Direction of Arrival Estimation by Employing Intra-block Correlations in Sparse Bayesian Learning Through Covariance Model

Raghu K, Prameela Kumari N

Abstract— **Estimation of the arriving signal directions at the receiver side is of utmost important in the field areas of array signal processing. The proposed technique in this paper involves two major steps, in that the first step is a C-step where, we deduce the covariance model for Direction of Arrival (DOA) estimation and through which, the noise variance of the model will be estimated. In the second step, i.e. L-step, the covariance model deduced in the C-step will be used along with the noise statistics to estimate the variance of sparse DOA spectrum, which is unknown. In this step, Sparse Bayesian Learning with Expectation maximization framework is extended to exploit the property of intra-block correlations in the unknown DOA spectrum. The variance of sparse DOA spectrum, which is estimated in L-step indicates the locations of non-zero values in the spectrum, hence resulting in directions of the signal sources. In the results section, it can be seen that the increase in accuracy and performance of the proposed algorithm is one of the result of exploiting intra-block correlations. The covariance modelling in C-step results in high probability of true DOA estimation in the case where number of signal sources is less than the antenna elements in the Uniform Linear Array (ULA) with lesser number of snapshots required. It is also shown in the simulation results that an acceptable estimation accuracy is achieved in the case where number of signal sources is greater than or equal to the antenna elements, but with larger snapshots required.**

Index Terms—**Direction of Arrival Estimation, Sparse Bayesian Learning, Intra-block correlations, Covariance model.**

I. Introduction

In the fields of RADAR, SONAR, seismology, wireless communication, navigation etc, it is very important to estimate the direction of signal arriving from the space around the receiver [27]-[28]. Several algorithms are developed and are being developed until present day for DOA estimation. All of these algorithms mainly focus on achieving good performance like high resolution, accuracy, less complex, fast estimation, highest probability of true DOA estimation and so on.

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Over a decade, such DOA estimation algorithms are broadly classified into two main categories such as sub-space based and sparse compressive sensing based algorithms. The subspace based algorithms include Schmidt's MUSIC [1] where, the DOAs can be retrieved from the second order statistics data, ESPRIT [2], in which retrieving can be done through rotational invariant techniques. Before these subspace based methods, there exist a conventional (Bartlett) beam former, which uses simple Fourier spectral analysis of received data. Capon's beam former [3] was also later proposed to improve the accuracy of estimation in the case of closely spaced signal sources.

All of these subspace based methods require a prior knowledge on the number of signal sources and also requires a sufficient number of data snapshots to obtain an acceptable accuracy value. The performance analysis of all these subspace based algorithms can be found in one of our previous paper [4]. In recent years, the sparse based DOA estimation algorithms are proposed by various researchers and are proved to be more accurate, efficient with minimum mean square error. These algorithms are famous for their simple data model, high resolution in the estimated results and the fact that they do not require the prior knowledge of source number and can even be applied in cases where there is only a single snapshot available.

In $\ell_{2,0}$ optimization [5], joint sparsity property of the data model is exploited, but it is NP-hard to solve. The tightest convex relaxation on $\ell_{2,0}$ optimization led to a bunch of sparse based DOA estimation algorithms. An $\ell_{2,1}$ optimization [6] approach solves a BPDN problem using the LASSO model, which led to increase in complexity. Malioutov, first proposes a dimensionality reduction technique for $\ell_{2,1}$ optimization based on the conventional subspace techniques, which gave rise to $\ell_{2,1}$-SVD [7]. This paper is one of the major turning point in the field of DOA estimation research, which exhibits better accuracy, resolution even in the case of correlated signal sources. The only disadvantage in this paper is that it requires source number knowledge though it is not very much sensitive to this parameter. Tuning the parameters is another major difficulty in $\ell_{2,1}$-SVD [7].

In [8], D Wipf proposed an iterative re-weighted ℓ_1 and ℓ_2 methods for sparse signal recovery that presents a good sparse solution accuracy compared to $\ell_{2,1}$-SVD. In one of our previously published paper [9], an algorithm which is based on adaptive compressive sensing is presented for underdetermined DOA estimation, which achieves good resolution but suffers from high complexity.

To overcome the difficulties of $\ell_{2,1}$-SVD, Baye's theorem based learning algorithms were proposed in most recent years. M Tipping was the first to propose Sparse Bayesian Learning (SBL) technique for linear regression and classification problems [10], [25]. In [11], David Wipf extended the Bayesian learning algorithm for multiple measurement vector (MMV) along with Expectation Maximization (EM) framework for updating hyper-parameters.

The extension of basic SBL-EM method to incorporate temporal correlations of sparse vector for finding sparse solution was first proposed in [12] by Z Zhang and B D Rao to improve the recovery performance in MMV case. The same framework was extended for SMV case to recover block sparse signals [13]. All of these proposed algorithms are presented for sparse signal recovery problem and the same can be extended and applied for the problem of DOA estimation due to the similarity between both the mentioned problems [29].

This Bayesian learning scheme along with maximum-a-posteriori of hyper-parameters for on-grid DOA estimation was proposed in [14], which results in increased accuracy and resolution. In this paper, we extend and apply the block sparse Bayesian learning technique along with expectation maximization framework by exploiting the block correlation property of DOA spectrum. To eliminate the effect of number of snapshots on the complexity of the algorithm, unlike the existing sparse based algorithms, a covariance model is proposed in this paper, where the DOA estimation problem is reframed as the problem of covariance estimation of the unknown DOA spectrum.

In this paper, bold letters are used to represent vectors and matrices, with uppercase letters for matrices and lowercase letters for vectors.

II. The C-Step: Covariance model for DOA estimation problem

Consider a uniform linear array (ULA) with M number of array sensor elements. Let us assume K number of signal sources impinging on the ULA with actual DOAs of $\{\theta_1, \theta_2 \dots . \theta_K\}$ with respect to normal axis. This ULA model is similar to the model described in [4] and [9]. Considering an MMV case with L number of snapshots, the DOA estimation problem can be modelled as in (1).

$$\mathbf{Y} = \mathbf{AX} + \mathbf{W} \tag{1}$$

Where, $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2 \dots . \mathbf{y}_L]$ is the received signal matrix at the ULA, with each $\mathbf{y}_i$, $i \in [1, 2 \dots L]$ representing a particular snapshot measurement vector i.e. $\mathbf{y}_i = [y_1, y_2, \dots . y_M]^T{}_i$. $\mathbf{A} \in \mathbb{R}^{M\times N}$ (M<N) is the array steering/vandermonde matrix, which is a function of DOAs. By defining a N-point grid which consists of all possible DOAs from -90° to 90°, steering matrix $\mathbf{A}$ can be constructed and is a known parameter. From these known parameters of $\mathbf{Y}$ and $\mathbf{A}$, the DOA estimation problem is to estimate the spatial DOA spectrum $\mathbf{X} \in \mathbb{R}^{N\times L}$, which is a K-sparse matrix. The model is considered along with the addition of array sensor measurement noise matrix $\mathbf{W} \in \mathbb{R}^{M\times L}$. The model in (1) can be re-written for each single i^{th} snapshot as given in (2).

$$\mathbf{y}_i = \mathbf{A}\mathbf{x}_i + \mathbf{w}_i \;\; ; \; \forall\, i \in [1, 2, \dots L] \tag{2}$$

As explained in [14], applying Baye's theorem inference to (2) and assuming Gaussian probability distribution to obtain likelihood function of $\mathbf{y}_i \sim \mathcal{N}(\mathbf{A}\mathbf{x}_i, \sigma^2\mathbf{I})$ as given in (3).

$$P(\mathbf{y}_i/\mathbf{x}_i\,; \sigma^2) = \frac{1}{(2\pi)^{M/2}(\sigma^2)^{1/2}} \exp\left\{-\frac{\|\mathbf{y}_i - \mathbf{A}\mathbf{x}_i\|_2^2}{2\sigma^2}\right\} \tag{3}$$

Where, σ^2 is the noise variance and $\mathbf{A}\mathbf{x}_i$ is the mean of the likelihood function. It can be shown that using ML estimate the mean of $P(\mathbf{y}_i/\mathbf{x}_i\,; \sigma^2)$ is $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}_i$ and the covariance matrix is $\sigma^2(\mathbf{A}^T\mathbf{A})^{-1}$ [16]. Therefore, we get (4).

$$P(\mathbf{y}_i/\mathbf{x}_i\,; \sigma^2) = \mathcal{N}_{y/x}((\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}_i\,, \sigma^2(\mathbf{A}^T\mathbf{A})^{-1}) \tag{4}$$

Let us assume the prior of $\mathbf{x}_i$ as Gaussian distribution with zero mean and a covariance matrix of $\mathbf{\Gamma}$ as given in (5).

$$P(\mathbf{x}_i\,; \gamma_j) = \mathcal{N}_x(\mathbf{0}\,, \mathbf{\Gamma}) \quad ; \forall\, i \in [1, 2, \dots L] \tag{5}$$

Where, $\mathbf{\Gamma} = \mathrm{diag}\{\gamma_1, \gamma_2 \dots . \gamma_N\}$. From Baye's theorem inference, the posterior of $\mathbf{x}_i$ and the prior of $\mathbf{y}_i$ can be obtained using Gaussian product lemma [17] as given in (6).

