2}$, $\hat{\mathbb{T}}^{g_1}_{\mathcal{C}_2}$, and $\mathcal{Z}^{g_1}_{\mathcal{C}_2}$.

We establish a simulation by considering two scenarios for the network topology. In the first scenario, the network topology is fixed for all of the time as g_4. Hence, scenario 1 corresponds to the centralized MPC scheme since all agents form a grand coalition. Then, in scenario 2, the network topology is periodically modified using Algorithm 2 at every multiple of τ_g with initial topology $g(0) = g_4$. Therefore, the set of switching instants $\mathcal{T}$ is $\mathcal{T} = \{\tau_g k : k \in \mathbb{Z}_+\}$.

To set the initial state of each subsystem, we firstly determine the feasibility region $\mathbb{X}^{g_4,N_p}_{\mathcal{N}}$ by employing Algorithm 4, and then project it to the state space of each subsystem.

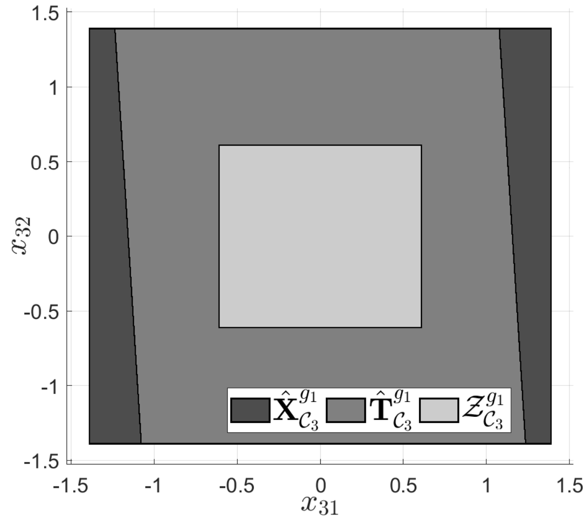

Fig. 4. The sets $\hat{\mathbb{X}}^{g_1}_{\mathcal{C}_3}$, $\hat{\mathbb{T}}^{g_1}_{\mathcal{C}_3}$, and $\mathcal{Z}^{g_1}_{\mathcal{C}_3}$.

Fig. 5 below depicts the feasibility region of subsystem 1. Since the feasibility regions of subsystems 2 and 3 are same as the feasibility region of subsystem 1, we don't provide the figure of those regions. By setting the initial state of each subsystem as $x_1(0) = (1.5, -1.5)$, $x_2(0) = (1.5, -1.5)$ and $x_3(0) = (-1.5, 1.5)$, and the cost of an enabled link as $c = 0.1$, the simulation results for 10 time steps are depicted in Fig.6, Fig. 7, and Fig. 8.

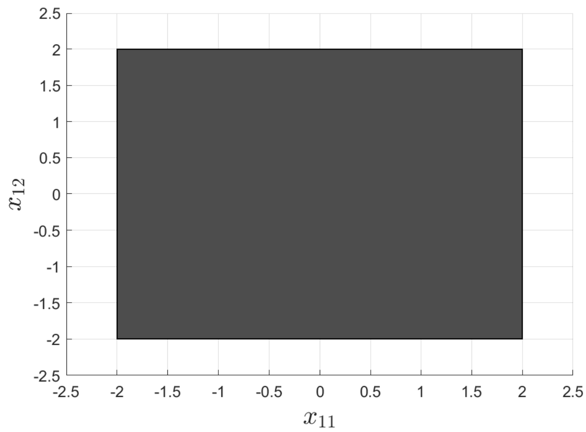

Fig. 5. The feasibility region of subsystem 1

Fig. 6 and Fig. 7 show the state and input trajectories of all subsystems for both scenarios. We can observe that both scenarios result in an asymptotically stable closed-loop system and a feasible system. In addition, the state trajectories of both scenarios are very close. Particularly, the trajectories $\{u_i(k)\}^2_{k=0}$ and $\{x_i(k)\}^3_{k=0}$, for all $i \in \{1, 2, 3\}$, are equal for both scenarios since the network topology for both scenarios are equal to g_4 for $k = 0, 1, 2$. Fig. 8 depicts the topologies for scenario two that are active during the simulation. Based on the set $\mathcal{T}$ above, the topology is revised at time steps 3, 6, and 9. We can see from Fig. 8 that $g(3) = g_3$ and $g(6) = g(9) = g_1$. This shows that when the state of all subsystems is around the origin, the topology g_1 is activated. We can observe from Fig. 6 that the state of

