Third Derivative Generalized Enright-Type Methods for Stiff Systems

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Abstract—A class of third derivative generalized Enright-type methods (TDGEMs) is derived. This class of methods is an extension of the GSDLMME from Ogunfeyitimi and Ikhile and a generalization of the method from Longe and Adeniran. The proposed TDGEMs which incorporate third derivative terms have the advantage of better accuracy and stability properties compared with the GSDLMME. The new class of methods is implemented as boundary value methods (BVMs) for the numerical solution of stiff ordinary differential equations (ODEs). The numerical results obtained show that the methods developed can compete with the existing ones in the literature.

***Index Terms*—Linear Multistep Formulae, Boundary Value Methods, $0_{v,k-v}$-stable and $A_{v,k-v}$-stable**

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I. Introduction

IN the past decades, considerable attention has been given to the development of methods with good stability properties for the numerical solutions of stiff initial value problems (IVPs) in ordinary differential equations (ODEs)

$$y'(x) = f(x, y(x)),\ x \in (x_0, X),\ y(x_0) = y_0,\ y(x) \in R^d,\ f(x, y(x)) \in R^d,\ x \in R,\ d = 1, 2, \cdots . \quad (1)$$

For a method to be suitable for problems in (1), the concept of A-stability of a numerical method is required and for high accuracy, higher order methods are preferable. However, the use of high order linear multistep methods (LMMs) for (1) is restricted by the Second Dahlquist Barrier Theorem see [1] which stated that the order of A-stable LMMs cannot exceed 2. Hence, there is a need to obtain methods with a higher degree of accuracy. Bickart and Rubin [2] stated that; to circumvent Dahlquist's Barrier, the conventional LMM should be modified into another class of methods. Several authors have introduced methods to overcome the Dahlquist Barrier Theorem, for example, second derivative methods were introduced (see Cash [3], Enright [4], Jia-Xiang and Jiao-Xun [5], Gupta [6], Hairer and Wanner [7] and Hojjati et al [8]). Again, exponentially fitted methods are considered (see Jackson and Kenue [9], Cash [10], Okunuga [11]). Also considered are Hybrid methods (see Gragg and Stetter [12], Gear [13], Butcher [14], Kohfeld and Thompson [15] and Lambert [16]) and many others. Recently, boundary value methods (BVMs) were proposed by Amodio et al [17], and Brugnano and Trigiante ([18], [19]). These methods approximate IVP (1) by means of a discrete boundary value problem (BVP) by fixing the initial $k_1(< k)$ number of solution values and the last $k_2(= k - k_1)$ number of solution values using the main method, which is a $k-$step linear multistep formula (LMF), and is of the form,

$$\sum_{j=-k_1}^{k_2} \alpha_j y_{n+j} = h \sum_{j=-k_1}^{k_2} \beta_j f_{n+j},\quad k_1 + k_2 = k,\ \ n = 0, 1, \cdots ,\quad y_1, y_2, \cdots , y_{k_1-1},\quad y_N, \cdots , y_{N+k_2-1}\ (fixed) . \quad (2)$$

Here, k_1 is the number of roots lying inside the unit circle and k_2 is the number of roots lying outside the unit circle, of the stability polynomial of the main method in (2) (see, [17], [18]). The implementation of these methods as BVMs overcomes the limitations of the well-known Dahlquist order and stability barrier for an $A-$stable LMM and all approximations of the solution of (1) are simultaneously generated on the entire interval. Axelsson and Verwer [20], Jator and Sahi [21], Ehigie et al [22], Nwachukwu et al [23], Nwachukwu and Okor [24], [25], Nwachukwu and Mokwunyei [26] and Okor and Nwachukwu [27] also considered boundary value techniques on the IVP (1).

The second derivative linear multistep method (SDLMM) of Enright [4] of the form

$$y_{n+k} = y_{n+k-1} + h \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \delta_k g_{n+k}\ , \quad (3)$$

provides $0-$stable methods up to $k = 7$. It is $A-$stable for $k = 1, 2$, $A(\alpha)-$stable for $k = 3(1)7$ and becomes $0-$unstable when $k \geq 8$. Ehigie et al [22] derived the Enright's second derivative formula which is $A-$stable up to order four using the multistep collocation method. They improved on the accuracy of the Enright's scheme [4] by adopting the boundary value technique. In Ogunfeyitimi and Ikhile [28], the Adams-type second derivative LMM of Enright [4] was generalized to a class of BVMs for the numerical solution of IVPs in ODEs. The class of generalized second derivative linear multistep methods based on the methods of Enright (GSDLMME) which is $A-$stable for all $k \geq 1$ and of order $p = k + 2$ is given as

$$y_{n+v} - y_{n+v-1} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \gamma_v g_{n+v}\ , \quad (4)$$

where

$$v = \begin{cases} \frac{k+1}{2}; & \text{k odd} \\ \frac{k}{2}; & \text{k even} \end{cases} \qquad k = 1, 2, 3 \ldots\ .$$

Longe and Adeniran [29] proposed the Enright's third deriva-

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tive method which is $A-$stable for step numbers $k = 2$ and 3, and is defined by

$$y_{n+k} = y_{n+k-1} + h\sum_{j=0}^{k} \beta_j f_{n+j} + h^2\delta_k g_{n+k} + h^3\gamma_k e_{n+k} \ , \tag{5}$$

where $y_{n+j} \approx y(x_n + jh)$, $f_{n+j} \equiv f(x_n + jh, y(x_n + jh))$, $g_{n+k} \equiv \frac{df(x,y(x))}{dx}\Big|_{y=y_{n+k}}^{x=x_{n+k}}$, $e_{n+k} \equiv \frac{dg(x,y(x))}{dx}\Big|_{y=y_{n+k}}^{x=x_{n+k}}$, x_n is a discrete point at node point n, β_j, δ_k and γ_k are coefficients and h is the chosen step-length. This method was used to generate the main method and the complementary methods to solve problems via boundary value techniques.

The aim of this paper is to develop a class of third derivative generalized Enright-type methods (TDGEMs) which is an extension of the GSDLMME of Ogunfeyitimi and Ikhile [28] and a generalization of method of Longe and Adeniran [29]. The purpose of adding the third derivative function into the GSDLMME in [28] is to obtain higher order methods with better stability properties.

This paper is organized as follows: In section II, we present the properties and the stability of the third derivative boundary value methods (TDBVMs). The construction of the TDGEMs where the stability properties are discussed, is provided in section III. The implementation technique is shown in section IV. In section V, some numerical experiments are considered. In section VI, we present the conclusion of the paper.

II. The Third Derivative Boundary Value Methods (TDBVMs)

The general k-step third derivative linear multistep method (TDLMM) can be written in the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} + h^2\sum_{j=0}^{k}\gamma_j g_{n+j} + h^3\sum_{j=0}^{k}\phi_j m_{n+j}, \quad k \geq 1 \ , \tag{6}$$

where y_{n+j} is the numerical approximation to the analytical solution $y(x_{n+j})$, $f_{n+j} = f(x_{n+j}, y_{n+j}) = y'(x_{n+j})$, $g_{n+j} = g(x_{n+j}, y_{n+j}) = y''(x_{n+j})$, $m_{n+j} = m(x_{n+j}, y_{n+j}) = y'''(x_{n+j})$, $\alpha_j, \beta_j, \gamma_j$ and ϕ_j are parameters and h is the chosen step-length. A generalization of this method to a class of third derivative boundary value methods (TDBVMs) is given in section III. The IVP (1) can be approximated by the following third derivative k-step LMF

$$\begin{gathered}\sum_{j=-k_1}^{k_2} \alpha_j y_{n+j} = h\sum_{j=-k_1}^{k_2} \beta_j f_{n+j} + h^2\sum_{j=-k_1}^{k_2}\gamma_j g_{n+j} + h^3\sum_{j=-k_1}^{k_2}\phi_j m_{n+j}, \\ k_1 + k_2 = k, \\ y_1, y_2, \cdots, y_{k_1-1}, \qquad y_N, \cdots, y_{N+k_2-1} (fixed),\end{gathered} \tag{7}$$

of order p with k_1 initial conditions and k_2 final conditions at the boundary of interest (see [17], [18]). y_n is the discrete approximation of the solution $y(x_n)$, $x_n = x_0 + nh$ denotes the uniform point with equal spacing h, $f_n = f(x_n, y_n)$, $g_n = g(x_n, y_n) = \frac{df(x,y(x))}{dx}\Big|_{y=y_n}^{x=x_n}$, $m_n = m(x_n, y_n) = \frac{d^2 f(x,y(x))}{dx^2}\Big|_{y=y_n}^{x=x_n}$, $\alpha_j, \beta_j, \gamma_j$ and ϕ_j are parameters.