$$P(\mathbf{x}_i/\mathbf{y}_i)P(\mathbf{y}_i) = \mathcal{N}_{x/y}(\boldsymbol{\mu}_{\mathbf{xi}}\,, \mathbf{\Sigma}_{\mathbf{xi}})\mathcal{N}_{y}(\mathbf{0}\,, \mathbf{\Sigma}_{\mathbf{yi}}) \tag{6}$$

Where, $\boldsymbol{\mu}_{\mathbf{xi}}$ and $\mathbf{\Sigma}_{\mathbf{xi}}$ are the posterior mean and covariance given by (7) and (8) respectively.

$$\boldsymbol{\mu}_{\mathbf{xi}} = \sigma^{-2}\mathbf{\Sigma}_{\mathbf{xi}}\mathbf{A}^T\mathbf{y}_i \tag{7}$$

$$\mathbf{\Sigma}_{\mathbf{xi}} = [\sigma^{-2}\mathbf{A}^T\mathbf{A} + \mathbf{\Gamma}^{-1}]^{-1} \tag{8}$$

$\mathbf{\Sigma}_{\mathbf{yi}}$ is the covariance matrix of the array measured/ received vector given by (9). This $\mathbf{\Sigma}_{\mathbf{yi}}$ can be averaged for all the available L number of snapshots to get a single covariance matrix $\mathbf{\Sigma}_{\mathbf{y}}$ for the array received signal matrix $\mathbf{Y}$ given by (9).

$$\mathbf{\Sigma}_{\mathbf{y}} = \mathbf{A}\mathbf{\Gamma}\mathbf{A}^T + \sigma^2\mathbf{I} \tag{9}$$

This covariance matrix of the measured vector $\mathbf{Y}$ depends on the $\mathbf{\Gamma}$, i.e., the variance of unknown parameter $\mathbf{X}$ [24]. This variance of $\mathbf{X}$ gives the support of $\mathbf{X}$ i.e. if the variance γ_j is zero, then it indicates that it's corresponding j^{th} row of $\mathbf{X}$ is also zero and if the variance γ_j is non-zero, then it's corresponding j^{th} row of $\mathbf{X}$ is also non-zero. This sparse support information of $\mathbf{X}$ given by $\mathbf{\Gamma}$ is sufficient for estimating the direction of arriving signal. Hence, the problem of DOA estimation has now been formulated into covariance estimation. To estimate $\mathbf{\Gamma}$ in (9), one should know $\mathbf{\Sigma}_{\mathbf{y}}$ and σ^2. The noise variance σ^2 can be estimated through ℓ_1- norm optimization as given in (10) and (11).

$$\min_{\hat{\mathbf{x}}_i} \|\hat{\mathbf{x}}_i\|_1 \quad \text{s.t.} \quad \|\mathbf{y}_i - \mathbf{A}\hat{\mathbf{x}}_i\|_2^2 \le \varepsilon \;\; ; \forall\, i \in [1, 2, \dots L] \tag{10}$$

$$\hat{\sigma}^2 = \frac{1}{ML}\sum_{i=1}^{L} \|\mathbf{y}_i - \mathbf{A}\hat{\mathbf{x}}_i\|_2^2 \tag{11}$$

Where, ε is a very small acceptable value of error. Equation (10) is a simple ℓ_1-optimization method of estimating $\mathbf{x}_i$ [18]. The estimated $\mathbf{x}_i$'s in this step also represents the DOA spectrum, but suffers with less accuracy and low resolution [21]. The covariance matrix of $\mathbf{Y}$ in (9) i.e. $\mathbf{\Sigma}_{\mathbf{y}}$, whose true values are not available (for L→∞ case), can be estimated by sample covariance matrix for finite L case as given in (12).

$$\hat{\mathbf{\Sigma}}_{\mathbf{y}} \triangleq \frac{1}{L}\sum_{i=1}^{L} \mathbf{y}_i\mathbf{y}_i^T \tag{12}$$

This sample covariance matrix is a sufficient statistic for estimating $\mathbf{\Gamma}$ but not "exact" hence, considering the covariance estimation noise $\mathbf{E}$ arising because of finite sample approximation, we get equation (13).

$$\hat{\mathbf{\Sigma}}_{\mathbf{y}} - \mathbf{\Sigma}_{\mathbf{y}} = \mathbf{E} \tag{13}$$

Now, first let us estimate the covariance noise statistics from (13), which in turn will be used for the $\mathbf{\Gamma}$ estimation in

(9). By combining (9) and (13) and vectorising, we get equation (14).

$$\mathrm{vec}(\widehat{\boldsymbol{\Sigma}}_y) = \mathrm{vec}(\mathbf{A}\boldsymbol{\Gamma}\mathbf{A}^T) + \mathrm{vec}(\sigma^2\mathbf{I}) + \mathrm{vec}(\mathbf{E})$$
$$\mathrm{vec}(\widehat{\boldsymbol{\Sigma}}_y) - \mathrm{vec}(\sigma^2\mathbf{I}) = (\mathbf{A}\odot\mathbf{A})\boldsymbol{\gamma} + \mathbf{e}$$
$$\mathbf{r} = (\mathbf{A}\odot\mathbf{A})\boldsymbol{\gamma} + \mathbf{e}$$
$$\mathbf{r} = \boldsymbol{\Phi}\boldsymbol{\gamma} + \mathbf{e} \quad (14)$$

Where $\mathbf{r} = \mathrm{vec}(\widehat{\boldsymbol{\Sigma}}_y) - \mathrm{vec}(\sigma^2\mathbf{I})$ is a known vector related to variance of the array received signal, $\odot$ denotes Khatri-Rao product [19], $\boldsymbol{\gamma} = [\gamma_1, \gamma_2 \dots \gamma_N]^T$, $\boldsymbol{\Phi} = (\mathbf{A}\odot\mathbf{A})$ and $\mathbf{e} = \mathrm{vec}(\mathbf{E})$ is the vectorized form of covariance noise matrix.

A. Covariance Noise Statistics

It can be easily shown that mean of the covariance noise $\mathbf{e}$ is zero i.e, $\mathbb{E}\{\mathbf{e}\} = 0$. To estimate the covariance of the noise $\mathbf{e}$, let us state the following Lemma 1 [15].

Lemma 1: Let $\widehat{\boldsymbol{\Sigma}}_y$ denote the sample covariance matrix and $\boldsymbol{e} = vec(\widehat{\boldsymbol{\Sigma}}_y - \boldsymbol{\Sigma}_y)$. Consider a normal random process $\boldsymbol{z}\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{I})$ in $\mathbb{R}^{N\times 1}$ space such that $\boldsymbol{F} = cov(vec(\boldsymbol{z}\boldsymbol{z}^T))$ and let $\boldsymbol{C}$ be a matrix such that $\boldsymbol{y}=\boldsymbol{C}\boldsymbol{z}$ and $\boldsymbol{\Sigma}_y = \boldsymbol{C}\boldsymbol{C}^T$. If $\boldsymbol{D} = \boldsymbol{A}^\dagger\boldsymbol{\Sigma}_y\boldsymbol{A}^{\dagger^T}$ then,

$$cov(\boldsymbol{e}) = \boldsymbol{\Sigma}_e \cong \frac{1}{L}\left[(\boldsymbol{A}\otimes\boldsymbol{A})(\boldsymbol{D}^{1/2} \otimes \boldsymbol{D}^{1/2})\boldsymbol{F}(\boldsymbol{D}^{1/2}\otimes\boldsymbol{D}^{1/2})^T(\boldsymbol{A}\otimes\boldsymbol{A})^T\right]$$

where $\otimes$ denotes the Kronecker product and $(.)^\dagger$ denotes the pseudo-inverse of a matrix.

Proof: From equation (13), we can write,

$$\mathrm{cov}(\mathbf{E}) = \mathrm{cov}\left\{\frac{1}{L}\sum_{i=1}^{L}\mathbf{y}_i\mathbf{y}_i^T - \boldsymbol{\Sigma}_y\right\}$$

By considering the fact that $\left(\sum_{i=1}^{L}\mathbf{y}_i\mathbf{y}_i^T - \boldsymbol{\Sigma}_y\right)$ are independent of i, $\mathrm{cov}(\mathbf{E}) \cong \frac{1}{L}\mathrm{cov}(\mathbf{y}\mathbf{y}^T)$ and hence vectorising, we get,

$$\mathrm{cov}(\mathrm{vec}(\mathbf{E})) \cong \frac{1}{L}\mathrm{cov}(\mathrm{vec}(\mathbf{y}\mathbf{y}^T)) \quad (15)$$
$$\mathrm{cov}(\mathbf{e}) \cong \frac{1}{L}\mathrm{cov}(\mathrm{vec}(\mathbf{C}\mathbf{z}\mathbf{z}^T\mathbf{C}^T))$$
$$\boldsymbol{\Sigma}_\mathbf{e} \cong \frac{1}{L}\mathrm{cov}((\mathbf{C}\otimes\mathbf{C})\,\mathrm{vec}(\mathbf{z}\mathbf{z}^T))$$
$$\boldsymbol{\Sigma}_\mathbf{e} \cong \frac{1}{L}\left((\mathbf{C}\otimes\mathbf{C})\mathrm{cov}(\,\mathrm{vec}(\mathbf{z}\mathbf{z}^T))(\mathbf{C}\otimes\mathbf{C})^T\right)$$
$$\boldsymbol{\Sigma}_\mathbf{e} \cong \frac{1}{L}[(\mathbf{C}\otimes\mathbf{C})\mathbf{F}(\mathbf{C}\otimes\mathbf{C})^T] \quad (16)$$

