An Fixed Point Iterative Method for Tensor Complementarity Problems

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***Abstract*—Nonlinear fixed point iterative method for finding a solution of the tensor complementarity problem (TCP) are proposed in this paper. Theoretical analysis shows that tensor complementarity problems are equivalent to a fixed point equation with monotonic increasing odd function. A fixed point iterative method is proposed based on the fixed point equation, and corresponding convergence results are studied. The computer-simulation results further substantiate that the proposed fixed point iterative method can find a solution of the TCP.**

***Index Terms*—Tensor complementarity problem, Fixed point, Iterative method, Convergence.**

I. Introduction

NOWADAYS, nonlinear complementarity problem (NCP) [1] is a popular research topic in the optimization fields [2]. Tensor complementarity problem (TCP) is a subset of the nonlinear complementarity problem (NCP) [1]. Tensor complementarity comes from many application problems, such as the n-person game theory [3], [4], [5], [6], hypergraph clustering, etc [1], [7], [8], [9].

For convenience, we use $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^{[m,n]}$ denote the real field, complex field and all order m and dimensional n real tensor, respectively. Tensors are a high-order extension of matrices, which have the following form

$$\mathcal{B} = (b_{i_1 i_2 \ldots i_m}),\ b_{i_1 i_2 \ldots i_m} \in \mathbb{R},\ i_k \in [n],\ k \in [m].$$

We call above multi-array $\mathcal{A}$ as an order m and dimensional n real tensor. Let $\mathbf{z} \in \mathbb{R}^n$, then $\mathcal{B}\mathbf{z}^{m-1}$ is a n dimensional vector, which can be defined as [10]:

$$(\mathcal{B}\mathbf{z}^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} b_{ii_2\ldots i_m} z_{i_2} \ldots z_{i_m},\ i \in [n], \tag{I.1}$$

here z_i is the ith entry of the vector $\mathbf{z}$. $\mathcal{B}\mathbf{z}^{m-2}$ is an $n \times n$ matrix, which can be defined by

$$(\mathcal{B}\mathbf{z}^{m-2})_{ij} = \sum_{i_3,\ldots,i_m}^{n} b_{iji_3,\ldots,i_m} z_{i_3} \ldots z_{i_m},\ i,j \in [n].$$

For any $\mathcal{B} \in \mathbb{R}^{[m,n]}$ and $\mathbf{y} \in \mathbb{R}^n$, the tensor complementarity problem, simplified by TCP$(\mathcal{B}, \mathbf{y})$, can be stated as: finding $\mathbf{z} \in \mathbb{R}^n$ and $\mathbf{z}$ holds

$$\mathbf{z} \geq \mathbf{0},\ \mathcal{B}\mathbf{z}^{m-1} + \mathbf{y} \geq \mathbf{0},\ \mathbf{z}^\top(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}) = 0. \tag{I.2}$$

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If $\mathcal{B} \in \mathbb{R}^{[2,n]}$ and $\mathbf{y} \in \mathbb{R}^n$, then TCP$(\mathcal{B}, \mathbf{y})$ reduces to the linear case and called as linear complementarity problem LCP$(\mathbf{B}, \mathbf{y})$, that is

$$\mathbf{z} \geq \mathbf{0},\ \mathbf{B}\mathbf{z} + \mathbf{y} \geq \mathbf{0},\ \mathbf{z}^\top(\mathbf{B}\mathbf{z} + \mathbf{y}) = 0.$$

Therefore, TCP$(\mathcal{B}, \mathbf{y})$ is a extension of linear case. So far a large number of researchers have focused on this topic [4], [6], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22].

Theoretical results of the TCP$(\mathcal{B}, \mathbf{y})$ have been reported in [4], [6], [11], [12], [13], [14], [15], [16], [17], [19], [20], [21], [22], [23], [24] and the references therein.