In order to implement (7) as a TDBVM and since y_0 is given in the IVP (1), $k_1 - 1$ initial solution values: $y_1, y_2, \cdots, y_{k_1-1}$ and k_2 final solution values: $y_N, \cdots, y_{N+k_2-1}$ are needed. It then follows that the $k_1 - 1$ initial solution values and k_2 final solution values can be generated from the following initial additional formula (8) and final additional formula (9) respectively.

$$\sum_{i=0}^{k} \alpha_i^{(j)} y_i = h\sum_{i=0}^{k} \beta_i^{(j)} f_i + h^2\sum_{i=0}^{k}\gamma_i^{(j)} g_i + h^3\sum_{i=0}^{k}\phi_i^{(j)} m_i, \quad j = 1(1)k_1 - 1, \tag{8}$$

$$\begin{gathered}\sum_{i=0}^{k}\alpha_{k-i}^{(j)} y_{N-i} = h\sum_{i=0}^{k} \beta_{k-i}^{(j)} f_{N-i} + h^2\sum_{i=0}^{k}\gamma_{k-i}^{(j)} g_{N-i} + h^3\sum_{i=0}^{k}\phi_{k-i}^{(j)} m_{N-i}, \\ j = (N - k_2) + 1(1)N.\end{gathered} \tag{9}$$

The composite scheme ((7),(8) and (9)) is a TDBVM assumed to have uniform order p. Thus, the method (7) which is assumed to be $0_{k_1k_2}-$stable, $A_{k_1k_2}-$stable is used with $(k_1, k_2)-$boundary conditions. To generalize the concept of zero-stability (0−stability) and $A-$stability of TDLMM from the theory of third derivative initial value method (IVM) in (6) to TDBVM (7), we let

$$\rho(z) = \sum_{j=0}^{k}\alpha_j z^j, \quad \sigma(z) = \sum_{j=0}^{k}\beta_j z^j, \quad \xi(z) = \sum_{j=0}^{k}\gamma_j z^j, \quad \omega(z) = \sum_{j=0}^{k}\phi_j z^j, \tag{10}$$

be first, second, third and fourth characteristics polynomials associated with (6) respectively. Here,

$$\pi(z, q) = \rho(z) - q\sigma(z) - q^2\xi(z) - q^3\omega(z), \quad q = h\lambda, \tag{11}$$

is the stability polynomial when (6) is applied on $y' = \lambda y$, $y'' = \lambda^2 y$, $y''' = \lambda^3 y$, $Re(\lambda) < 0$. We now have the following definitions.

Definition *2.1:* A polynomial $\rho(z)$ in (10) of degree $k = k_1 + k_2$ is an $S_{k_1,k_2}-$polynomial, if its roots $\{z_j\}_{j=1}^{k}$ are such that $|z_1| \leq |z_2| \leq \cdots \leq |z_{k_1}| < 1 < |z_{k_1+1}| \leq \cdots \leq |z_k|$.

Definition *2.2:* A polynomial $\rho(z)$ in (10) of degree $k = k_1 + k_2$ is an $N_{k_1,k_2}-$polynomial, if its roots $\{z_j\}_{j=1}^{k}$ are such that $|z_1| \leq |z_2| \leq \cdots \leq |z_{k_1}| \leq 1 < |z_{k_1+1}| \leq \cdots \leq |z_k|$ with simple zeros of unit modulus.

If $k_1 = k$, $k_2 = 0$, an N_{k_1,k_2}-polynomial reduces to a Von-Neumann polynomial and an S_{k_1,k_2}-polynomial reduces to a schur-polynomial.

Definition *2.3:* A TDBVM (7) with (k_1, k_2)-boundary conditions, where $k = k_1 + k_2$ is:

(a) $0_{k_1,k_2}$-stable if the corresponding polynomial $\rho(z)$ in (10) is an N_{k_1,k_2}-polynomial.

(b) (k_1, k_2)-absolute stable for a given $q \in \mathbb{C}$, if the polynomial $\pi(z, q)$ in (11) is an S_{k_1,k_2}-polynomial.

(c) The region $D_{k_1,k_2} = \{q \in \mathbb{C} : \pi(z, q)$ is an S_{k_1,k_2}-polynomial$\}$ is said to be the region of (k_1, k_2)-absolute stability. Here $\pi(z, q)$ is a polynomial of type $(k_1, 0, k_2)$.

(d) A_{k_1,k_2}-stable if $\mathbb{C}^- \subseteq D_{k_1,k_2}$.

For definitions 2.1, 2.2 and 2.3, see [17], [18].

III. Third Derivative Generalized Enright-type Methods (TDGEMs)

The Enright's third derivative method is based on the TDLMM (6) and can be defined generally as

$$y_{n+k} - y_{n+k-1} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\gamma_k g_{n+k} + h^3\phi_k m_{n+k} \quad . \tag{12}$$

Following Brugnano and Trigiante [18], [19], [30], (12) can be written as

$$y_{n+i} - y_{n+i-1} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\gamma_i g_{n+i} + h^3\phi_i m_{n+i} \quad , \tag{13}$$

where $i = 0(1)k$.

For $i \neq k$, we can choose the values of i which provide methods with the best stability properties for all values of the step number $k \geq 1$. Practically, we get the best stability properties for the choice of $i = v$ such that

$$v = \begin{cases} \frac{k+1}{2}; & \text{for odd } k \\ \frac{k+2}{2}; & \text{for even } k \end{cases} \quad , k = 1, 2, 3 \dots \quad . \tag{14}$$

Therefore, (13) becomes

$$y_{n+v} - y_{n+v-1} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\gamma_v g_{n+v} + h^3\phi_v m_{n+v}, \tag{15}$$

(15) is our main method in this paper. The class of methods (15) of maximum order $p = k + 3$ is found to be $0_{v,k-v}-$stable and $A_{v,k-v}-$stable for all values of $k \geq 1$ and must be used with $(v, k - v)$-boundary conditions (i.e with v number of roots inside the unit circle and $k - v$ number of roots outside the unit circle). The methods (15) shall be referred to as third derivative generalized Enright-type methods (TDGEMs). Rewriting (15) in the form

$$y(x_n + vh) - y(x_n + (v-1)h) - h\sum_{j=0}^{k}\beta_j y'(x_n + jh) - h^2\gamma_v y''(x_n + vh) - h^3\phi_v y'''(x_n + vh) = 0, \tag{16}$$

expanding in Taylor's series and applying the method of undefined coefficient, we obtained the coefficients of the methods (15) for $k = 1(1)10$ as shown in Table I, Table II and Table III.

We now consider the order, consistency and stability of our TDGEMs. In the spirit of Fatunla [31] and Lambert [32], we define the local truncation error (LTE) associated with (15) as the linear difference operator $\mathscr{L}[y(x_n); h]$ such that

$$\mathscr{L}[y(x_n); h] = y(x_n + vh) - y(x_n + (v-1)h) - h\sum_{j=0}^{k}\beta_j y'(x_n + jh) - h^2\gamma_v y''(x_n + vh) - h^3\phi_v y'''(x_n + vh) \quad . \tag{17}$$

Assuming that $y(x_n)$ is continuously differentiable, we can find the Taylor series expansion of the terms in (17) about the point x_n to obtain the expression,

$$\mathscr{L}[y(x_n); h] = C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \cdots + C_q h^q y^q(x_n) + \cdots , \tag{18}$$

where

$$\begin{aligned} C_0 &= 0 \\ C_1 &= 1 - \sum_{j=0}^{k}\beta_j \\ C_2 &= \tfrac{v^2-(v-1)^2}{2!} - \sum_{j=0}^{k} j\beta_j - \gamma_v \\ C_3 &= \tfrac{v^3-(v-1)^3}{3!} - \sum_{j=0}^{k}\tfrac{j^2}{2!}\beta_j - v\gamma_v - \phi_v \\ &\vdots \\ C_q &= \tfrac{v^q-(v-1)^q}{q!} - \sum_{j=0}^{k}\tfrac{j^{q-1}}{(q-1)!}\beta_j - \tfrac{v^{q-2}}{(q-2)!}\gamma_v \\ &\quad - \tfrac{v^{q-3}}{(q-3)!}\phi_v, \text{ for } q = 1, 2 \dots \; . \end{aligned} \tag{19}$$

Thus, the TDGEMs (15) is of order p if

$$C_j = 0, \quad j = 0(1)p, \quad C_{p+1} \neq 0, \tag{20}$$

where C_{p+1} is the error constant of the methods (15) and its principal LTE is given as

$$C_{p+1}h^{p+1}y^{(p+1)}(x_n) + \mathcal{O}(h^{p+2}), \;\; C_{p+1} \neq 0,$$

$$C_{p+1} = \frac{v^{p+1} - (v-1)^{p+1}}{(p+1)!} - \sum_{j=0}^{k}\frac{j^p}{p!}\beta_j - \frac{v^{p-1}}{(p-1)!}\gamma_v - \frac{v^{p-2}}{(p-2)!}\phi_v \; . \tag{21}$$

The TDGEMs (15) is consistent, if it has an order of $p \geq 1$. The order conditions defined by (19) is equivalent to (27). The order p and the error constant C_{p+1} of the TDGEMs (15) are presented in Table III, for $k = 1(1)10$. In Fig.1, we have the plot of absolute value of error constant against step number of the TDGEMs (15), the GSDLMME in [28] and the method of Enright [4]. As it can be observed, our new methods show a sharp decrease in error constant.

Now, we analyse the stability of the proposed methods (15). According to Hairer and Wanner [33], the stability analysis is carried out through linearisation with the usual test equation

$$y' = \lambda y, \quad y'' = \lambda^2 y, \quad y''' = \lambda^3 y, \ \ Re(\lambda) < 0, \qquad (22)$$

which is applied to (15) to yield the characteristic equation

$$\pi(\zeta, z) = \zeta^v - \zeta^{v-1} - z \sum_{j=0}^{k} \beta_j \zeta^j - \zeta^v \left(z^2 \gamma_v + z^3 \phi_v\right) \quad (23)$$

$$z = \lambda h, z \in \mathbb{C}.$$

Letting $\zeta = \exp^{i\theta}$, $\theta \in [0, 2\pi]$, we then plot the stability regions. These are given in Fig.4 and Fig.6, for odd and even values of k, respectively. The new methods are $0_{v,k-v}$-stable and $A_{v,k-v}$-stable with $(v, k-v)$-boundary conditions for $k \geq 1$. One can see that, as the value of k increases, the boundary locus plot of the TDGEMs (15) reduces and as a result the region of absolute stability (exterior of the closed curve) increases. However, for the GSDLMME in [28], as k (even) increases, the boundary locus plot increases, that is the region of absolute stability reduces. Therefore the TDGEMs have better stability properties than the GSDLMME, see Fig.2-Fig.6.