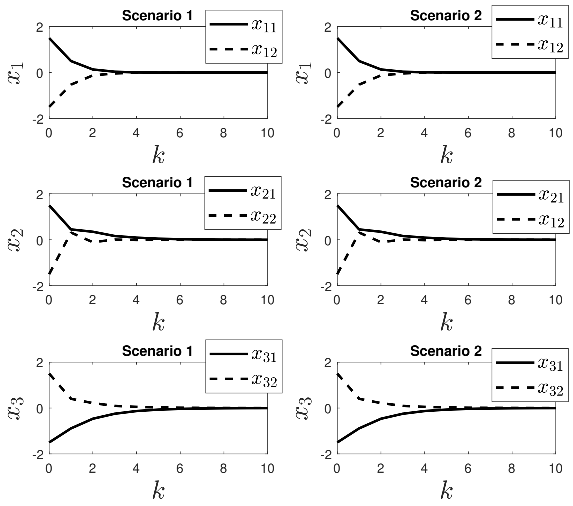

Fig. 6. The state trajectories of subsystems

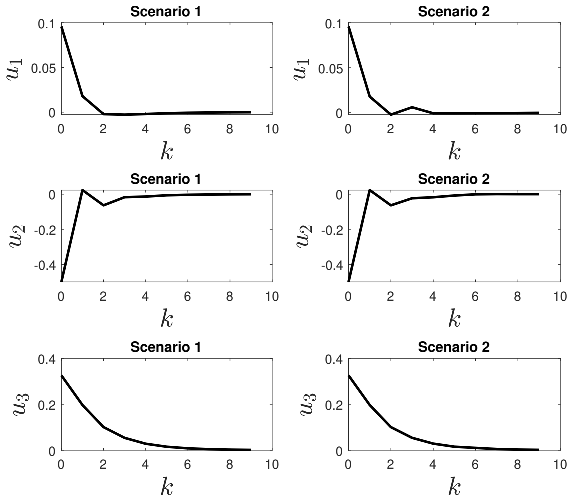

Fig. 7. The input trajectories of subsystems

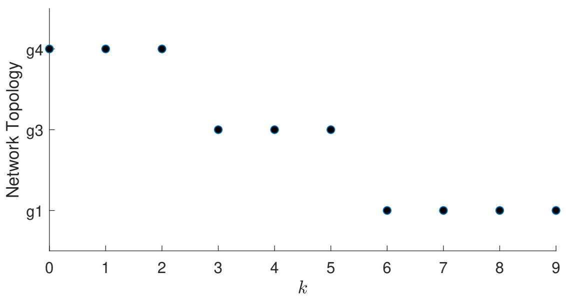

Fig. 8. The dynamic of network topology of Scenario 2

all subsystems is around the origin is at time step 5. Table II presents the total cost for both scenarios that are defined as

$$\sum_{k=0}^{10}(J_{\mathcal{N}}^{g(k),N_p^*}(x_{\mathcal{N}}(k))+c|g(k)|).$$

It can be observed that the total cost for scenario 2 is lower than the total cost for scenario 1. This means that our proposed control scheme has a control performance similar to the centralized MPC scheme with a lower cost.

TABLE II
TOTAL COST FOR BOTH SCENARIOS

	Scenario 1 (Centralized MPC)	Scenario 2 (Coalitional MPC)
Total Cost	196.7172	193.6126

V. CONCLUSION

In this paper, we investigate a coalitional MPC approach for a system consisting of interconnected subsystems. The approach seeks the optimal trade-off between the control performance and the communicational cost by periodically altering the network topology, which induces a collection of coalitions. Each coalition computes the value of its corresponding input variable by employing a decentralized MPC scheme. The recursive feasibility of the system is guaranteed under the assumption that DS-RCI sets exist. It is shown that the recursive feasibility implies the stability of the closed-loop system. Moreover, the closed-loop system is guaranteed to be asymptotically stable if the network topology is finally fixed. Simulation results show that the proposed control approach provides a control performance similar to the centralized MPC approach with a lower cost.

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Zaiful Ulum received his M.Sc. at Mathematics Department, Universitas Gadjah Mada, Yogyakarta, Indonesia, in 2015. Currently, he is a doctoral student at Mathematics Department, Universitas Gadjah Mada, Yogyakarta, Indonesia. He has been a lecturer at Program Studi Pendidikan Matematika, Sekolah Tinggi Keguruan dan Ilmu Pendidikan (STKIP) PGRI Bangkalan, Bangkalan 69116, Indonesia, since 2015. His research interest includes Control Theory, Distributed Model Predictive Control, and Game Theory.