Recently, many numerical algorithms for finding the solution of the TCPs have also been provided, such as nonsmooth Newton's method [25], smoothing type algorithms [4], Kojima-Megiddo-Mizuno type continuation method [26], gradient dynamic approach [27], inexact Levenberg-Marquardt method [16], mixed integer programming model [28]. Dai [29] proposed a fixed iterative method for solving TCP$(\mathcal{B}, \mathbf{y})$. The fixed point method has shown many advantages in dealing with nonlinear problems [30], [31], [32]. In this paper, we prove that TCP$(\mathcal{B}, \mathbf{y})$ (I.2) is equivalent to a fixed point equation, which is a extension of the fixed point equation in [29].

This paper arrange the rest parts as follows. In Section II, we review some preliminary background and give some definitions and results. We show that the TCP$(\mathcal{B}, \mathbf{y})$ is equivalent to a fixed point equation in Section III. Some numerical results are considered in Section IV.

II. Preliminaries

We denote the set $\{1, 2, , \ldots, n\}$ by $[n]$ and denote tensor with capital calligraphic letters, for example $\mathcal{B} \in \mathbb{R}^{[m,n]}$. We denote the entries of a tensor $\mathcal{B} \in \mathbb{R}^{[m,n]}$ by $b_{i_1 i_2 \ldots i_m}$, where $i_j \in [n]$ and $j \in [m]$. We denote matrices and vectors by bold capital and bold lower case letters, respectively. Let $\mathbf{A}, \mathbf{B}, \mathbf{T} \in \mathbb{R}^{[2,n]}$ with $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$ and $\mathbf{T} = (t_{ij})$, $\mathbf{T} = \max\{\mathbf{A}, \mathbf{B}\}$ means $t_{ij} = \max\{a_{ij}, b_{ij}\}$.

Firstly, we review a useful Lemma as follows.

Lemma II.1. [29] *Suppose $\mathbf{z} \in \mathbb{R}^n$, we define vector $\mathbf{z}_+$ as $(\mathbf{z}_+)_k = \max\{0, z_k\}$, $n \in [n]$. Thus for all $\mathbf{w}, \mathbf{v} \in \mathbb{R}^n$, the following inequalities hold:*

1) $(\mathbf{w} + \mathbf{v})_+ \leq \mathbf{w}_+ + \mathbf{v}_+$;
2) $\mathbf{w}_+ - \mathbf{v}_+ \leq (\mathbf{w} - \mathbf{v})_+$;
3) $|\mathbf{w}| = \mathbf{w}_+ + (-\mathbf{w})_+$;
4) $\mathbf{w} \leq \mathbf{v}$ *implies* $\mathbf{w}_+ \leq \mathbf{v}_+$.

We recall the definitions of eigenvalues and eigenvectors of tensors, which has been introduced by Lim [33] and Qi [10], independently.

Definition II.1. ([10], [33]) *Set $\mathcal{B} \in \mathbb{R}^{[m,n]}$. If there exists a vector $\mathbf{z} \in \mathbf{C}^n \setminus \{\mathbf{0}\}$ and $\mathbf{z}$ holds*

$$\mathcal{B}\mathbf{z}^{m-1} = \lambda \mathbf{z}^{[m-1]},$$

we call $\lambda \in \mathbb{C}$ as an eigenvalue of $\mathcal{B}$, where the entries of the vector $\mathbf{z}^{[m-1]}$ are given by

$$z_k^{[m-1]} = z_k^{m-1} \; for \; k \in [n].$$

This nonzero vector $\mathbf{z}$ is the eigenvector associated to eigenvalue λ.

We recall the following lemma.

Lemma II.2. ([34]) *Suppose $\mathbf{u} \in \mathbb{C}^n \backslash \{\mathbf{0}\}$ and $\mathbf{y} \in \mathbb{C}^n \backslash \{0\}$. Then $\frac{1}{\mathbf{z}^H \mathbf{z}} \mathbf{y}\mathbf{z}^H$ is a solution of the linear system $\mathbf{A}\mathbf{z} = \mathbf{y}$.*

III. Fixed point iterative method for TCP

We will prove firstly that TCP($\mathcal{B}, \mathbf{y}$) is equivalent to a fixed point equation, furthermore, we provide the fixed point iterative method to solve TCP($\mathcal{B}, \mathbf{y}$) in this section.