If $\mathbf{D} = \mathbf{A}^\dagger\boldsymbol{\Sigma}_y\mathbf{A}^{\dagger^T}$, then from matrix algebra it can be shown that $\mathbf{C} = \mathbf{A}\mathbf{D}^{1/2}$. Therefore, equation (16) becomes:

$$\boldsymbol{\Sigma}_\mathbf{e} \cong \frac{1}{L}\left[(\mathbf{A}\mathbf{D}^{1/2}\otimes\mathbf{A}\mathbf{D}^{1/2})\mathbf{F}(\mathbf{A}\mathbf{D}^{1/2}\otimes\mathbf{A}\mathbf{D}^{1/2})^T\right]$$
$$\boldsymbol{\Sigma}_\mathbf{e} \cong \frac{1}{L}\left[(\mathbf{A}\otimes\mathbf{A})\left(\mathbf{D}^{\frac{1}{2}}\otimes\mathbf{D}^{\frac{1}{2}}\right)\mathbf{F}\left(\mathbf{D}^{\frac{1}{2}}\otimes\mathbf{D}^{\frac{1}{2}}\right)^T(\mathbf{A}\otimes\mathbf{A})^T\right] \quad (17)$$

Where $\mathbf{D} = \mathbf{A}^\dagger\boldsymbol{\Sigma}_y\mathbf{A}^{\dagger^T}$ can also be re-written as

$$\mathbf{D} = \left[\boldsymbol{\Gamma} + \sigma^2\mathbf{A}^\dagger\mathbf{A}^{\dagger^T}\right] \quad (18)$$

It can be seen from (17) and (18) that, to estimate covariance of noise, we require $\boldsymbol{\Gamma}$ information, which is unknown again. This requires an iterative algorithm to sequentially estimate $\boldsymbol{\Sigma}_\mathbf{e}$ from the previous estimate of $\boldsymbol{\Gamma}$ and for the first step, $\boldsymbol{\Gamma}$ can be initialized to a suitable value [22]-[23]. The parameters in model (14) are all vectorized versions of MxM symmetric matrices in $\mathbb{R}^{M^2\times 1}$ space. Out of these M^2 equations, only $\frac{M(M+1)}{2}$ equations are linearly independent. By pre-multiplying the equation (14) by a selection matrix $\mathbf{S} \in \mathbb{R}^{\frac{M(M+1)}{2}\times M^2}$, we can concentrate only on these linearly independent equations. Where, this $\mathbf{S}$ matrix is formed by a subset of rows of $\mathbf{I}_{M^2}$ that will select $\frac{M(M+1)}{2}$ independent row entries of equation (14) to result in equation (19).

$$\mathbf{r}_s = \boldsymbol{\Phi}_s\boldsymbol{\gamma} + \mathbf{e}_s \quad (19)$$

Where $\mathbf{r}_s = \mathbf{S}\mathbf{r}$, $\boldsymbol{\Phi}_s = \mathbf{S}\boldsymbol{\Phi}$, $\mathbf{e}_s = \mathbf{S}\mathbf{e}$ and $\mathbf{e}_s\sim\mathcal{N}_\mathbf{e}(\mathbf{0}, \boldsymbol{\Sigma}_{es})$ with $\boldsymbol{\Sigma}_{es} = \mathbf{S}\boldsymbol{\Sigma}_e\mathbf{S}^T$.

The model in equation (19) is similar to that in equation (2), which indicates that it can be solved for unknown parameter $\boldsymbol{\gamma}$ using any of the sparse signal recovery algorithm. In this paper, we apply sparse Bayesian learning with expectation maximization framework [13], [14], [20]. Due to correlative nature of $\boldsymbol{\gamma}$, it can be processed and estimated block-wise. This intra-block correlations in $\boldsymbol{\gamma}$ is exploited in this paper to increase the accuracy of the estimated DOA's [20]. But, unlike in paper [13], this paper eliminates the update of the hyper-parameter noise variance in the maximization step.

III. The L-Step: Block Sparse Bayesian Learning

Consider the model in equation (19), where $\boldsymbol{\gamma} \in \mathbb{R}^{N\times 1}$ is the sparse vector to be recovered. Let us partition the $\boldsymbol{\gamma}$ into 'g' number of blocks, each with length d_i , $i \in \{1, 2, \dots g\}$ and not necessarily be same for all i's.

$$\boldsymbol{\gamma} = \left[\underbrace{\gamma_1, \gamma_2, \dots\gamma_{d_1}}_{\boldsymbol{\Upsilon}_1^T} \dots\dots\dots \underbrace{\gamma_{d_{g-1}+1} \dots\dots \gamma_{d_g}}_{\boldsymbol{\Upsilon}_g^T}\right]^T$$

Each block of $\boldsymbol{\gamma}$ i.e, $\boldsymbol{\Upsilon}_i \; \forall\, i \in \{1, 2, \dots g\}$ is assumed to be gaussian random vector with zero mean and covariance of $\xi_i\boldsymbol{B}_i$.

$$P(\boldsymbol{\Upsilon}_i\,;\,\xi_i, \mathbf{B}_i)\sim\mathcal{N}(\mathbf{0}, \xi_i\mathbf{B}_i) \quad ;\; \forall\, i \in \{1, 2, \dots g\}$$

The covariance matrix of each $\boldsymbol{\Upsilon}_i$ is modelled as product of a non-negative parameter that control the block sparsity i.e, ξ_i and a positive definite matrix $\mathbf{B}_i$, capturing the correlation structure of i^{th} block of $\boldsymbol{\gamma}$. By assuming the mutual un-correlation between the $\boldsymbol{\Upsilon}_i$ blocks, the covariance matrix of $\boldsymbol{\gamma}$ is written as $\boldsymbol{\Sigma}_\mathbf{0} = diag\{\xi_1\boldsymbol{B}_1, \xi_2\boldsymbol{B}_2 \dots\dots \xi_g\boldsymbol{B}_g\}$. By applying Bayesian learning scheme [14], we get the posterior of $\boldsymbol{\gamma}$ as in equation (20).

$$P\left(\boldsymbol{\gamma}/\mathbf{r}_s\,;\,\boldsymbol{\Sigma}_{es}, \{\xi_i, \boldsymbol{B}_i\}_{i=1}^{g}\right)\sim\mathcal{N}(\boldsymbol{\mu}_\gamma, \boldsymbol{\Sigma}_\gamma) \quad (20)$$

Where, $\boldsymbol{\mu}_\gamma$ *and* $\boldsymbol{\Sigma}_\gamma$ are the posterior mean and covariance given by equation (21) and (22) respectively.

$$\boldsymbol{\mu}_\gamma = \boldsymbol{\Sigma}_{es}^{-1}\,\boldsymbol{\Sigma}_\gamma\boldsymbol{\Phi}_s^T\,\mathbf{r}_s \quad (21)$$
$$\boldsymbol{\Sigma}_\gamma = [\boldsymbol{\Sigma}_{es}^{-1}\boldsymbol{\Phi}_s^T\boldsymbol{\Phi}_s + \boldsymbol{\Sigma}_0^{-1}]^{-1} \quad (22)$$

The $\boldsymbol{\mu}_\gamma$ itself is the MAP estimate of $\boldsymbol{\gamma}$, which can be plotted as the DOA spatial spectrum. The non-zero locations in $\boldsymbol{\gamma}$ correspond to the DOA's, when mapped to the on-grid points.

A. Updation of Hyper-parameters

The hyper-parameters in equation (21) and (22) are ξ_i and $\mathbf{B}_i \; \forall\, i \in \{1, 2, \dots g\}$. These parameters can be updated in every iterative step using Expectation Maximization technique. The E-step in EM algorithm involves defining of expectation of the joint probability of $\mathbf{r}_s$ and $\boldsymbol{\gamma}$ as a function of the hyper-parameters $\xi_i, \mathbf{B}_i, \boldsymbol{\Sigma}_{es}$.

$$i.e, \quad Q(\xi_i, \mathbf{B}_i, \mathbf{\Sigma}_{es}) = \mathbb{E}_{\boldsymbol{\gamma}/\mathbf{r_s}}\{P(\mathbf{r_s}, \boldsymbol{\gamma}\,;\, \mathbf{\Sigma}_{es}, \xi_i, \mathbf{B}_i)\}$$
$$Q(\xi_i, \mathbf{B}_i, \mathbf{\Sigma}_{es}) = \mathbb{E}_{\boldsymbol{\gamma}/\mathbf{r_s}}\{\log(P(\mathbf{r_s}, \boldsymbol{\gamma}\,;\, \mathbf{\Sigma}_{es}, \xi_i, \mathbf{B}_i))\}$$
$$Q(\xi_i, \mathbf{B}_i, \mathbf{\Sigma}_{es}) = \mathbb{E}_{\boldsymbol{\gamma}/\mathbf{r_s}}\left\{\log\left(P\left(\frac{\mathbf{r_s}}{\boldsymbol{\gamma}}; \mathbf{\Sigma}_{es}\right)\right)\right\} + \mathbb{E}_{\boldsymbol{\gamma}/\mathbf{r_s}}\{\log(P(\boldsymbol{\gamma}\,;\, \xi_i, \mathbf{B}_i))\} \quad (23)$$