The discrete problem generated by the k-step TDGEMs (15) with $(v, k-v)$-boundary conditions can be written in compact form, as follows:

$$AY - hBF - h^2CG - h^3DM = \begin{bmatrix} y_{v-1} + h\sum_{j=0}^{v-1}\beta_j f_j \\ h\sum_{j=0}^{v-2}\beta_j f_j \\ \vdots \\ h\beta_0 f_{v-1} \\ 0 \\ \vdots \\ 0 \\ h\beta_k f_N \\ \vdots \\ h\sum_{j=1}^{k-v}\beta_{v+j} f_{N-1+j} \end{bmatrix} \qquad (24)$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \ddots & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & -1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{(N-v)\times(N-v)}$$

$$B = \begin{bmatrix} \beta_v & \cdots & \beta_k & & & \\ \vdots & \ddots & & \ddots & & \\ \beta_0 & & \ddots & & \ddots & \\ & \ddots & & \ddots & & \beta_k \\ & & \ddots & & \ddots & \vdots \\ & & & \beta_0 & \cdots & \beta_v \end{bmatrix}_{(N-v)\times(N-v)}$$

$$C = \begin{bmatrix} \gamma_v & & & & & \\ 0 & \ddots & 0 & & & \\ \vdots & \ddots & \ddots & & & \\ \vdots & & 0 & \gamma_v & & \\ & & & \ddots & \ddots & \\ 0 & & & & 0 & \gamma_v \end{bmatrix}_{(N-v)\times(N-v)}$$

$$D = \begin{bmatrix} \phi_v & & & & & \\ 0 & \ddots & 0 & & & \\ \vdots & \ddots & \ddots & & & \\ \vdots & & 0 & \phi_v & & \\ & & & \ddots & \ddots & \\ 0 & & & & 0 & \phi_v \end{bmatrix}_{(N-v)\times(N-v)}$$

The A, B, C and D are Toeplitz matrices (T-matrices) of the same dimension and

$$Y = (y_v, \cdots, y_{N-1})^T, \ F = (f_v, \cdots, f_{N-1})^T,$$
$$G = (g_v, \cdots, g_{N-1})^T, \ M = (m_v, \cdots, m_{N-1})^T$$

are the solution, function and derivative function vectors. The coefficient matrices A, B, C and D in (24) are T-matrices having lower band with v (number of initial conditions) and upper band with $k - v$ (number of final conditions).

IV. Implementation Technique

The TDGEMs (15) will be implemented using the BVM technique as discussed in [18], [22]–[27], [30] so that the numerical solution, $(y_1, y_2, \cdots, y_N)^T$ of the IVP (1) is given simultaneously at all the grid points. The main methods (15) are to be used with $(v, k-v)$-boundary conditions or, equivalently, they are conveniently coupled with the following set of $v-1$ initial additional methods (y_0 is already provided by the initial value defining the ODE (1)),

$$y_i - y_{i-1} = h\sum_{j=0}^{k}\beta_j f_j + h^2\gamma_i g_i + h^3\phi_i m_i \qquad (25)$$
$$i = 1, \cdots v-1$$

and $k-v$ final additional methods,

$$y_{N+i}-y_{N+i-1}=h\sum_{j=0}^{k}\beta_j f_{N+j}+h^2\gamma_i g_{N+i} \\ +h^3\phi_i m_{N+i} \qquad i=v+1,\cdots,k \tag{26}$$

Therefore, for $k=2$, the main method,

$$y_{n+2}-y_{n+1}=h\left(-\frac{f_n}{160}+\frac{3f_{n+1}}{10}+\frac{113f_{n+2}}{160}\right) \\ -h^2\frac{17g_{n+2}}{80}+h^3\frac{7m_{n+2}}{240},$$

is coupled with the following initial additional method,

$$y_1-y_0=h\left(\frac{9f_0}{40}+\frac{4f_1}{5}-\frac{f_2}{40}\right)+h^2\frac{g_1}{4}-h^3\frac{m_1}{15}.$$

For $k=3$, the main method,

$$y_{n+2}-y_{n+1}=h\left(-\frac{f_n}{288}+\frac{4f_{n+1}}{15}+\frac{359f_{n+2}}{480}-\frac{f_{n+3}}{90}\right) \\ -h^2\frac{11g_{n+2}}{48}+h^3\frac{11m_{n+2}}{240},$$

is coupled with the following initial additional method,

$$y_1-y_0=h\left(\frac{19f_0}{90}+\frac{409f_1}{480}-\frac{f_2}{15}+\frac{f_3}{288}\right) \\ +h^2\frac{11g_1}{48}+h^3\frac{7m_1}{80}$$

and final additional method,

$$y_{N+3}-y_{N+2}= \\ h\left(\frac{f_N}{810}-\frac{7f_{N+1}}{480}+\frac{f_{N+2}}{3}+\frac{8813f_{N+3}}{12960}\right) \\ +h^2\frac{83g_{N+3}}{432}+h^3\frac{17m_{N+3}}{720}.$$

For $k=4$, the main method,

$$y_{n+3}-y_{n+2}= \\ h\left(\frac{23f_n}{45360}-\frac{9f_{n+1}}{1120}+\frac{247f_{n+2}}{840}-\frac{65321f_{n+3}}{90720}\right. \\ \left.-\frac{11f_{n+4}}{1680}\right)-h^2\frac{647g_{n+3}}{3024}+h^3\frac{37m_{n+3}}{1008},$$

is coupled with the following two initial additional methods,

$$y_1-y_0=h\left(\frac{113f_0}{560}+\frac{82471f_1}{90720}-\frac{103f_2}{840}+\frac{43f_3}{3360}-\frac{47f_4}{45360}\right) \\ -h^2\frac{599g_1}{3024}+h^3\frac{107m_1}{1008},$$

$$y_2-y_1=h\left(-\frac{47f_0}{20160}+\frac{313f_1}{1260}+\frac{219f_2}{280}-\frac{37f_3}{1260}\right. \\ \left.+\frac{23f_4}{20160}\right)-h^2\frac{11g_2}{48}+h^3\frac{5m_1}{84}$$

and one final additional method,

$$y_{N+4}-y_{N+3}= \\ h\left(-\frac{11f_N}{26880}+\frac{47f_{N+1}}{11340}-\frac{41f_{N+2}}{1680}+\frac{151f_{N+3}}{420}\right. \\ \left.+\frac{479833f_{N+4}}{725760}\right)-h^2\frac{2159g_{N+4}}{12096}+h^3\frac{41m_{N+4}}{2016}.$$

For $k=5$, the main method,

$$y_{n+3}-y_{n+2}= \\ h\left(\frac{7f_n}{259920}-\frac{433f_{n+1}}{80640}+\frac{2749f_{n+2}}{10080}\right. \\ \left.+\frac{271819f_{n+3}}{362880}-\frac{347f_{n+4}}{20160}+\frac{43f_{n+5}}{80640}\right) \\ -h^2\frac{191g_{n+3}}{864}+h^3\frac{191m_{n+3}}{4032},$$

is coupled with the following two initial additional methods,

$$y_1-y_0=h\left(\frac{3929f_0}{20160}+\frac{2819077f_1}{2903040}-\frac{1931f_2}{10080}+\frac{173f_3}{5760}\right. \\ \left.-\frac{883f_4}{181440}+\frac{139f_5}{322560}\right)-h^2\frac{1111g_1}{6912}+h^3\frac{995m_1}{8064},$$

$$y_2-y_1=h\left(-\frac{139f_0}{80640}+\frac{4763f_1}{20160}+\frac{295829f_2}{362880}-\frac{541f_3}{10080}\right. \\ \left.+\frac{337f_4}{80640}-\frac{7f_5}{25920}\right)-h^2\frac{191g_2}{864}+h^3\frac{289m_2}{4032}$$

and two final additional methods,

$$y_{N+4}-y_{N+3}= \\ h\left(-\frac{43f_N}{322560}+\frac{307f_{N+1}}{181440}-\frac{77f_{N+2}}{5760}\right. \\ \left.+\frac{3179f_{N+3}}{10080}+\frac{2034589f_{N+4}}{2903040}-\frac{89f_{N+5}}{20160}\right) \\ -h^2\frac{1399g_{N+4}}{6912}+h^3\frac{253m_{N+4}}{8064},$$

$$y_{N+5}-y_{N+4}= \\ h\left(\frac{89f_N}{504000}-\frac{577f_{N+1}}{322560}+\frac{821f_{N+2}}{90720}\right. \\ \left.-\frac{1429f_{N+3}}{40320}+\frac{1099f_{N+4}}{2880}-\frac{46913609f_{N+5}}{72576000}\right) \\ -h^2\frac{29101g_{N+5}}{172800}+h^3\frac{731m_{N+5}}{40320}.$$