A. Odd and monotonically increasing functions

Let $\mathbf{W} = (w_{ks})$, the function $\mathcal{G}(\mathbf{W})$ is defined as $(g(w_{ks}))$, $k, s \in [n]$, where $g(\cdot)$ is an odd and monotonically increasing function (OMIF). We usually take $g(\cdot)$ as the following functions .

Linear function (LF)

$$g_{\mathrm{lf}}(z) = z;$$

Bipolar-sigmoid function (BsF)

$$g_{\mathrm{bsf}}(z, r) = \frac{1 + \exp(-r)}{1 - \exp(-r)} \frac{1 - \exp(-rz)}{1 + \exp(-rz)}, \; r > 2;$$

Smooth power-sigmoid function (SpsF)

$$g_{\mathrm{spsf}}(z, r, s) = \frac{1}{2} z^r + \frac{1 + \exp(-r)}{1 - \exp(-r)} \frac{1 - \exp(-sz)}{1 + \exp(-sz)}, \quad r \geq 3, \; s > 2.$$

In all, any OMIF $g(\cdot)$ can be used for building of the fixed pint iterative method [35].

B. Fixed point equation for TCP($\mathcal{B}, \mathbf{y}$)

We transform the TCP($\mathcal{B}, \mathbf{y}$) into a fixed point iterative method in this subsection. Dai [29] proposed a fixed equation of the TCP($\mathcal{B}, \mathbf{y}$) and have the following lemma.

Lemma III.1. [29] *Suppose $\mathcal{B} \in \mathbb{R}^{[m,n]}$, $\mathbf{y} \in \mathbb{R}^n$, parameter $\alpha > 0$ and $\mathbf{\Omega} = diag(d_1, d_2, \ldots, d_n)$ with $d_k > 0, \; k = 1, 2, \ldots, n$. Then TCP($\mathcal{B}, \mathbf{y}$) (I.2) is equivalent to the following equation*

$$\mathbf{z}^{[m-1]} = \left(\mathbf{z}^{[m-1]} - \alpha \mathbf{\Omega} (\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}) \right)_+, \qquad \text{(III.1)}$$

where $(\mathbf{y})_+ = \max(\mathbf{y}, \mathbf{0})$.

Notice that, positive diagonal matrix $\mathbf{\Omega}$ is a projection operator in equation (III.1). The convergence properties of fixed iterative method (III.1) is determined by $\mathbf{\Omega}$. Hence, we extend $\mathbf{\Omega}$ in (III.1) to more common forms.

We extend the matrix $\mathbf{\Omega}$ to an OMIF $g(\cdot)$ and obtain TCP($\mathcal{B}, \mathbf{y}$) and fixed point equation are equivalent.

Theorem III.1. *Suppose $\mathcal{B} \in \mathbb{R}^{[m,n]}$, $\mathbf{y} \in \mathbb{R}^n$, $\alpha > 0$ and $g(\cdot)$ be a OMIF. Then TCP($\mathcal{B}, \mathbf{y}$) (I.2) is equivalent to the following equation*

$$\mathbf{z}^{[m-1]} = \left(\mathbf{z}^{[m-1]} - \alpha \mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}) \right)_+. \qquad \text{(III.2)}$$

Proof: Because $g(\cdot)$ is an odd function, we have $g(-z_i) = -g(z_i)$ and

$$g(z_i) \begin{cases} \geq 0, \; for \; z_i \geq 0; \\ < 0, \; for \; z_i < 0. \end{cases}$$

Suppose that $\mathbf{z}$ is a solution of TCP($\mathcal{B}, \mathbf{y}$). If $z_i = 0$ and $(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y})_i \geq 0$, integrate with $g(\cdot)$ is a monotonically increasing function, we immediately obtain

$$\begin{aligned} & z_i^{m-1} - \left(z_i^{m-1} - \alpha (\mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}))_i \right)_+ \\ & = - \left(-\alpha (\mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}))_i \right)_+ = 0. \end{aligned}$$