In the M-step of EM algorithm, the function $Q(\xi_i, \mathbf{B}_i, \mathbf{\Sigma}_{es})$ in equation (23) is maximized with respect to the required hyper-parameters $\xi_i, \mathbf{B}_i$, and $\mathbf{\Sigma}_{es}$. Maximizing the first term in (23) is not essential as, $\mathbf{\Sigma}_e$ has already been updated in (17) and $\mathbf{\Sigma}_{es} = \mathbf{S}\mathbf{\Sigma}_e\mathbf{S}^T$. To estimate ξ_i , maximizing the second term in (23) with respect to ξ_i , we get:

$$\frac{\partial Q(\xi_i, \mathbf{B}_i)}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\left(\mathbb{E}_{\boldsymbol{\gamma}/\mathbf{r_s}}\{\log(P(\boldsymbol{\gamma}\,;\, \xi_i, \mathbf{B}_i))\}\right)$$
$$\frac{\partial Q(\xi_i, \mathbf{B}_i)}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\left(\mathbb{E}_{\boldsymbol{\gamma}/\mathbf{r_s}}\left\{\log\left[\frac{1}{(2\pi)^{N/2}|\mathbf{\Sigma_0}|^{1/2}}\exp\left\{-\frac{1}{2}\boldsymbol{\gamma}^T\mathbf{\Sigma_0}^{-1}\boldsymbol{\gamma}\right\}\right]\right\}\right)$$
$$\frac{\partial Q(\xi_i, \mathbf{B}_i)}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\left(\mathbb{E}_{\boldsymbol{\gamma}/\mathbf{r_s}}\left\{-\frac{1}{2}\log|\mathbf{\Sigma_0}| - \frac{1}{2}\boldsymbol{\gamma}^T\mathbf{\Sigma_0}^{-1}\boldsymbol{\gamma}\right\}\right) \quad (24)$$

Assuming different $\mathbf{B}_i$ for each 'i' may result in overfitting. If each block size is same, then constraining $\mathbf{B} = \mathbf{B}_i\ (\forall\, i)$ will be the best effective strategy. With this, $\mathbf{\Sigma_0} = diag\{\xi_1\boldsymbol{B}_1, \xi_2\boldsymbol{B}_2 \ldots\ldots \xi_g\boldsymbol{B}_g\}$ can be re-written as $\mathbf{\Sigma_0} = \boldsymbol{Z}\otimes\boldsymbol{B}$, where, $\boldsymbol{Z} = diag\{\xi_1, \xi_2 \ldots\ldots \xi_g\}$. Therefore, equation (24) becomes:

$$\frac{\partial Q(\xi_i, \mathbf{B}_i)}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\left(\mathbb{E}_{\boldsymbol{\gamma}/\mathbf{r_s}}\left\{-\frac{1}{2}\log|\boldsymbol{Z}\otimes\boldsymbol{B}| - \frac{1}{2}\boldsymbol{\gamma}^T(\boldsymbol{Z}\otimes\boldsymbol{B})^{-1}\boldsymbol{\gamma}\right\}\right)$$
$$\frac{\partial Q(\xi_i, \mathbf{B}_i)}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\left(-\frac{g}{2}log|\boldsymbol{Z}| - \frac{g}{2}log|\boldsymbol{B}| - \frac{1}{2}Tr\left[(\boldsymbol{Z}^{-1}\otimes\boldsymbol{B}^{-1})\left(\mathbf{\Sigma}_{\boldsymbol{\gamma}} + \boldsymbol{\mu}_{\boldsymbol{\gamma}}\boldsymbol{\mu}_{\boldsymbol{\gamma}}^T\right)\right]\right)$$

Simplifying and equating to zero, we get,

$$\xi_i = \frac{Tr\left[\boldsymbol{B}^{-1}\left(\mathbf{\Sigma}_{\boldsymbol{\gamma}}^i + \boldsymbol{\mu}_{\boldsymbol{\gamma}}^i\boldsymbol{\mu}_{\boldsymbol{\gamma}}^{i\,T}\right)\right]}{g} \quad (25)$$

$\boldsymbol{\mu}_{\boldsymbol{\gamma}}^i$ and $\mathbf{\Sigma}_{\boldsymbol{\gamma}}^i$ represents posterior mean and covariance of i^{th} block .

In order to estimate $\mathbf{B}$, maximizing the second term in equation (23) with respect to $\mathbf{B}$, we get,

$$\mathbf{B} = \frac{1}{g}\sum_{i=1}^{g}\frac{\left(\mathbf{\Sigma}_{\boldsymbol{\gamma}}^i + \boldsymbol{\mu}_{\boldsymbol{\gamma}}^i\boldsymbol{\mu}_{\boldsymbol{\gamma}}^{i\,T}\right)}{\xi_i} \quad (26)$$

The proposed COV-BSBL-EM algorithm steps are summarized in the Table 1.

IV. Results and Discussions

This section exhibits the experimental results of the proposed algorithm. MATLAB R2013a platform is utilized for the experiments. Considering an ULA with number of array elements M=10, number of snapshots L=5 and number of on-grid points N=361, with -20° and 31° as the true DOAs of two uncorrelated sources, Figure 1 shows the DOA spectrum for this case. The proposed COV-BSBL-EM algorithm is compared with standard and popular DOA estimation algorithms such as MUSIC [1], $\ell_{2,1}$-SVD [7] and one of our previous work SBL-MAP-H [14]. Figure 1 indicates the sharp peak at true DOAs which is the result of proposed algorithm when compared to MUSIC results, which shows a non-sharp spectrum. Figure 2, shows the results for the same above mentioned parametric case but with increased number of snapshots L=100. As there is an increase in number of snapshots, all the algorithms performs satisfactorily well with proposed algorithm being the best among all. A case with single snapshot (L=1) is shown in Figure 3, indicating the better performance of proposed

Table 1. The Proposed COV-BSBL-EM DOA Estimation Algorithm

Input Parameters: $\mathbf{Y}_{(M\times L)}$, $\mathbf{A}_{(M\times N)}$
Output Parameters: $\boldsymbol{\gamma}_{(N\times 1)}$

1. Initialize $\varepsilon = 10^{-5}$ (or to any suitable minimum value), set the value for number of blocks 'g' such that N/g is an integer. Initialize a normal random process $\mathbf{z}\sim\mathcal{N}(\mathbf{0}, \mathbf{I})$ in $\mathbb{R}^{N\times 1}$ space, $= \mathbf{I}_{N\times N}$, DOA search grid, $\mathbf{Z} = \mathbf{I}_{g\times g}$ and $\mathbf{B} = \text{ones}\left(\frac{N}{g}, \frac{N}{g}\right)$.
2. Form a selection matrix $\mathbf{S} \in \mathbb{R}^{\frac{M(M+1)}{2}\times M^2}$ by a subset of rows of $\mathbf{I}_{M^2}$.
3. $\mathbf{F} = \text{cov}(\text{vec}(\mathbf{z}\mathbf{z}^T))$
4. Obtain l_1-norm estimation of $\hat{\mathbf{x}}_i$ by: $\min_{\hat{\mathbf{x}}_i}\|\hat{\mathbf{x}}_i\|_1 \quad \text{s.t.} \quad \|\mathbf{y}_i - \mathbf{A}\hat{\mathbf{x}}_i\|_2^2 \leq \varepsilon\ ;\ \forall\, i \in [1, 2, \ldots L]$
5. Estimate the noise variance: $\hat{\sigma}^2 = \frac{1}{ML}\sum_{i=1}^{L}\|\mathbf{y}_i - \mathbf{A}\hat{\mathbf{x}}_i\|_2^2$
6. $\hat{\mathbf{\Sigma}}_{\mathbf{y}} \triangleq \frac{1}{L}\sum_{i=1}^{L}\mathbf{y}_i\mathbf{y}_i^T$
7. $\mathbf{r} = \text{vec}(\hat{\mathbf{\Sigma}}_y) - \text{vec}(\sigma^2\mathbf{I})$
8. $\mathbf{\Phi} = (\mathbf{A}\odot\mathbf{A})$
9. $\mathbf{D} = \left[\mathbf{\Gamma} + \sigma^2\mathbf{A}^{\dagger}\mathbf{A}^{\dagger T}\right]$
10. $\mathbf{\Sigma_e} \cong \frac{1}{L}\left[(\mathbf{A}\otimes\mathbf{A})\left(\mathbf{D}^{\frac{1}{2}}\otimes\mathbf{D}^{\frac{1}{2}}\right)\mathbf{F}\left(\mathbf{D}^{\frac{1}{2}}\otimes\mathbf{D}^{\frac{1}{2}}\right)^T(\mathbf{A}\otimes\mathbf{A})^T\right]$
11. $\mathbf{r}_s = \mathbf{S}\mathbf{r}$, $\mathbf{\Phi}_s = \mathbf{S}\mathbf{\Phi}$ and $\mathbf{\Sigma}_{es} = \mathbf{S}\mathbf{\Sigma}_e\mathbf{S}^T$
12. $\mathbf{\Sigma_0} = \mathbf{Z}\otimes\mathbf{B}$
13. $\mathbf{\Sigma}_{\boldsymbol{\gamma}} = [\mathbf{\Sigma}_{es}^{-1}\mathbf{\Phi}_s^T\mathbf{\Phi}_s + \mathbf{\Sigma}_0^{-1}]^{-1}$
14. $\boldsymbol{\mu}_{\boldsymbol{\gamma}} = \mathbf{\Sigma}_{es}^{-1}\,\mathbf{\Sigma}_{\boldsymbol{\gamma}}\mathbf{\Phi}_s^T\,\mathbf{r}_s$
15. Update $\xi_i = \frac{Tr\left[\mathbf{B}^{-1}\left(\mathbf{\Sigma}_{\boldsymbol{\gamma}}^i + \boldsymbol{\mu}_{\boldsymbol{\gamma}}^i\boldsymbol{\mu}_{\boldsymbol{\gamma}}^{i\,T}\right)\right]}{g}$ & $\mathbf{B} = \frac{1}{g}\sum_{i=1}^{g}\frac{\left(\mathbf{\Sigma}_{\boldsymbol{\gamma}}^i + \boldsymbol{\mu}_{\boldsymbol{\gamma}}^i\boldsymbol{\mu}_{\boldsymbol{\gamma}}^{i\,T}\right)}{\xi_i}$
16. $\boldsymbol{\gamma} \xleftarrow{\text{assign}} \boldsymbol{\mu}_{\boldsymbol{\gamma}}$
17. $\mathbf{Z} = \text{diag}\{\xi_1, \xi_2 \ldots\ldots \xi_g\}$ and $\mathbf{\Gamma} = \text{diag}\{\gamma_1, \gamma_2 \ldots.\gamma_N\}$
18. If any row of $\boldsymbol{\gamma}$ is less than a threshold, then equate the row of $\boldsymbol{\gamma}$ to zero and delete the particular corresponding column in $\mathbf{A}$ matrix for the next iteration.
19. Repeat from step 8 to step 18 until a stopping criteria is achieved.
20. Plot $\boldsymbol{\gamma}$ v/s the DOA search grid points and locate the peaks to estimate the direction of arrival.