For $k=7$, the main method,

$$y_{n+4}-y_{n+3}= \\ h\left(-\frac{149f_n}{4838400}+\frac{8959f_{n+1}}{16329600}-\frac{1583f_{n+2}}{241920}\right. \\ +\frac{6673f_{n+3}}{24192}+\frac{19599451f_{n+4}}{26127360}-\frac{12697f_{n+5}}{604800} \\ \left.+\frac{437f_{n+6}}{403200}-\frac{163f_{n+7}}{3265920}\right) \\ -h^2\frac{2497g_{n+4}}{11520}+h^3\frac{2497m_{n+4}}{51840},$$

is coupled with the following three initial additional methods,

$$y_1-y_0=h\Big(\frac{12437f_0}{67200}+\frac{3620781881f_1}{3265920000}-\frac{220919f_2}{604800}+\frac{23141f_3}{241920}-\frac{20267f_4}{653184}+\frac{39901f_5}{4838400}-\frac{21941f_6}{15120000}+\frac{4001f_7}{32659200}\Big)-h^2\frac{192697g_1}{2592000}+h^3\frac{40187m_1}{259200},$$

$$y_2-y_1=h\Big(-\frac{4001f_0}{3628800}+\frac{79963f_1}{362880}+\frac{11876503f_2}{13440000}-\frac{42929f_3}{362880}+\frac{13409f_4}{725760}-\frac{241f_5}{67200}+\frac{307f_6}{580608}-\frac{1807f_7}{45360000}\Big)-h^2\frac{6133g_2}{32000}+h^3\frac{2687m_2}{28800},$$

$$y_3-y_2=h\Big(\frac{1807f_n}{16329600}-\frac{3743f_1}{1209600}+\frac{150137f_2}{604800}+\frac{20882341f_3}{26127360}-\frac{653f_4}{13440}+\frac{5483f_5}{1209600}-\frac{7967f_6}{16329600}+\frac{149f_7}{4838400}\Big)-h^2\frac{2497g_3}{11520}+h^3\frac{3391m_3}{51840}$$

and three final additional methods,

$$y_{N+5}-y_{N+4}=h\Big(\frac{163f_N}{9072000}-\frac{3707f_{N+1}}{14515200}+\frac{1177f_{N+2}}{604800}-\frac{9313f_{N+3}}{725760}+\frac{22301f_{N+4}}{72576}+\frac{17226557f_{N+5}}{24192000}-\frac{15271f_{N+6}}{1814400}+\frac{673f_{N+7}}{3628800}\Big)-h^2\frac{11833g_{N+5}}{57600}+h^3\frac{41m_{N+5}}{1152},$$

$$y_{N+6}-y_{N+5}=h\Big(-\frac{673f_N}{32659200}+\frac{143f_{N+1}}{560000}-\frac{7517f_{N+2}}{4838400}+\frac{21431f_{N+3}}{3265920}-\frac{1249f_{N+4}}{48384}+\frac{210967f_{N+5}}{604800}+\frac{2201987399f_{N+6}}{3265920000}-\frac{1501f_{N+7}}{604800}\Big)-h^2\frac{481337g_{N+6}}{2592000}+h^3\frac{6533m_{N+6}}{259200},$$

$$y_{N+7}-y_{N+6}=h\Big(\frac{1501f_N}{29635200}-\frac{3737f_{N+1}}{6531840}+\frac{45889f_{N+2}}{15120000}-\frac{367f_{N+3}}{35840}+\frac{84473f_{N+4}}{3265920}-\frac{73253f_{N+5}}{1209600}+\frac{50599f_{N+6}}{120960}+\frac{99876260699f_{N+7}}{160030080000}\Big)-h^2\frac{19568837g_{N+7}}{127008000}+h^3\frac{27719m_{N+7}}{1814400}.$$

V. Numerical Experiments

We consider some standard linear and non-linear stiff problems. We intend to experimentally examine the accuracy of the TDGEMs (15). The numerical computations were carried out in this paper using MATLAB Programme.

Problem 1: Consider the mildly stiff linear problem solved by [25], [34]

$$\begin{aligned} y_1' &= 998y_1 + 1998y_2, & y_1(0) &= 1\\ y_2' &= -999y_1 - 1999y_2, & y_2(0) &= 1\,. \end{aligned}$$

and the exact solution is given by the sum of two decaying exponential components

$$y_1 = 4e^{-x} - 3e^{-1000x}$$

$$y_2 = -2e^{-x} + 3e^{-1000x}$$

with stiffness ratio 1:1000.

Problem 1 was solved using TDGEM of order $p = 5$ in the interval $[0, 100]$ using step-length $h = 0.1$ and the absolute errors $|y_i - y(x_i)|$ are presented in Table IV. From Table IV, it is interesting to note that our TDGEM with order $p = 5$ has superior accuracy when comapred with the methods of Yakubu and Markus [34] and the extended generalized Adams-type second derivative boundary value method (EGASDBVM) of Nwachukwu and Okor [25] which are both of order $p = 8$.

Problem 2: we consider the stiff system

$$\begin{aligned} y_1' &= -y_1 - 15y_2 + 15e^{-t} \;;& y_1(0) &= 1\\ y_2' &= 15y_1 - y_2 - 15e^{-t} \;;& y_2(0) &= 1\,. \end{aligned}$$

Its exact solution is $y_1(t) = y_2(t) = e^{-t}$.

This system has eigenvalues of large modulus lying close to the imaginary axis $-1 \pm 15i$. The 2-step TDGEM was applied to this problem and the absolute errors $|y_i - y(x_i)|$ were compared with that of the 2-step second derivative multistep method (SDMM) in Hojjatti et al [8] and the 2-step continuous third derivative block method (CTDBM) of Akinfenwa et al [35]. Clearly, from Table V, it is observed that the newly derived method performs better than the SDMM and the CTDBM for the same step number, $k = 2$.

Problem 3 Consider the non-linear system proposed by [36] and solved by [22], [28], [29]

$$\begin{aligned} y_1' &= -1002y_1 + 1000y_2^2 \;,& y_1(0) &= 1\\ y_2' &= y_1 - y_2(1 + y_2) \;,& y_2(0) &= 1\,. \end{aligned}$$

$0 \le t \le T$, the smaller t is, the more serious the stiffness of the system. The exact solution is $y_1(t) = y_2^2(t)$, $y_2(t) = e^{-t}$.

We solved this problem with the 2-step and 3-step TDGEMs for $h = 0.008, 0.01, 0.02$ on the range of $0 \le t \le 5$ and the maximum errors $\max\|y_i - y(t_i)\|$ were presented in Table VI. We found from Table VI that the 2-step TDGEM is superior to the 2-step boundary value method (BVM2) of Ehigie et

al [22] and the 2-step third derivative continuous multistep method (TCM2) of Longe and Adeniran [29]. Also, from Table VI we can observe that the 3-step TDGEM shows superiority over the 3-step GSDLMMEs3 of Ogunfeyitimi and Ikhile [28], the 3-step TCM3 of Longe and Adeniran [29] and the 3-step BVM3 of Ehigie et al [22] (where N is the number of integration steps given as $N = \frac{b-a}{h}$).
In Table VII, we made a comparison of the absolute errors of the TDGEMs for different orders. Table VII shows that as the order increases, the new class of methods (15) performs better as expected.

Problem 4 We consider the stiffly nonlinear system

$$\begin{aligned} y_1' &= -(\epsilon^{-1}+2)y_1 + \epsilon^{-1}y_2^2, & y_1(0) &= 1 \\ y_2' &= y_1 - y_2 - y_2^2, & y_2(0) &= 1 \ . \end{aligned}$$

It's exact solution is given by $y_1 = y_2^2, \quad y_2 = e^{-t}$. According to [35] and [37], the smaller the value ϵ is, the more serious the stiffness of the system.
In Table VIII and Table IX, we present absolute errors $y_i = |y_i - y(t_i)|$, $i = 1, 2$ for $\epsilon = 10^{-3}$ and $\epsilon = 10^{-4}$ respectively. From Table VIII, using $\epsilon = 10^{-3}$ we see that the TDGEM of order 5 performs better than the CTDBM of Akinfenwa et al [35] and the SDMM of Hojjati et al [8].
Also, we observe that in Table IX using $\epsilon = 10^{-4}$ the TDGEM of order 6 is superior to the EGASDBVM of Nwachukwu and Okor [25] and the second derivative generalised Adams-type method (SDGAM) of Nwachukwu and Mokwunyei [26] both of order 10. The TDGEM shows superiority despite its lower order when compared to the EGASDBVM and the SDGAM.

Problem 5 The following problem was suggested by [38],

$$\begin{aligned} y_1' &= -0.013y_2 - 1000y_1y_2 - 2500y_1y_3; & y_1(0) &= 0 \\ y_2' &= -0.013y_2 - 1000y_1y_2; & y_2(0) &= 1 \\ y_3' &= -2500y_1y_3; & y_3(0) &= 1 \ . \end{aligned}$$

Problem 5 was solved using the TDGEM of order 6 and the results were compared with the solution from the Ode15s in MATLAB. It is observed from Fig.8 that the new method is very comparable with the Ode15s in MATLAB.

Problem 6 The linear stiff test solved by Brugnano and Trigiante [18], [39]

$$\begin{aligned} y_1' &= -21y_1 + 19y_2 - 20y_3; & y_1(0) &= 1 \\ y_2' &= 19y_1 - 21y_2 + 20y_3; & y_2(0) &= 0 \\ y_3' &= 40y_1 - 40y_2 - 40y_3; & y_3(0) &= -1 \ . \end{aligned}$$

The theoretical solution is given by:

$$\begin{aligned} y_1(t) &= \tfrac{1}{2}(e^{-2t} + e^{-40t}(\cos(40t) + \sin(40t))) \\ y_2(t) &= \tfrac{1}{2}(e^{-2t} - e^{-40t}(\cos(40t) + \sin(40t))) \\ y_3(t) &= -\tfrac{1}{2}(2e^{-40t}(\sin(40t) + \cos(40t))) \ . \end{aligned}$$

The TDGEMs of order $6, 8,$ and 10 were applied to problem 6 and the numerical results were reported in Table X. From Table X, we have the following observations. The TDGEM of order 6 is more accurate than the top order method (TOM) of order 6 and the high order extended boundary value method (HEBVM) of order 7 for the same step number, $k = 3$. The TDGEM of order 8 is of higher accuracy than the TOM of order 10 and the HEBVM of order 9 for the same step number, $k = 5$. The TDGEM of order 10 performs better when compared with the TOM of order 14 and the HEBVM of order 11 for the same step number, $k = 7$. Furthermore, in Fig.9, we have the numerical results for Problem 6 using the Tenth order TDGEM. We notice that the graphs of the exact and numerical solutions in Fig.9 coincide. Hence, our method is very accurate.