If $(\mathcal{B}\mathbf{z}^{m-1} - \mathbf{y})_i = 0$ and $z_i \geq 0$, then

$$\begin{aligned} & z_i^{m-1} - \left(z_i^{m-1} - \alpha (\mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}))_i \right)_+ \\ & = z_i^{m-1} - z_i^{m-1} = 0. \end{aligned}$$

Conversely, assume that (III.2) holds. Notice that

$$\mathbf{z}^{[m-1]} = \left(\mathbf{z}^{[m-1]} - \alpha \mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}) \right)_+ \geq \mathbf{0},$$

we have $\mathcal{B}\mathbf{z}^{m-1} - \mathbf{y} \geq \mathbf{0}$. In fact, if $(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y})_i < \mathbf{0}$ for some i, then

$$\begin{aligned} 0 & = z_i^{m-1} - \left(z_i^{m-1} - \alpha (\mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}))_i \right)_+ \\ & = \alpha (\mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}))_i < 0, \end{aligned}$$

which is a contradiction. For one hand, if

$$z_i^{m-1} - \alpha (\mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}))_i \geq 0,$$

then,

$$\begin{aligned} 0 & = z_i^{m-1} - \left(z_i^{m-1} - \alpha (\mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}))_i \right)_+ \\ & = \alpha (\mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}))_i. \end{aligned}$$

Based on $g(\cdot)$ is an odd function, we have $(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y})_i = 0$. Therefore

$$z_i (\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y})_i = 0.$$

For the other hand, if

$$z_i^{m-1} - \alpha (\mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}))_i < 0,$$

then,

$$0 = z_i^{m-1} - \left(z_i^{m-1} - \alpha (\mathcal{G}(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y}))_i \right)_+ = z_i^{m-1},$$

which shows

$$z_i (\mathcal{B}\mathbf{z}^{m-1} + \mathbf{y})_i = 0.$$

This completes the proof. ∎

C. Fixed point iterative method for TCP$(\mathcal{B}, \mathbf{y})$

In this subsection, we attain a general fixed point iterative method to solve TCP$(\mathcal{B}, \mathbf{y})$ based on fixed point equation (III.2) as follows.

$$\mathbf{z}_{k+1} = \left[\left(\mathbf{z}_k^{[m-1]} - \alpha\mathcal{G}(\mathcal{B}\mathbf{z}_k^{m-1} + \mathbf{y})\right)_+\right]^{[\frac{1}{m-1}]}. \quad \text{(III.3)}$$

Next, we consider the convergence properties on the proposed iterative method. We first have the following lemma for the fixed point iterative method (III.3).

Lemma III.2. *Let $\mathcal{B} \in \mathbb{R}^{[m,n]}$. For any $\mathbf{y} \in \mathbb{R}^n$, the iterative sequence $\{\mathbf{z}_k\}$ satisfy the recursive inequality*

$$\left|\mathbf{z}_{k+1}^{[m-1]} - \mathbf{z}_k^{[m-1]}\right| \le |\mathbf{I} - \alpha\mathbf{y}_k| \left|\mathbf{z}_k^{[m-1]} - \mathbf{z}_{k-1}^{[m-1]}\right|, \quad \text{(III.4)}$$

where $\mathbf{y}_k$ satisfies

$$\mathcal{G}(\mathcal{B}\mathbf{z}_k^{m-1} + \mathbf{y}) - \mathcal{G}(\mathcal{B}\mathbf{z}_{k-1}^{m-1} + \mathbf{y}) = \mathbf{y}_k\left(\mathbf{z}_k^{[m-1]} - \mathbf{z}_{k-1}^{[m-1]}\right).$$