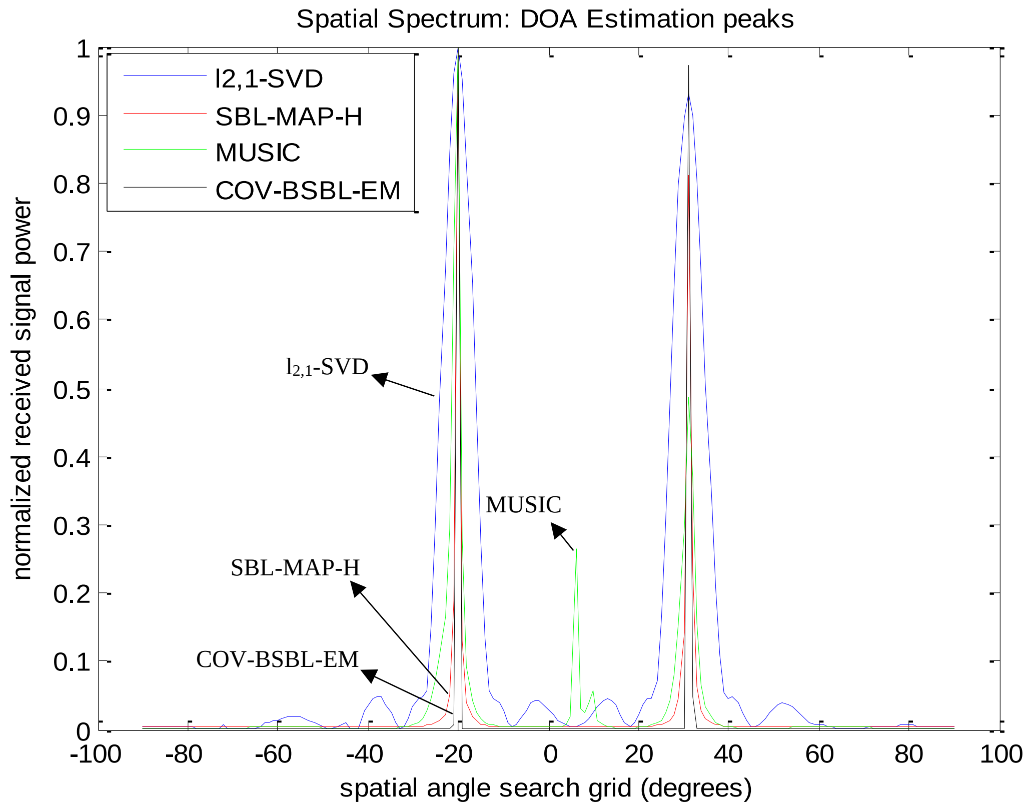

Fig. 1. DOA spectrum for L=5

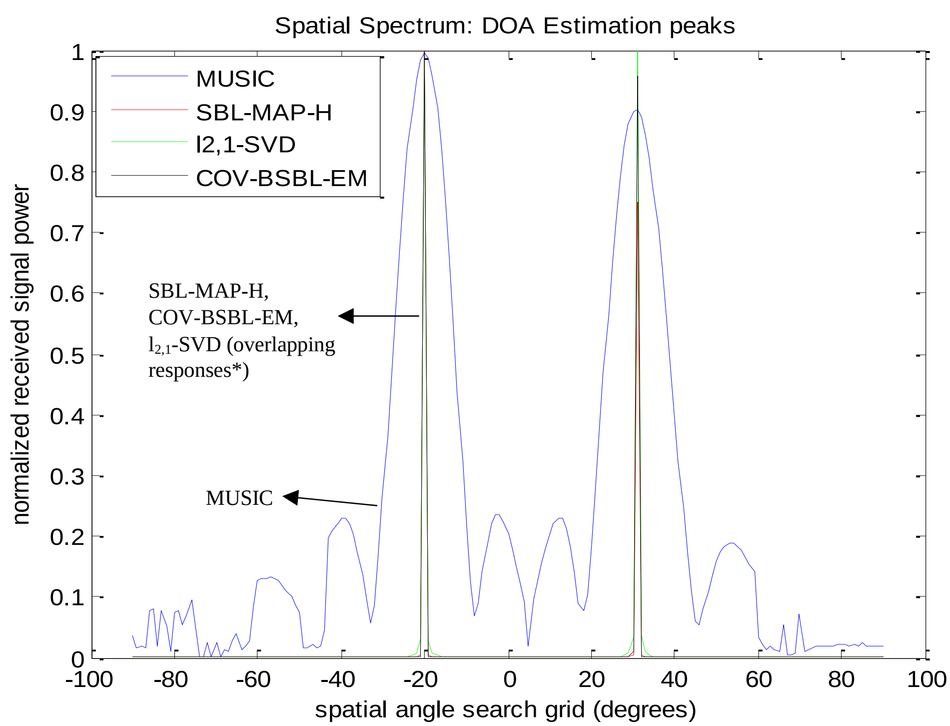

Fig. 2. DOA spectrum for L=100

*Note: All the three algorithms: SBL-MAP-H, COV-BSBL-EM and $l_{2,1}$-SVD exhibits the same peak which are overlapping, hence all the three are shown using a single pointer