Problem 7 Robertson's Equation (nonlinear problem), see [6]

$$\begin{aligned} y_1' &= -0.04y_1 + 10^4y_2y_3; & y_1(0) &= 1 \\ y_2' &= 0.04y_1 - 10^4y_2y_3 - 3\times 10^7y_2^2; & y_2(0) &= 0 \\ y_3' &= 3\times 10^7y_2^2; & y_3(0) &= 0 \ . \end{aligned}$$

The numerical results for Robertson's equation using the sixth order and eighth order TDGEMs are given in Fig.10 and Fig.11 respectively. From the figures, it is obvious that the new methods are very comparable with the Ode15s in MATLAB, Higham and Higham [40].

Problem 8 Van der Pol Equation (nonlinear problem), see [6]

$$\begin{aligned} y_1' &= y_2; & y_1(0) &= 2 \\ y_2' &= -y_1 + 10y_2(1 - y_1^2); & y_2(0) &= 0 \ . \end{aligned}$$

The numerical results for Van der Pol equations in Fig.12 and Fig.13 show that our methods coincide with the Ode15s in MATLAB, Higham and Higham [40].

VI. CONCLUSION

This paper presents a class of TDGEMs for solving stiff IVPs. The proposed class of methods is an extension of the GSDLMME of Ogunfeyitimi and Ikhile [28] and a generalization of the method of Longe and Adeniran [29] which is A-stable for $k = 2$ and 3. The new methods are $0_{v,k-v}$-stable and $A_{v,k-v}$-stable with $(v, k - v)$-boundary conditions and of order $p = k + 3$ for values of step-number $k \geq 1$. In the new class of methods, there is no limit concerning the maximum order attainable. It was shown that this class of methods is of higher order and has superior accuracy and stability properties when compared with the GSDLMME of [28], see Fig.1 - Fig.6. The linear and non-linear stiff problems considered, show that our class of methods is more accurate than some existing methods and also very comparable with Ode15s of MATLAB, these are shown in Table IV - Table X and Fig.8-Fig.13.

VII. ACKNOWLEDGEMENT

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Table I: The coefficients, Error Constants C_{p+1} and order p of TDGEMs (15).

k	β_0	β_1	β_2	β_3	β_4	β_5
1	$\frac{1}{4}$	$\frac{3}{4}$				
2	$-\frac{1}{160}$	$\frac{3}{10}$	$\frac{113}{160}$			
3	$-\frac{1}{288}$	$\frac{4}{15}$	$\frac{359}{480}$	$-\frac{1}{90}$		
4	$\frac{23}{45360}$	$-\frac{9}{1120}$	$\frac{247}{840}$	$\frac{65321}{90720}$	$-\frac{11}{1680}$	
5	$\frac{7}{25920}$	$-\frac{433}{80640}$	$\frac{2749}{10080}$	$\frac{271819}{362880}$	$-\frac{347}{20160}$	$\frac{43}{80640}$
6	$-\frac{1709}{29030400}$	$\frac{611}{680400}$	$-\frac{4307}{483840}$	$\frac{2645}{9072}$	$\frac{12676459}{17418240}$	$-\frac{437}{37800}$
7	$-\frac{149}{4838400}$	$\frac{8959}{16329600}$	$-\frac{1583}{24192}$	$\frac{19599451}{26127360}$	$-\frac{12697}{604800}$	$-\frac{12697}{604800}$
8	$-\frac{4093}{498960000}$	$-\frac{21289}{159667200}$	$\frac{71051}{59875200}$	$-\frac{375799}{39916800}$	$\frac{1158347}{3991680}$	$\frac{8774922643}{11975040000}$
9	$-\frac{37}{8709120}$	$-\frac{198553}{2554675200}$	$\frac{2707}{3421440}$	$-\frac{1171697}{159667200}$	$\frac{22172443}{79833600}$	$\frac{115113679}{153280512}$
10	$-\frac{742823}{581188608000}$	$\frac{12342923}{544864320000}$	$-\frac{96305309}{464950886400}$	$\frac{7653601}{5448643200}$	$-\frac{16196893}{1660538880}$	$\frac{107241247}{370656000}$

Table II: The coefficients, Error Constants C_{p+1} and order p of TDGEMs (15) (continuation).

k	β_6	β_7	β_8	β_9	β_{10}
1					
2					
3					
4					
5					
6	$\frac{2161}{7257600}$				
7	$\frac{437}{403200}$	$-\frac{163}{3265920}$			
8	$-\frac{304397}{19958400}$	$-\frac{26891}{39916800}$	$-\frac{29}{1069200}$		
9	$-\frac{104477}{4435200}$	$\frac{49927}{31933440}$	$-\frac{60349}{479001600}$	$\frac{47357}{7664025600}$	
10	$\frac{1833901084727}{2490808320000}$	$-\frac{3639469}{201801600}$	$\frac{121565617}{116237721600}$	$-\frac{326329}{4358914560}$	$\frac{23038649}{6974263296000}$

Table III: The coefficients, Error Constants C_{p+1} and order p of TDGEMs (15) (continuation).

k	v	γ_v	ϕ_v	C_{p+1}	p
1	1	$-\frac{1}{4}$	$\frac{1}{24}$	$\frac{1}{480}$	4
2	2	$-\frac{17}{80}$	$\frac{7}{240}$	$\frac{1}{1800}$	5
3	2	$-\frac{11}{48}$	$\frac{11}{240}$	$-\frac{23}{151200}$	6
4	3	$-\frac{647}{3024}$	$\frac{37}{1008}$	$-\frac{43}{846720}$	7
5	3	$-\frac{191}{864}$	$\frac{191}{4032}$	$\frac{1709}{101606400}$	8
6	4	$-\frac{44131}{207360}$	$\frac{1393}{34560}$	$-\frac{163}{26127360}$	9
7	4	$-\frac{2497}{11520}$	$\frac{2497}{51840}$	$-\frac{4093}{1796256000}$	10
8	5	$-\frac{670303}{3168000}$	$\frac{40321}{950400}$	$-\frac{47357}{52690176000}$	11
9	5	$-\frac{14797}{69120}$	$\frac{14797}{304128}$	$\frac{742823}{2131024896000}$	12
10	6	$-\frac{132424231}{628992000}$	$\frac{5512813}{125798400}$	$\frac{4867481}{33999533568000}$	13

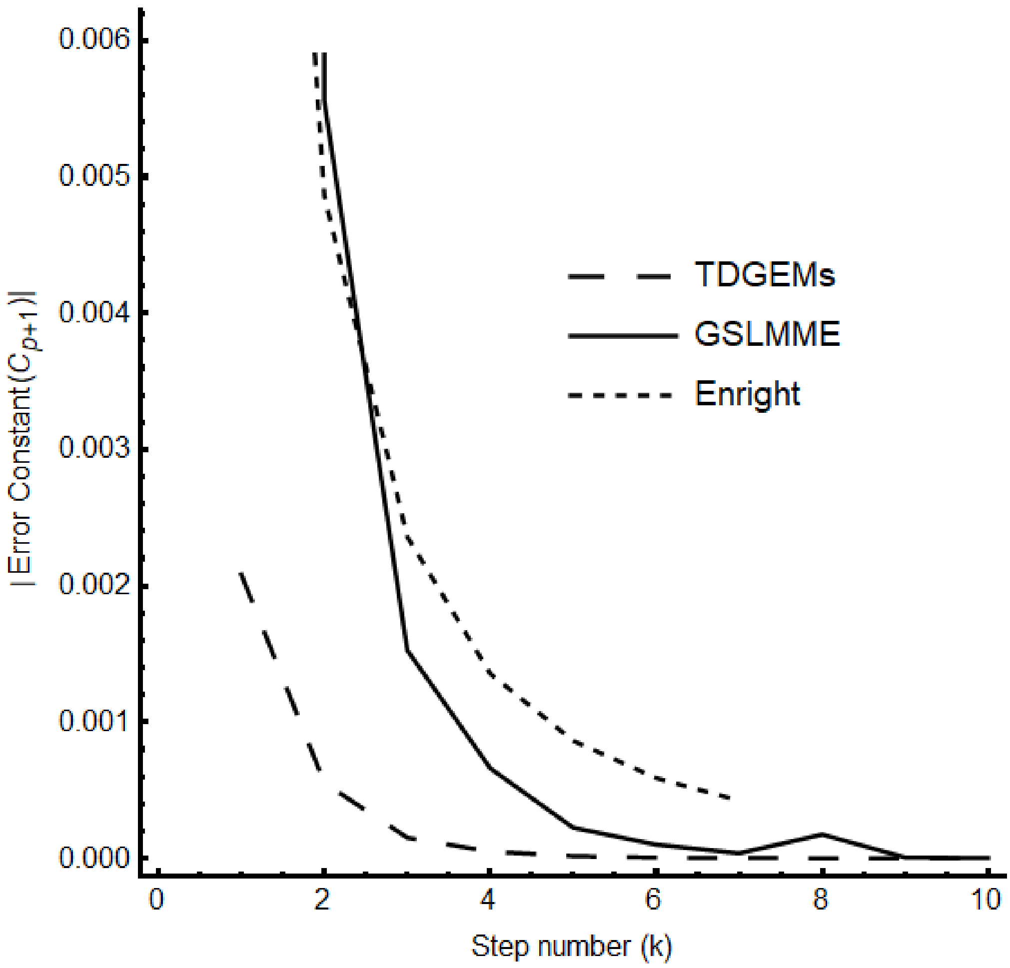

Figure 1: The plot of $|\text{error constant}|$ against step number k of the TDGEMs (15), the GSDLMME [28] and the method of Enright [4].