Proof: Based on fixed point iterative formula (III.3), we obtain the following relation

$$\begin{aligned} \mathbf{z}_{k+1}^{[m-1]} - \mathbf{z}_k^{[m-1]} &= \left(\mathbf{z}_k^{[m-1]} - \alpha\mathcal{G}(\mathcal{B}\mathbf{z}_k^{m-1} + \mathbf{y})\right)_+ \\ &\quad - \left(\mathbf{z}_{k-1}^{[m-1]} - \alpha\mathcal{G}(\mathcal{B}\mathbf{z}_{k-1}^{m-1} + \mathbf{y})\right)_+ \\ &\le (\mathbf{z}_k^{[m-1]} - \mathbf{z}_{k-1}^{[m-1]} - \alpha\mathcal{G}(\mathcal{B}\mathbf{z}_k^{m-1} \\ &\quad +\mathbf{y}) + \alpha\mathcal{G}(\mathcal{B}\mathbf{z}_{k-1}^{m-1} + \mathbf{y}))_+ \\ &\le \mathbf{z}_k^{[m-1]} - \mathbf{z}_{k-1}^{[m-1]} - \alpha[\mathcal{G}(\mathcal{B}\mathbf{z}_k^{m-1} \\ &\quad +\mathbf{y}) - \mathcal{G}(\mathcal{B}\mathbf{z}_{k-1}^{m-1} + \mathbf{y})]_+. \end{aligned}$$

By Lemma II.2, we have

$$\begin{aligned} \mathbf{z}_{k+1}^{[m-1]} - \mathbf{z}_k^{[m-1]} &\le \Big(\mathbf{z}_k^{[m-1]} - \mathbf{z}_{k-1}^{[m-1]} \\ &\quad -\alpha\mathbf{y}_k(\mathbf{z}_k^{[m-1]} - \mathbf{z}_{k-1}^{[m-1]})\Big)_+ \\ &= \left((\mathbf{I} - \mathbf{y}_k)(\mathbf{z}_k^{[m-1]} - \mathbf{z}_{k-1}^{[m-1]})\right)_+ \end{aligned}$$

and then

$$\left(\mathbf{z}_{k+1}^{[m-1]} - \mathbf{z}_k^{[m-1]}\right)_+ \le \left((\mathbf{I} - \mathbf{y}_k)(\mathbf{z}_k^{[m-1]} - \mathbf{z}_{k-1}^{[m-1]})\right)_+. \quad \text{(III.5)}$$

Analogously, we have

$$\left(\mathbf{z}_k^{[m-1]} - \mathbf{z}_{k+1}^{[m-1]}\right)_+ \le \left((\mathbf{I} - \mathbf{y}_k)(\mathbf{z}_{k-1}^{[m-1]} - \mathbf{z}_k^{[m-1]})\right)_+. \quad \text{(III.6)}$$

Since $|\mathbf{z}| = (\mathbf{z})_+ + (-\mathbf{z})_+$, combining (III.5) and (III.6), we obtain

$$\begin{aligned} |\mathbf{z}_{k+1}^{[m-1]} - \mathbf{z}_k^{[m-1]}| &\le |(\mathbf{I} - \mathbf{y}_k)(\mathbf{z}_k^{[m-1]} - \mathbf{z}_{k-1}^{[m-1]})| \\ &\le |\mathbf{I} - \mathbf{y}_k||\mathbf{z}_k^{[m-1]} - \mathbf{z}_{k-1}^{[m-1]}|. \end{aligned} \quad \text{(III.7)}$$

We complete the proof. ■

Based on Lemma III.2, we can attain the convergence result on the proposed fixed point iterative method.