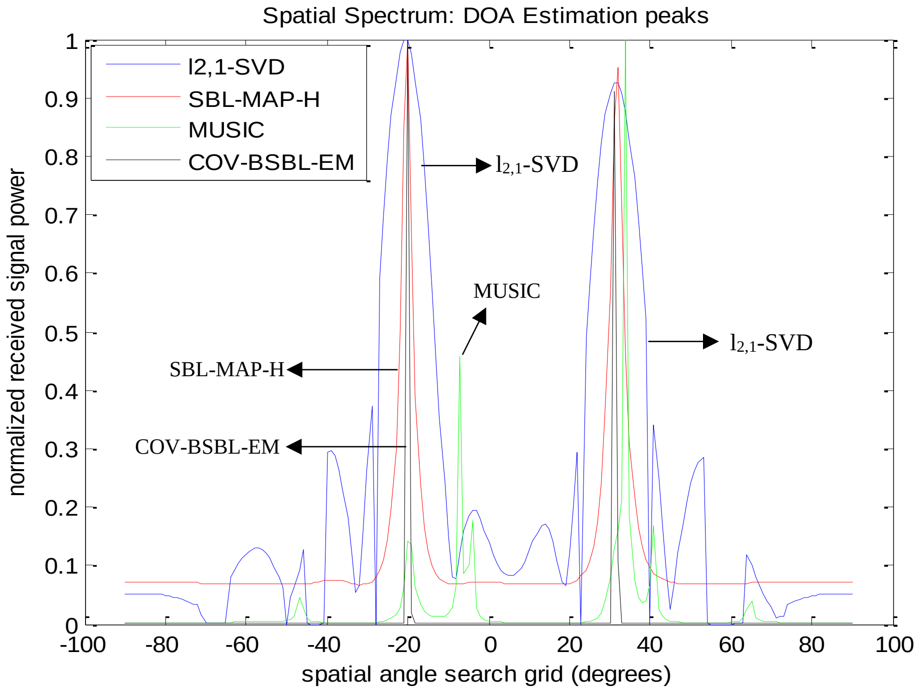

Fig. 3. DOA spectrum for L=1

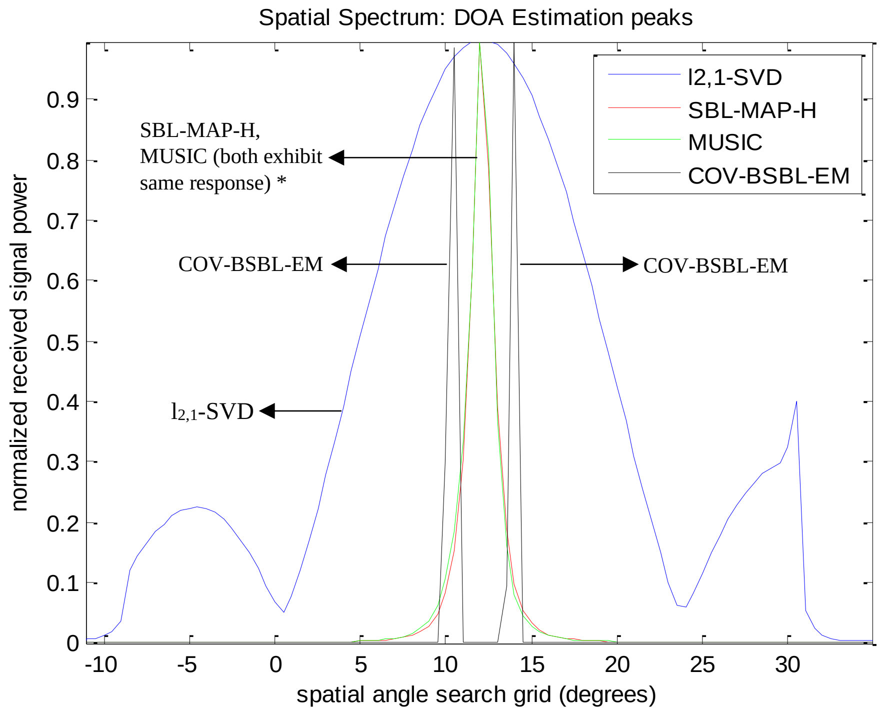

Fig. 4. DOA spectrum indicating resolution

*Note: The two algorithms: SBL-MAP-H and MUSIC exhibits the same peak which are overlapping, hence both are shown using a single pointer