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ v-1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ (v-1)^2 & 0 & 2 & 2*2 & 2*3 & 2*4 & \cdots & 2*k & 2 & 0 \\ (v-1)^3 & 0 & 3 & 3*2^2 & 3*3^2 & 3*4^2 & \cdots & 3*k^2 & 3*2*v & 3*2 \\ (v-1)^4 & 0 & 4 & 4*2^3 & 4*3^3 & 4*4^3 & \cdots & 4*k^3 & 4*3*v^2 & 4*3*2*v \\ (v-1)^5 & 0 & 5 & 5*2^4 & 5*3^4 & 5*4^4 & \cdots & 5*k^4 & 5*4*v^3 & 5*4*3*v^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ (v-1)^q & 0 & q & q*2^{(q-1)} & q*3^{(q-1)} & q*4^{(q-1)} & \cdots & q*k^{(q-1)} & q(q-1)v^{(q-2)} & q(q-1)(q-2)v^{(q-3)} \end{bmatrix} \begin{bmatrix} \text{-1} \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_k \\ \gamma_v \\ \phi_v \end{bmatrix} = \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \\ v^4 \\ v^5 \\ \vdots \\ v^q \end{bmatrix} \tag{27}$$

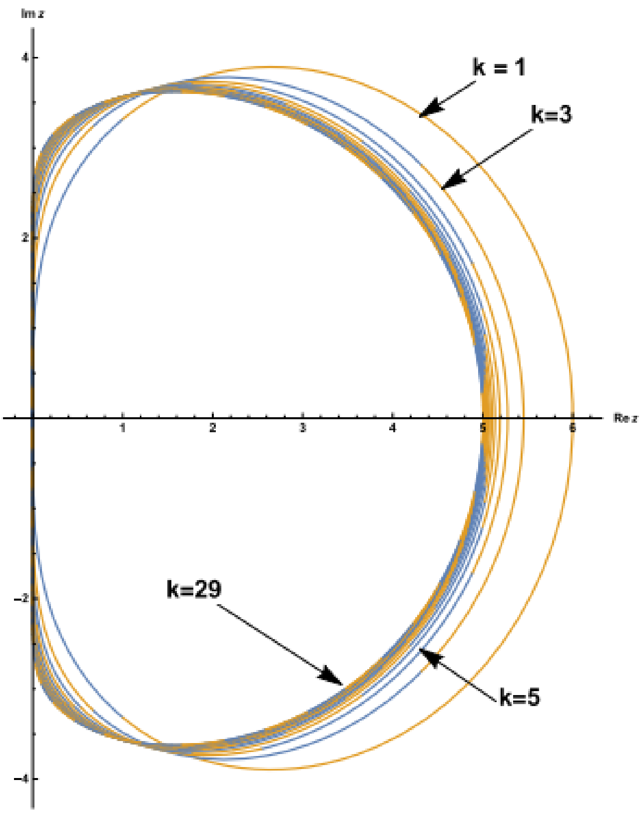

Figure 2: Stability region (exterior of closed curve) of GSDLMME in [28], $k = 1(2)29$. The curves are for $k = 1, 3, 5, 7, \cdots, 29$. However, the curves for $k = 7, 9, 11, \cdots, 29$ are so close to one another that they are indistinguishable.

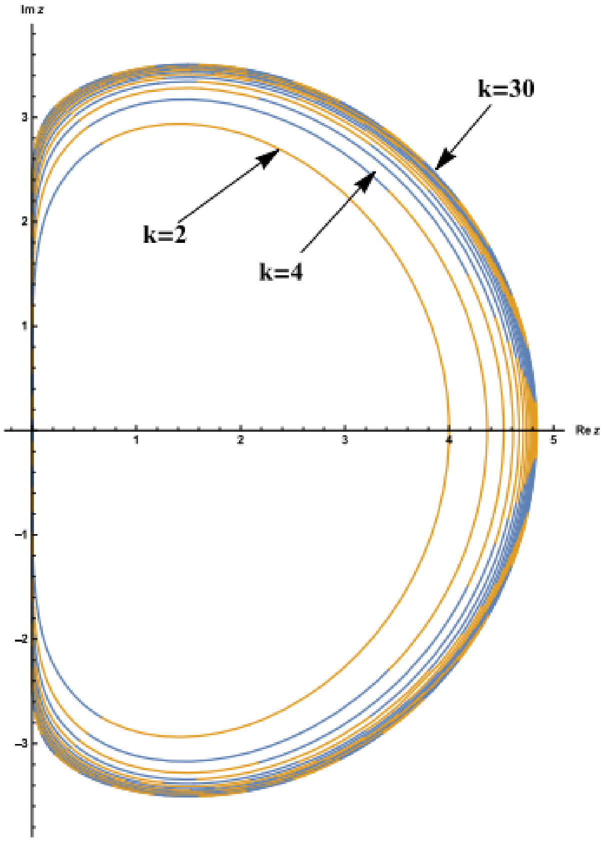

Figure 3: Stability region (exterior of closed curve) of GSDLMME in [28], $k = 2(2)30$. The curves are for $k = 2, 4, 6, 8, \cdots, 30$. However, the curves for $k = 6, 8, 10, \cdots, 30$ are so close to one another that they are indistinguishable.

Table IV: Absolute errors in the numerical solution of Problem 1, $h = 0.1$.

t	y_i	$TDGEM(15)$ $p = 5$	$EGASDBVM$ [25] $p = 8$	$Yakubu - Markus$ [34] $p = 8$
5	y_1	$2.559605895291472 \times 10^{-10}$	$3.2024170123130 \times 10^{-2}$	$1.96006208591687 \times 10^{-2}$
	y_2	$1.279802947645736 \times 10^{-10}$	$3.2602729149065 \times 10^{-2}$	$9.80025491509760 \times 10^{-1}$
40	y_1	$1.102792804033894 \times 10^{-24}$	$7.1981394073653299 \times 10^{-15}$	$3.81292881577727 \times 10^{-7}$
	y_2	$5.513964020169468 \times 10^{-25}$	$7.198139407651458 \times 10^{-15}$	$1.90646440788863 \times 10^{-7}$
70	y_1	$1.787044718505958 \times 10^{-37}$	$8.848025628743198 \times 10^{-26}$	$8.90990527186305 \times 10^{-12}$
	y_2	$8.935223588150732 \times 10^{-38}$	$8.848025628742016 \times 10^{-26}$	$4.45495263593152 \times 10^{-12}$
100	y_1	$2.378834730098595 \times 10^{-50}$	$1.087608242814579 \times 10^{-36}$	$2.08203236381127 \times 10^{-18}$
	y_2	$1.189417365547139 \times 10^{-50}$	$1.087608242814395 \times 10^{-36}$	$1.04101618190563 \times 10^{-18}$

Table V: Absolute errors in the numerical intergration of Problem 2, $h = 0.01$.

t	y_i	$TDGEM(15)$ k=2	$CTDBM$ [35] k=2	$SDMM$ [8] k=2
5	y_1	7.81×10^{-18}	3.73×10^{-17}	6.09×10^{-14}
	y_2	3.73×10^{-17}	9.54×10^{-18}	2.24×10^{-14}
10	y_1	2.71×10^{-20}	4.68×10^{-19}	7.17×10^{-16}
	y_2	8.13×10^{-20}	2.71×10^{-19}	1.12×10^{-16}
15	y_1	6.35×10^{-22}	3.92×10^{-21}	6.64×10^{-18}
	y_2	7.41×10^{-22}	2.44×10^{-21}	2.86×10^{-19}
20	y_1	9.10×10^{-24}	4.30×10^{-23}	5.32×10^{-20}
	y_2	8.27×10^{-25}	4.14×10^{-24}	1.31×10^{-20}

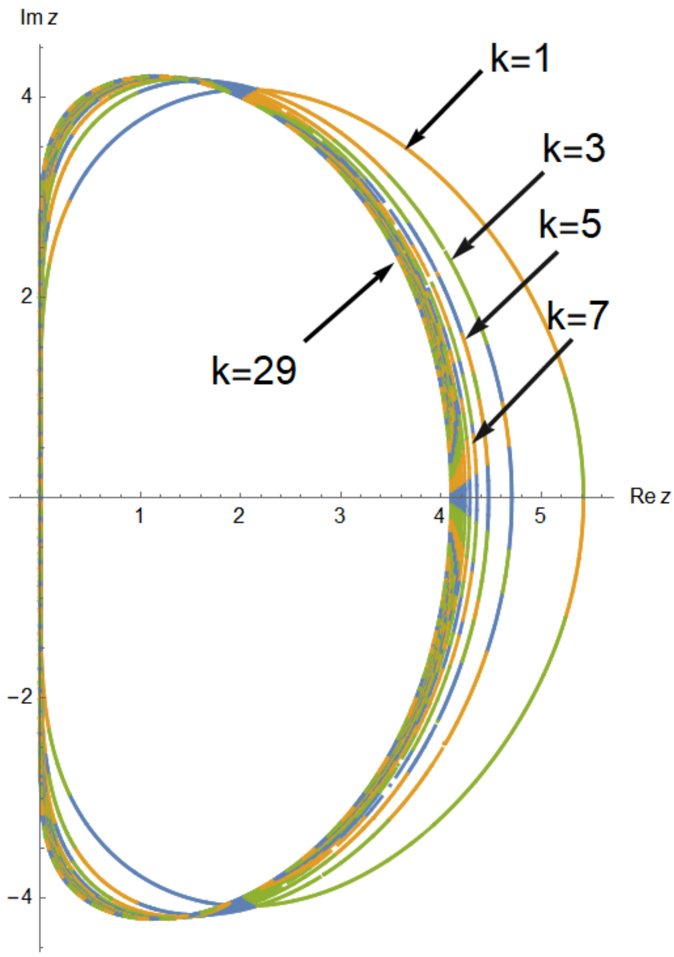

Figure 4: Stability region (exterior of closed curve) of TDGEMs (15), $k = 1(2)29$.