Theorem III.2. *Assume that $\mathcal{B} \in \mathbb{R}^{[m,n]}$. If $\rho(\mathbf{T}) < 1$, then the iterative sequence $\{\mathbf{z}_k\}$ generated by the fixed point iterative method (III.3) converges to a solution of the TCP$(\mathcal{B}, \mathbf{y})$ for any $\mathbf{y} \in \mathbb{R}^n$ and initial guess vector $\mathbf{z}_0 \in \mathbb{R}^n$,, where $\mathbf{T} = \max_{1\le k}\{\mathbf{T}_k\}$ and $\mathbf{T}_k = |\mathbf{I} - \mathbf{y}_k|$.*

Proof: Our purpose is to explain

$$\lim_{k\to+\infty} |\mathbf{z}_{k+1}^{[m-1]} - \mathbf{z}_k^{[m-1]}| = 0.$$

Set $\mathbf{T} = \max_{1\le k}\{|\mathbf{I} - \mathbf{y}_k|\}$. From

$$\begin{aligned} |\mathbf{z}_k^{[m-1]} - \mathbf{z}_0^{[m-1]}| &\le |\mathbf{z}_k^{[m-1]} - \mathbf{z}_{k-1}^{[m-1]}| + |\mathbf{z}_{k-1}^{[m-1]} \\ &\quad -\mathbf{z}_{k-2}^{[m-1]}| + \ldots + |\mathbf{z}_1^{[m-1]} - \mathbf{z}_0^{[m-1]}|. \end{aligned}$$

Together with inequality (III.4) in Lemma III.2 with $\mathbf{T} = \max_{1\le k}\{\mathbf{T}_k\}$ and $\mathbf{T}_k = |\mathbf{I} - \mathbf{y}_k|$, this lead to

$$\begin{aligned} |\mathbf{z}_k^{[m-1]} - \mathbf{z}_0^{[m-1]}| &\le (\mathbf{T}_{k-1}\mathbf{T}_{k-2}\ldots\mathbf{T}_2\mathbf{T}_1 + \mathbf{T}_{k-2}\ldots\mathbf{T}_2 \\ &\quad \mathbf{T}_1 + \ldots + \mathbf{T}_1 + \mathbf{I})|\mathbf{z}_1^{[m-1]} - \mathbf{z}_0^{[m-1]}| \\ &\le (\mathbf{T}^{k-1} + \ldots + \mathbf{I})|\mathbf{z}_1^{[m-1]} - \mathbf{z}_0^{[m-1]}| \\ &\le (\mathbf{I} - \mathbf{T})^{-1}|\mathbf{z}_1^{[m-1]} - \mathbf{z}_0^{[m-1]}|, \end{aligned}$$

it follows from $\mathbf{T}$ is a nonnegative matrix and $\rho(\mathbf{T}) < 1$.

Employing the same approach of the proof of Theorem 3.1 in [29], we can show the iterative sequence $\{\mathbf{z}_k\}$ converges to a solution of TCP$(\mathcal{B}, \mathbf{y})$. So it is omitted. The proof is completed. ■

IV. Test examples

We show the efficacy of the proposed iterative method for solving TCP$(\mathcal{B}, \mathbf{y})$ by using two numerical examples. We compare proposed fixed point iterative method and method I present in [29] on iterative numbers and CPU times in this section,. In all tables, we use 'LF', 'BsF', 'SpsF' and 'M-I' denote the proposed fixed point iterative method with LF, BsF, SpsF and method I present in [29], respectively.

Let

$$\text{Err} = \|\mathbf{z}_k - \mathbf{z}_{k-1}\|_2. \quad \text{(IV.1)}$$

In our numerical tests, we take the stopping criteria as Err $< 10^{-12}$ or iteration number is less than 1000 times.

In the following two tests, we take vector $\mathbf{y} \in \mathbb{R}^n$ with $y_i = (-1)^i i$ for $i \in [n]$ and initial vector $\mathbf{z}_0 = (1, 1, \ldots, 1)^\top$. We compare superiority of the proposed iterative method and method I [29] with $\alpha = 1$ for solving TCP$(\mathcal{B}, \mathbf{y})$ with different order m and dimension n, where BsF is $f_{bsf}(v, 3)$ and SpsF is $f_{spsf}(v, 5, 7)$.