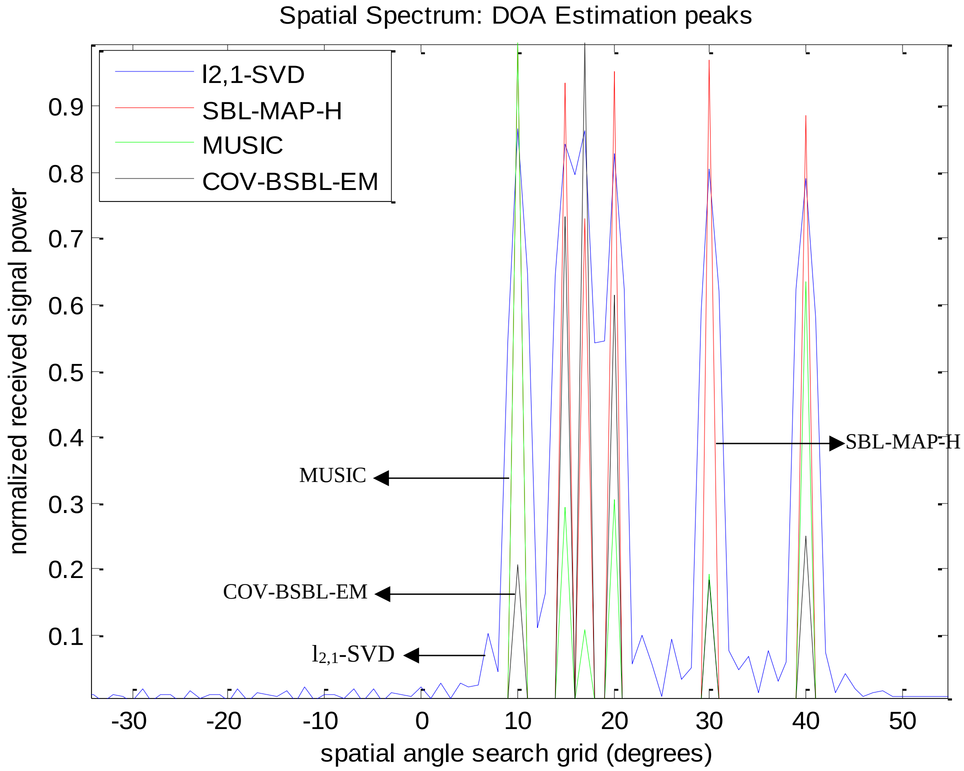

Fig. 5. DOA spectrum for underdetermined case

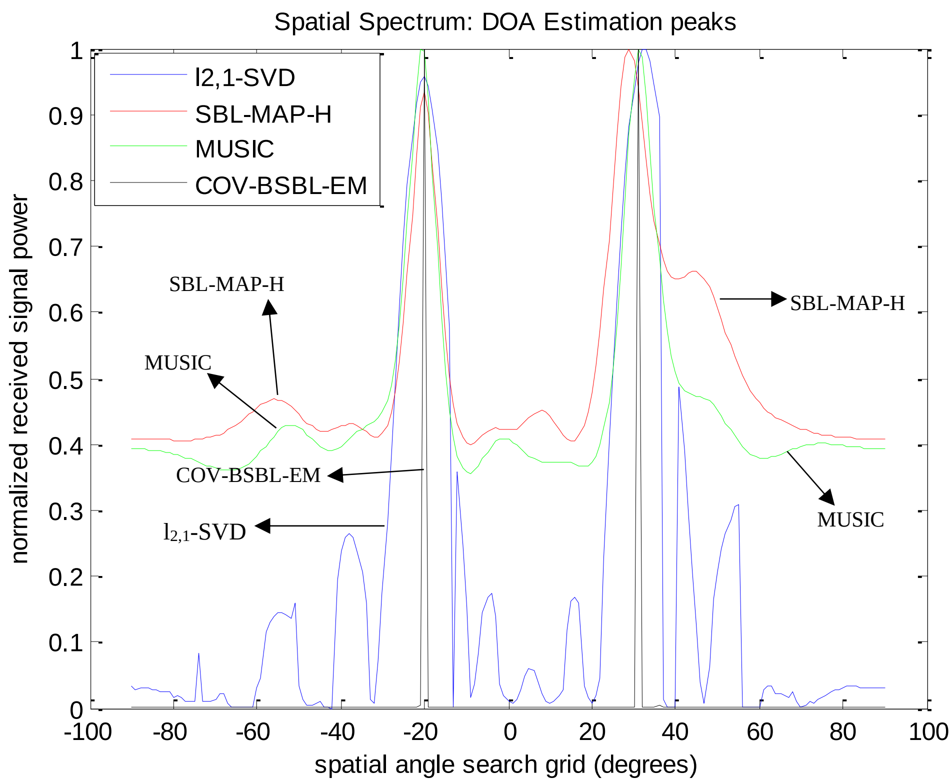

Fig. 6. DOA spectrum for a case of correlated sources

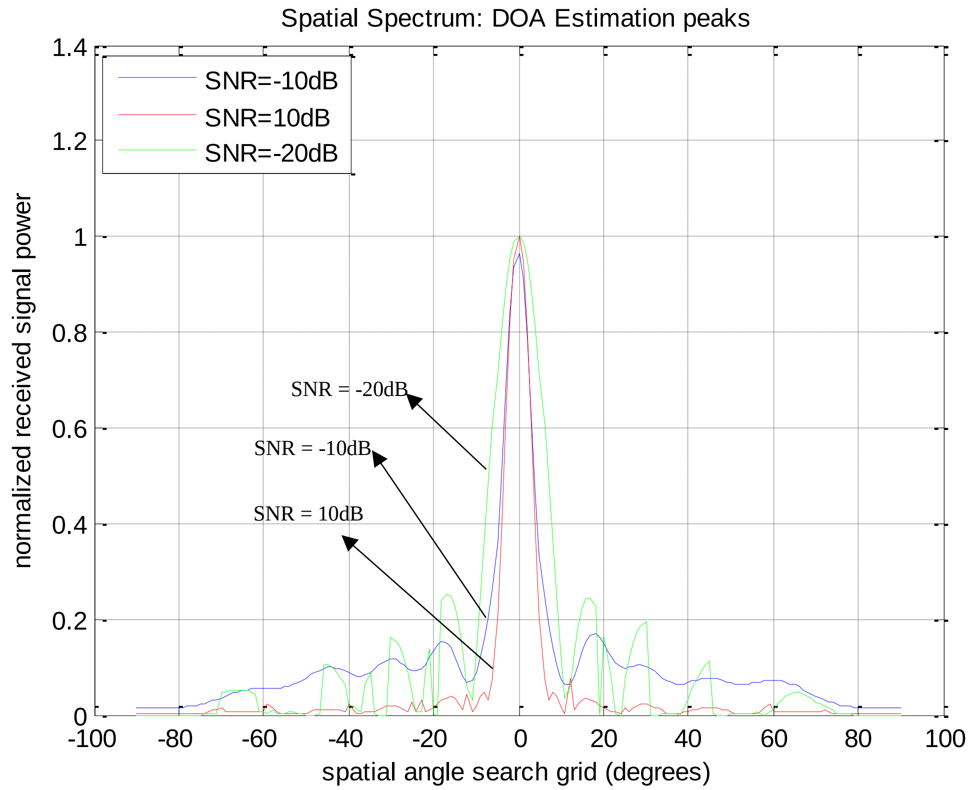

Fig. 7. Effect of SNR on the proposed algorithm

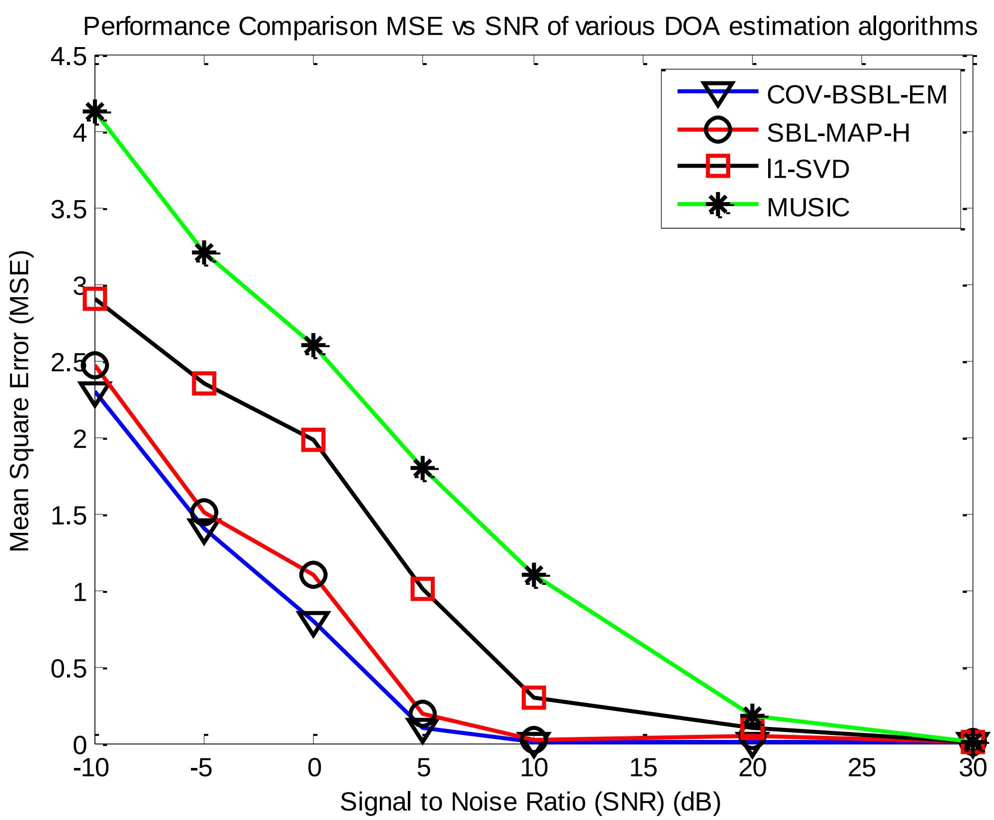

Fig. 8. Performance comparison: MSE v/s SNR

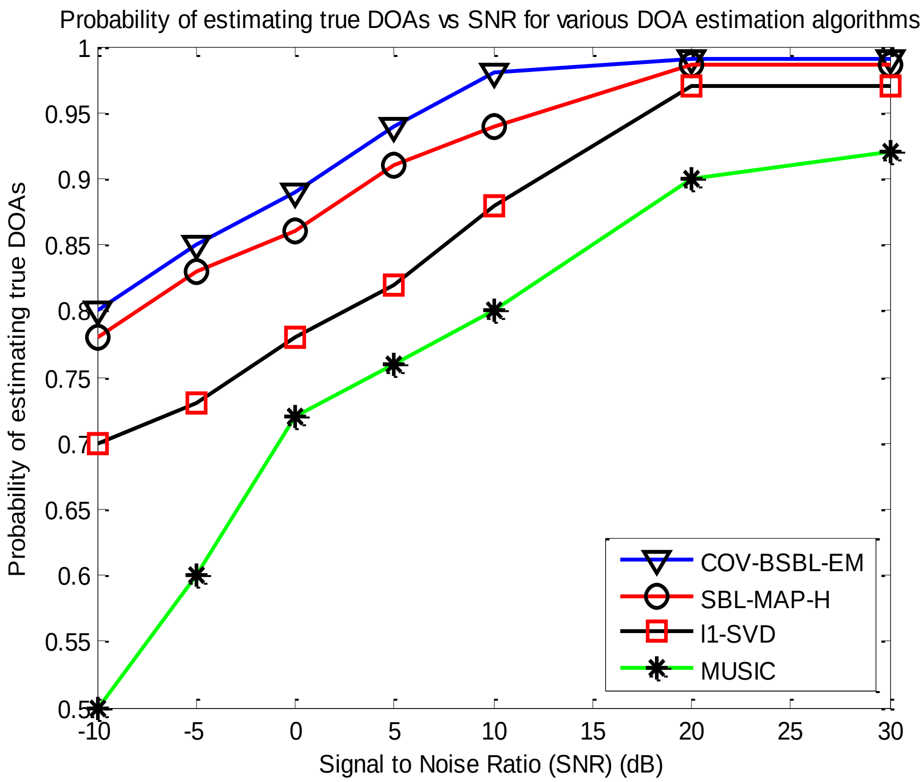

Fig. 9. Performance comparison: success rate v/s SNR

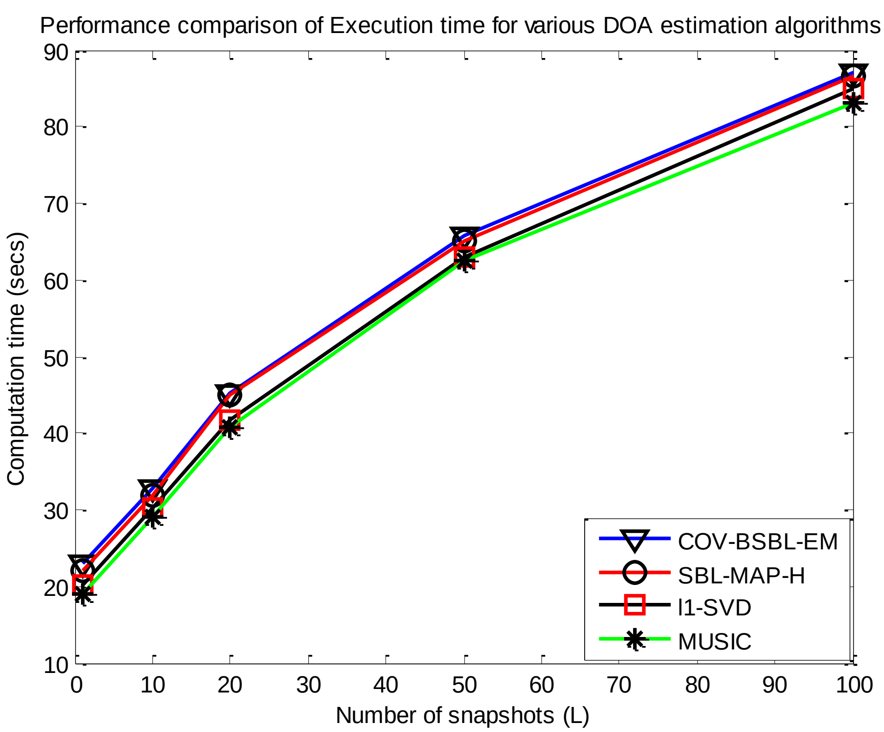

Fig. 10. Performance comparison: computation time v/s L

COV-BSBL-EM algorithm, even in the applications where only one snapshot is available.

Considering a case where the true DOAs are located very close to each other, say at 11^o and 14^o, Figure 4 exhibits the resolution accuracy of proposed algorithm, whereas, MUSIC and $\ell_{2,1}$-SVD fail to successfully identify two very closely spaced signal sources. The COV-BSBL-EM clearly differentiates two separate peaks; one at 11^o and the other one at 14^o.

Figure 5 shows the result of the case where, the number of signal sources are more than the antenna array elements with true DOAs of 10^o, 15^o, 17^o, 20^o, 30^o and 40^o and M=4 for a greater number of snapshot (L=100). The proposed algorithm gives satisfactory results even in underdetermined DOA estimation but with greater number of snapshots required. The other algorithms [26] suffer in exhibiting peaks at true DOAs for this case of undetermined DOA estimation. Figure 6 exhibits the case of correlated signal sources impinging the ULA. The true DOAs of -20^o and 31^o with a SNR of 10dB, array element size of M = 10 with N = 50 number of snapshots are considered for the experimentation purpose. It can be seen clearly from the Figure 6 that the proposed algorithm shows sharp peaks at the true DOAs. The other conventional algorithms exhibits less sharper peaks at true DOA's which affects the accuracy of the estimation. The effect of SNR on the proposed algorithm is depicted in Figure 7. Even in the low SNR range of -20dB also, the COV-BSBL-EM algorithm produces a considerable DOA peak in its spectrum. The input parameters considered for the result shown in Figure 7 are M=10, N=50 and a single true DOA at 0^o is considered.

The performance comparison of the proposed algorithm with other standard algorithms is represented in Figure 8, Figure 9 and Figure 10. Figure 8 gives the performance evaluation in-terms of mean square error (MSE) of the DOA estimation with respect to signal to noise ratio (SNR). As seen from Figure 8, the proposed algorithm achieves lesser MSE as compared to all other algorithms. As and when the SNR increases, the proposed COV-BSBL-EM algorithm performs similar MSE as that of SBL-MAP-H. In low SNR case, COV-BSBL-EM shows better results with a requirement of larger number of snapshots. For the performance comparison analysis, a fixed value for number of snapshots of L=100 is considered for all the algorithms in common.