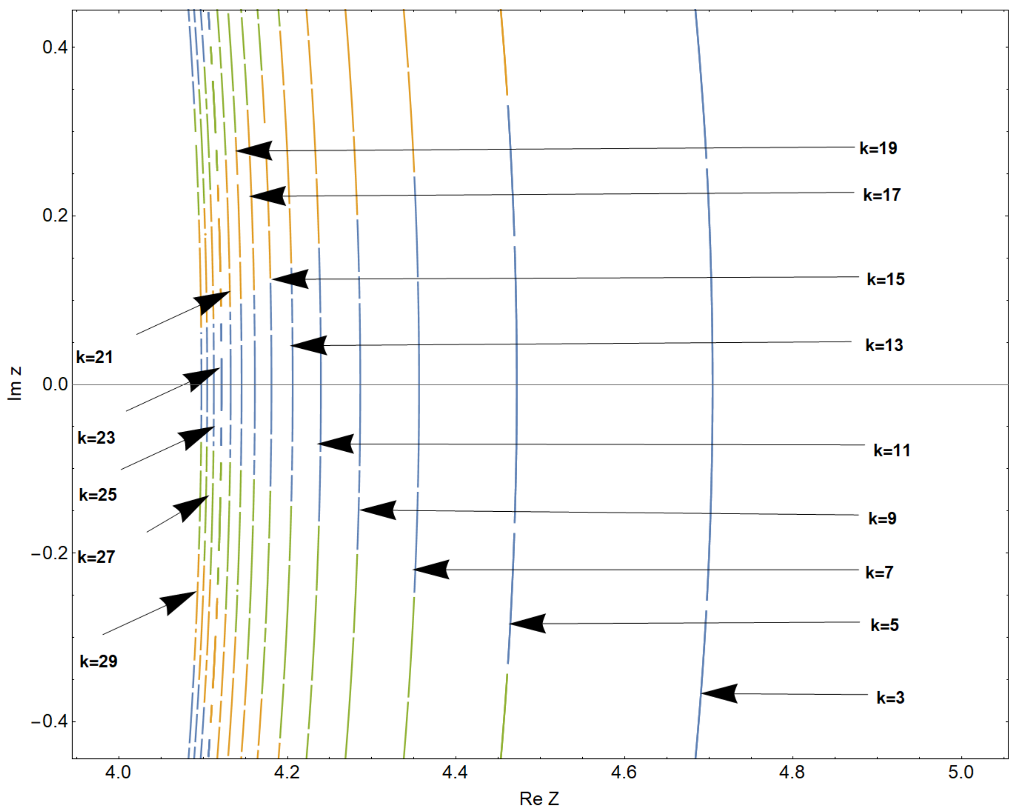

Figure 5: Stability region (exterior of closed curve) of TDGEMs (15), $k = 3(2)29$ zoomed in for clarity for Figure 4.

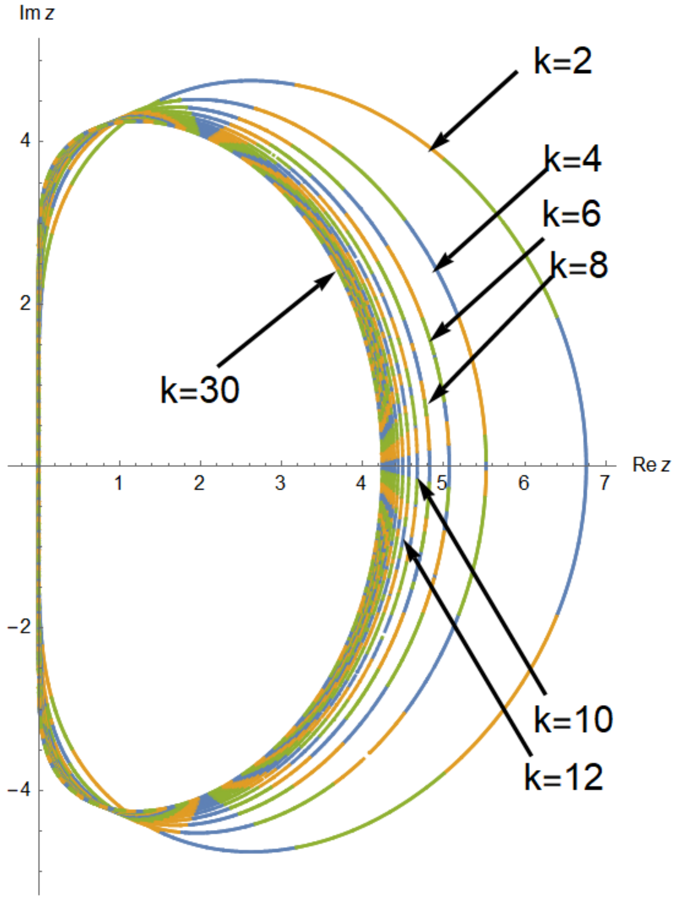

Figure 6: Stability region (exterior of closed curve) of TDGEMs (15), $k = 2(2)30$.

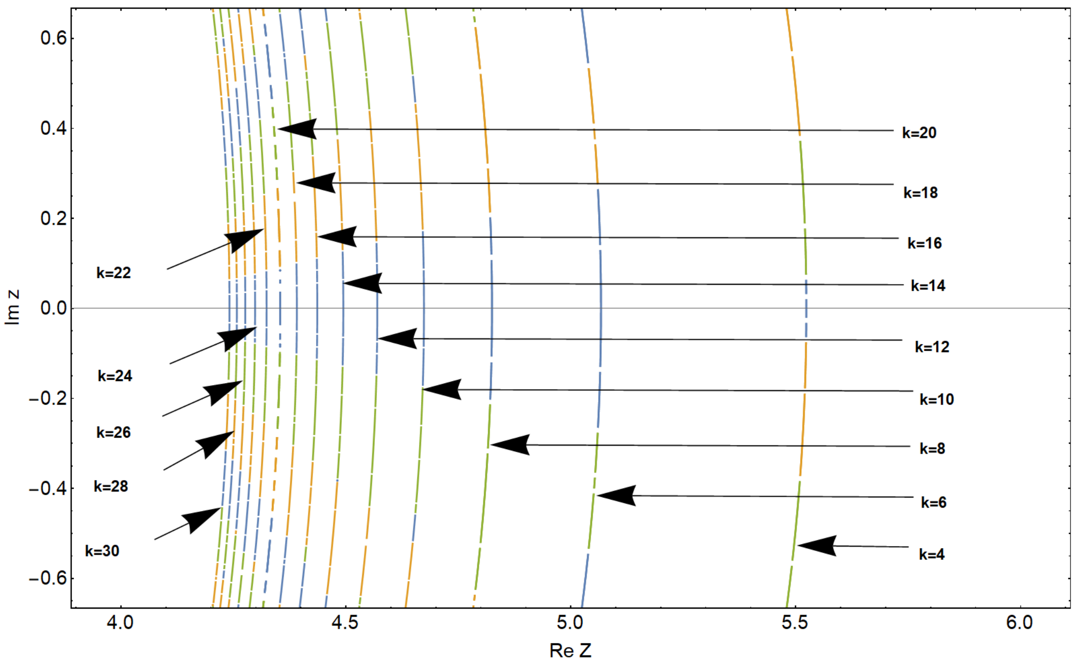

Figure 7: Stability region (exterior of closed curve) of TDGEMs (15), $k = 4(2)30$ zoomed in for clarity for Figure 6.

Table VI: Maximum Errors, max $\|y_i - y(x_i)\|$ for Problem 3 .