Example IV.1. *Let tensor $\mathcal{B} \in \mathbb{R}^{[n,n]}$ with*

$$b_{k_1\ldots k_m} = |\sin(k_1 + k_2 + \ldots + k_m)|,$$

then $\mathcal{B} = n^{m-1}\mathcal{I} - \mathcal{B}$ is a nonsingular tensor, it follows from [36], [37], *where $\mathcal{I}$ is a real unit tensor.*

Example IV.2. *Let $\mathcal{C} \in \mathbb{R}^{[m,n]}$ be a randomized tensor whose elements obey standard uniform distribution on $(0, 1)$. Suppose*

$$\lambda = (1 + \epsilon) \max_{i=1,2,\cdots,n} (\mathcal{C}\mathbf{u}^2)_i, \ \epsilon > 0,$$

where $\mathbf{u} = (1, 1, \cdots, 1)^\top$. Then $\mathcal{B} = |\lambda\mathcal{I} - \mathcal{C}|$ is a nonsingular $\mathcal{H}$-tensor [37].

The computer simulations show that the fixed point iterative method with nonlinear function can solve TCP effectively. Moreover, the convergence property of the proposed fixed point iterative method performs better than that of the corresponding the method I in [29].

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TABLE I
THE COMPARISON RESULTS ON PRESENT METHOD AND METHOD I [29] FOR EXAMPLE IV.1.

	LF			BsF			SpsF			M-I	
(m,n)	α	Ite	CPU	α	Ite	CPU	α	Ite	CPU	Ite	CPU
$(3,2)$	0.2	22	0.0239	0.162	15	0.0158	0.162	9	0.0110	9	0.0155
$(3,4)$	0.06	19	0.0169	0.035	19	0.0155	0.035	11	0.0106	21	0.0309
$(3,10)$	0.012	22	0.0167	0.005	17	0.0138	1E-4	10	0.0082	35	0.0458
$(3,15)$	0.006	28	0.0214	0.005	18	0.0139	1E-5	8	0.0074	26	0.0346
$(3,100)$	1E-4	33	0.0431	1E-4	20	0.0290	1E-8	8	0.0122	52	0.0654
$(4,4)$	0.017	18	0.0194	0.012	12	0.0113	1E-3	7	0.0061	20	0.0184
$(4,10)$	0.001	24	0.0203	0.01	11	0.0101	1E-5	5	0.0051	24	0.1754
$(4,15)$	3E-4	23	0.0183	0.01	10	0.0088	1E-5	7	0.0063	25	0.1564
$(4,100)$	1.2E-6	29	1.3950	1E-4	14	0.6764	1E-8	7	0.3567	34	2.8439

TABLE II
THE COMPARISON RESULTS ON PRESENT METHOD AND METHOD I [29] FOR EXAMPLE IV.2.

	Lin			Bs			Sps			M-I	
(m,n)	α	Ite	CPU	α	Ite	CPU	α	Ite	CPU	Ite	CPU
$(3,2)$	0.5	17	0.0184	0.6	10	0.0098	0.4	12	0.0121	17	0.0188
$(3,4)$	0.15	34	0.0324	0.1	25	0.0252	1E-3	29	0.0308	32	0.0624
$(3,10)$	0.02	32	0.0271	0.01	17	0.0145	1E-4	14	0.0131	52	0.0742
$(3,15)$	0.02	33	0.0338	0.015	18	0.0219	4E-5	14	0.0222	46	0.0689
$(3,100)$	2.5E-4	50	0.0595	1.5E-3	17	0.0223	4E-8	12	0.0179	78	0.1562
$(4,4)$	0.002	43	0.0367	0.015	11	0.0103	4E-4	9	0.0123	45	0.0598
$(4,10)$	0.001	24	0.0203	1.5E-3	11	0.0112	4E-5	6	0.0068	98	0.1632
$(4,15)$	6.5E-4	47	0.0391	1E-3	13	0.0113	2E-5	9	0.0087	125	0.2653
$(4,50)$	2E-5	74	0.2419	1.5E-4	13	0.0480	1.5E-7	7	0.0278	203	1.2598