The probability of estimating true DOAs with respect to SNR is plotted in Figure 9. It highlights the good success rate of the proposed algorithm almost similar to the $\ell_{2,1}$-SVD algorithm even in low SNR ranges. In Figure 10, the complexity and time of execution of the algorithms with respect to the number of snapshots is presented. The proposed algorithm is quite complex with higher execution time, when compared to all other algorithms, but outperforms the other algorithms in-terms of resolution and accuracy.

V. Conclusion

In this paper, a covariance-based DOA estimation algorithm is proposed by exploiting the intra-block correlation property of the DOA spatial spectrum. The covariance noise model developed in C-step helps in estimation of the noise statistics, which is a hyper-parameter required for the iterative procedure in L-step. This increases the performance in-terms of accuracy of the DOA estimation algorithm. The simulation results in the previous section indicates the better performance of the proposed algorithm when compared with the recent and standard DOA estimation algorithms. Exploitation of intra-block correlations of the DOA spectrum in sparse Bayesian learning technique results in good resolution of the proposed algorithm. The increased complexity of the proposed algorithm enlarges the execution time when compared to the other standard algorithms. In applications where, the complexity is not of a major concern, but the estimation accuracy is of utmost important, this proposed algorithm is the best suitable for DOA estimation. With further improvement in reducing the complexity and execution time of the proposed algorithm in future research, will lead into a remarkable work in the field of sparse DOA estimation.

References

[1] R. O. Schmidt, "Multiple emitter location and signal parameter estimation", IEEE Trans. Antennas Propag, vol. 34, no. 3, pp. 276-280, Mar. 1986.

[2] Richard Roy, Thonas Kailath, "ESPRIT-Estimation of Signal Parameters via Rotational Invariance Techniques", IEEE Trans on Acoustics Speech and Signal Processing, Vol. 37, No.7, pp 984-995, July 1989.

[3] Capon J, "High-resolution Frequency-wavenumber Spectrum Analysis", IEEE, Vol. 57, Issue 8, pp 1408-1418, 1969.

[4] R. K and P. K. N, "Performance Evaluation & Analysis of Direction of Arrival Estimation Algorithms using ULA," 2018 International Conference on Electrical, Electronics, Communication, Computer, and Optimization Techniques (ICEECCOT), Msyuru, India, 2018, pp. 1467-1473, doi: 10.1109/ICEECCOT43722.2018.9001455.

[5] M. M. Hyder and K. Mahata, "Direction-of-Arrival Estimation Using a Mixed $\ell _{2,0}$ Norm Approximation," in IEEE Transactions on Signal Processing, vol. 58, no. 9, pp. 4646-4655, Sept. 2010, doi: 10.1109/TSP.2010.2050477.

[6] S. F. Cotter, B. D. Rao, Kjersti Engan and K. Kreutz-Delgado, "Sparse solutions to linear inverse problems with multiple measurement vectors," in IEEE Transactions on Signal Processing, vol. 53, no. 7, pp. 2477-2488, July 2005, doi: 10.1109/TSP.2005.849172.

[7] D. Malioutov, M. Cetin and A. S. Willsky, "A sparse signal reconstruction perspective for source localization with sensor arrays," in IEEE Transactions on Signal Processing, vol. 53, no. 8, pp. 3010-3022, Aug. 2005, doi: 10.1109/TSP.2005.850882.

[8] D. Wipf and S. Nagarajan, "Iterative Reweighted ℓ_1 and ℓ_2 Methods for Finding Sparse Solutions," in IEEE Journal of Selected Topics in Signal Processing, vol. 4, no. 2, pp. 317-329, April 2010, doi: 10.1109/JSTSP.2010.2042413.

[9] Raghu K and Prameela Kumari N, "On-grid Adaptive Compressive Sensing framework for Underdetermined DOA Estimation by employing Singular Value Decomposition," International Journal of Innovative Technology and Exploring Engineering, ISSN: 2278-3075, vol. 8 Issue. 11, pp. 3076-3082, 2019.

[10] Michael E Tipping, "Sparse Bayesian Learning and the Relevance Vector Machine," Journal of Machine Learning Research, vol. 1, pp. 211-244, 2001.

[11] D. P. Wipf and B. D. Rao, "Sparse Bayesian learning for basis selection," IEEE Transactions on Signal Processing, vol. 52, no. 8, pp. 2153-2164, Aug. 2004, doi: 10.1109/TSP.2004.831016.

[12] Z. Zhang and B. D. Rao, "Sparse Signal Recovery With Temporally Correlated Source Vectors Using Sparse Bayesian Learning," in IEEE Journal of Selected Topics in Signal Processing, vol. 5, no. 5, pp. 912-926, Sept. 2011, doi: 10.1109/JSTSP.2011.2159773.

[13] Z. Zhang and B. D. Rao, "Extension of SBL Algorithms for the Recovery of Block Sparse Signals With Intra-Block Correlation," in IEEE Transactions on Signal Processing, vol. 61, no. 8, pp. 2009-2015, April15, 2013, doi: 10.1109/TSP.2013.2241055.

[14] Raghu K, Prameela Kumari N, "Bayesian learning scheme for sparse DOA estimation based on maximum-a-posteriori of

hyperparameters”, International Journal of Electrical and Computer Engineering (IJECE), Vol. 11, No. 4, August 2021, pp. 3049~3058, ISSN: 2088-8708, DOI: 10.11591/ijece.v11i4.pp3049-3058.

[15] L. Ramesh and C. R. Murthy, "Sparse Support Recovery Via Covariance Estimation," 2018 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), 2018, pp. 6633-6637, doi: 10.1109/ICASSP.2018.8462509.

[16] M. Al-Shoukairi, P. Schniter and B. D. Rao, "A GAMP-Based Low Complexity Sparse Bayesian Learning Algorithm," in IEEE Transactions on Signal Processing, vol. 66, no. 2, pp. 294-308, 15 Jan.15, 2018, doi: 10.1109/TSP.2017.2764855.

[17] S. Schoenecker and T. Luginbuhl, "Characteristic Functions of the Product of Two Gaussian Random Variables and the Product of a Gaussian and a Gamma Random Variable," in IEEE Signal Processing Letters, vol. 23, no. 5, pp. 644-647, May 2016, doi: 10.1109/LSP.2016.2537981.

[18] D. L. D. a. M. Elad, "Optimally sparse representation in general (nonorthogonal) dictionaries via l1 minimization," Proceedings of the National Academy of Sciences of the United States of America, vol. 100, no. 5, pp. 2197-2202, 2003.

[19] J. L. A. Q. XIE, "Inequalities Involving Khatri-Rao Products Of Positive Semidefinite Hermitian Matrices," International Journal of Information and System Sciences, vol. 4, no. 1, pp. 30-40, 2006.

[20] Z. Zhang and B. D. Rao, "Sparse Signal Recovery With Temporally Correlated Source Vectors Using Sparse Bayesian Learning," in IEEE Journal of Selected Topics in Signal Processing, vol. 5, no. 5, pp. 912-926, Sept. 2011, doi: 10.1109/JSTSP.2011.2159773.

[21] Z. M. Liu, Z. T. Huang, and Y. Y. Zhou, “Direction-of-arrival estimation of wideband signals via covariance matrix sparse representation”, IEEE Trans. Signal Process., vol. 59, no. 9, pp. 4256-4270, Sep. 2011.

[22] Q. Huang, G. Zhang and Y. Fang, "DOA Estimation Using Block Variational Sparse Bayesian Learning," in Chinese Journal of Electronics, vol. 26, no. 4, pp. 768-772, 7 2017, doi: 10.1049/cje.2017.04.004.

[23] Amgad A, Salama M, Omair Ahmad and M. N. S. Swamy, “Underdetermined DOA Estimation Using MVDR-Weighted LASSO”, MDPI-Sensors, Vol 16, 2016.

[24] N. Hu, B. Sun, J. Wang and J. Yang, "Covariance-based DOA estimation for wideband signals using joint sparse Bayesian learning," 2017 IEEE International Conference on Signal Processing, Communications and Computing (ICSPCC), Xiamen, 2017, pp. 1-5, doi: 10.1109/ICSPCC.2017.8242525.

[25] A.C. Faul and M.E. Tipping, “Analysis of sparse Bayesian learning,” Advances in Neural Information Processing Systems, vol. 14, pp. 383–389, 2002.

[26] Chen Miao, Hui Tang, Yue Ma, and Peishuang Ni, "Underdetermined DOA Estimation Based on Target Space Diversity," IAENG International Journal of Computer Science, vol. 48, no.1, pp80-85, 2021.

[27] Denis S. Radchenko, Anton B. Tokarev, Alexander A. Makarov, and Oleg V. Chernoyarov, "Detection of the Frequency Hopping Spread Spectrum Signal Sources and Estimation of Their Number by Direction Finding Data," Engineering Letters, vol. 30, no.2, pp797-805, 2022.

[28] Oleg V. Chernoyarov, Vladimir A. Ivanov, Alexandra V. Salnikova, and Marina A. Slepneva, "Intra-Period Signal Processing in a Synthetic Aperture Radar," Engineering Letters, vol. 30, no.1, pp66-72, 2022.

[29] Oleg V. Chernoyarov, Alexey N. Glushkov, Vladimir P. Litvinenko, Boris V. Matveev, and Alexander A. Makarov, "Algorithms and Devices for Noncoherent Digital Radio Signal Processing," Engineering Letters, vol. 28, no.4, pp1238-1248, 2020.

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