Method	N	h	t	y_1 $max\|y_1 - y(t_1)\|$	y_2 $max\|y_2 - y(t_2)\|$
TDGEM(15) $(k = 2)$	125	0.008	1	4.163×10^{-15}	5.773×10^{-15}
	50	0.008	0.4	1.798×10^{-14}	1.332×10^{-14}
	3	0.008	0.24	1.268×10^{-10}	8.982×10^{-14}
	500	0.01	5	1.68×10^{-17}	1.31×10^{-15}
	25	0.02	0.5	4.458×10^{-14}	3.575×10^{-14}
	3	0.02	0.06	3.624×10^{-10}	3.33×10^{-13}
TDGEM(15) $(k = 3)$	125	0.008	1	5.551×10^{-17}	5.551×10^{-17}
	50	0.008	0.4	1.11×10^{-16}	1.665×10^{-16}
	3	0.008	0.24	0.00	0.00
	500	0.01	5	$7. \times 10^{-19}$	4.25×10^{-17}
	25	0.02	0.5	1.81×10^{-14}	1.488×10^{-14}
	3	0.02	0.06	1.221×10^{-14}	6.328×10^{-15}
GSDLMMEs3 [28] $(k = 3)$	125	0.008	1	6.88×10^{-15}	3.33×10^{-15}
	50	0.008	0.4	1.02×10^{-14}	1.67×10^{-14}
	3	0.008	0.24	5.61×10^{-9}	1.66×10^{-9}
	25	0.02	0.5	1.46×10^{-12}	1.46×10^{-12}
	3	0.02	0.06	8.34×10^{-8}	1.73×10^{-8}
TCM2 [29] $(k = 2)$	500	0.01	5	4.22×10^{-13}	9.89×10^{-13}
TCM3 [29] $(k = 3)$	500	0.01	5	2.245×10^{-15}	5.00×10^{-14}
BVM2 [22] $(k = 2)$	125	0.008	1	6.61×10^{-12}	6.74×10^{-12}
	50	0.02	1	2.49×10^{-10}	2.64×10^{-9}
BVM3 [22] $(k = 3)$	125	0.008	1	3.88×10^{-14}	3.10×10^{-14}
	50	0.02	1	3.20×10^{-12}	3.02×10^{-12}

Table VII: Absolute Errors in the Numerical Solution of Problem 3.

h	y_i	$TDGEM(15)$ $k = 3; p = 6$	$TDGEM(15)$ $k = 4; p = 7$	$TDGEM(15)$ $k = 5; p = 8$
0.001	y_1	2.312×10^{-22}	4.136×10^{-24}	4.136×10^{-25}
	y_2	2.582×10^{-18}	4.066×10^{-20}	6.776×10^{-21}
0.01	y_1	8.479×10^{-24}	3.722×10^{-23}	3.309×10^{-24}
	y_2	1.220×10^{-19}	5.014×10^{-19}	4.066×10^{-20}
0.1	y_1	2.953×10^{-19}	9.607×10^{-21}	2.593×10^{-22}
	y_2	1.319×10^{-14}	5.241×10^{-16}	1.727×10^{-17}
0.5	y_1	8.208×10^{-20}	2.718×10^{-21}	7.470×10^{-23}
	y_2	9.913×10^{-13}	8.922×10^{-14}	6.665×10^{-15}

Table VIII: Numerical Solutions of Problem 4, $h = 0.01$ $\epsilon = 10^{-3}$.

t	y_i	$TDGEM(15)$ k=2,p=5	$CTDBM$ [35] k=2,p=5	$SDMM$ [8] k=3,p=5
5	y_1	1.50×10^{-17}	1.00×10^{-16}	3.92×10^{-16}
	y_2	1.25×10^{-15}	7.49×10^{-15}	3.72×10^{-14}
10	y_1	1.44×10^{-21}	9.11×10^{-21}	4.56×10^{-20}
	y_2	1.72×10^{-17}	1.00×10^{-16}	5.00×10^{-16}
20	y_1	5.26×10^{-30}	3.74×10^{-29}	1.28×10^{-28}
	y_2	1.38×10^{-21}	9.08×10^{-21}	4.56×10^{-20}

Table IX: Numerical Solutions of Problem 4, $h = 0.01$ $\epsilon = 10^{-4}$.

t	y_i	$TDGEM$(15) $p = 6$	$EGASDBVM$ [25] $p = 10$	$SDGAM$ [26] $p = 10$
2	y_1	1.388×10^{-17}	1.735×10^{-17}	2.038×10^{-13}
	y_2	6.939×10^{-17}	8.327×10^{-17}	7.087×10^{-13}
4	y_1	6.505×10^{-19}	1.247×10^{-18}	3.518×10^{-15}
	y_2	2.082×10^{-17}	3.469×10^{-17}	9.594×10^{-14}
6	y_1	6.776×10^{-21}	6.268×10^{-20}	6.428×10^{-17}
	y_2	1.735×10^{-18}	1.388×10^{-17}	1.297×10^{-14}
8	y_1	5.294×10^{-23}	2.475×10^{-21}	1.176×10^{-18}
	y_2	1.084×10^{-19}	4.066×10^{-18}	1.752×10^{-15}
10	y_1	2.068×10^{-25}	6.969×10^{-23}	2.150×10^{-20}
	y_2	0.000	8.403×10^{-19}	2.368×10^{-16}

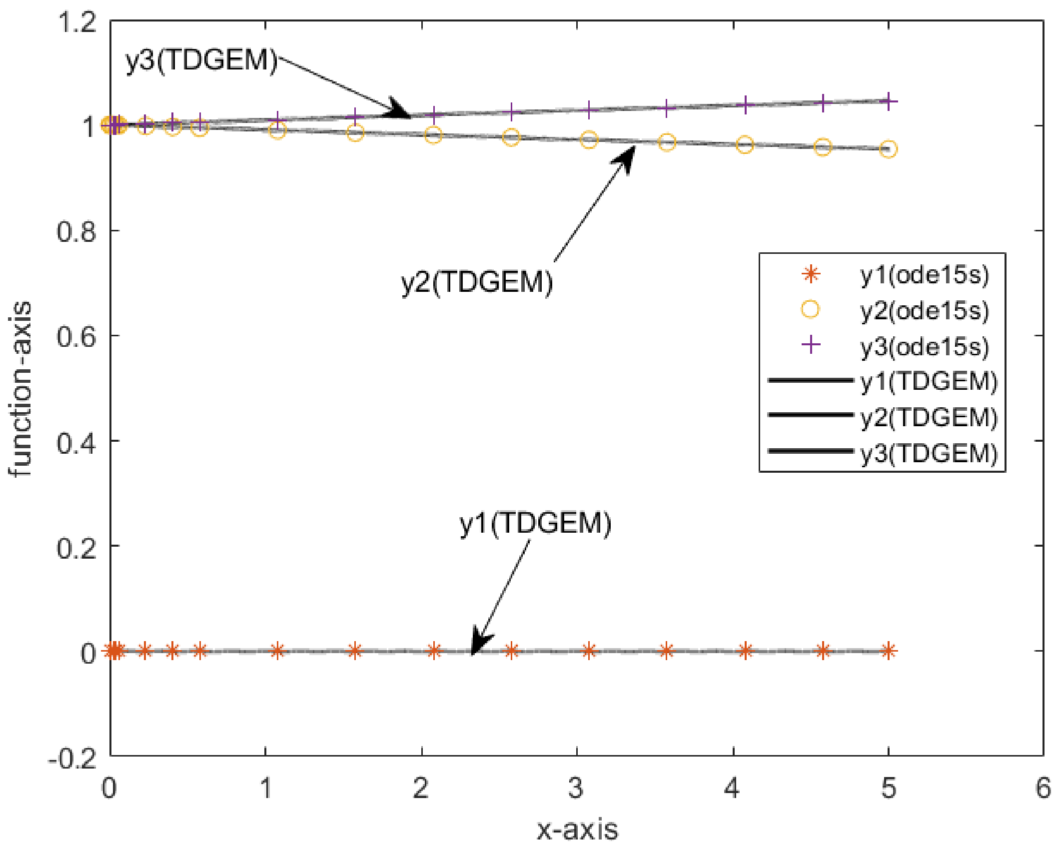

Figure 8: Numerical results for Problem 5 using the sixth order TDGEM, $h = 0.0001$.

Table X: A comparison of methods for problem 6, $Error = Max|y(t) - y|$, $Rate = \log_2\left(\frac{e^2h}{e^h}\right)$ where e^h is the maximum absolute error for $h, N = \frac{T-t_0}{h}$ and $0 \leq t \leq 1$.

h	N	$TOMs$ [18] $k=3, p=6$	$Rate$	$TOMs$ [18] $k=5, p=10$	$Rate$	$TOMs$ [18] $k=7, p=14$	$Rate$
2×10^{-2}	50	1.552×10^{-3}	–	1.523×10^{-4}	–	4.780×10^{-5}	–
1×10^{-2}	100	9.775×10^{-6}	7.31	2.504×10^{-7}	9.25	1.213×10^{-8}	11.94
5×10^{-3}	200	1.197×10^{-7}	6.35	7.490×10^{-11}	11.70	2.472×10^{-13}	15.58
2.5×10^{-3}	400	1.853×10^{-9}	6.01	3.009×10^{-14}	11.28	1.654×10^{-14}	3.90
h	N	$HEBVM3$ [27] $k=3, p=7$	$Rate$	$HEBVM5$ [27] $k=5, p=9$	$Rate$	$HEBVM7$ [27] $k=7, p=11$	$Rate$
2×10^{-1}	5	9.091×10^{-5}	–	1.596×10^{-3}	–	4.132×10^{-3}	–
1×10^{-1}	10	2.171×10^{-8}	12.03	3.204×10^{-7}	12.28	2.049×10^{-7}	14.30
5×10^{-2}	20	5.856×10^{-12}	11.86	6.150×10^{-12}	15.67	1.315×10^{-13}	20.57
2.5×10^{-2}	40	5.934×10^{-14}	6.62	1.110×10^{-16}	15.76	5.551×10^{-17}	11.21
h	N	$TDGEM(15)$ $k=3, p=6$	$Rate$	$TDGEM(15)$ $k=5, p=8$	$Rate$	$TDGEM(15)$ $k=7, p=10$	$Rate$
2×10^{-1}	5	2.938×10^{-9}	–	1.211×10^{-10}	–	1.215×10^{-9}	–
1×10^{-1}	10	5.409×10^{-11}	5.76	2.430×10^{-12}	5.64	1.554×10^{-14}	16.25
5×10^{-2}	20	7.163×10^{-13}	6.24	1.608×10^{-14}	7.23	4.510×10^{-17}	8.43
2.5×10^{-2}	40	5.163×10^{-15}	7.12	9.714×10^{-17}	7.37	6.939×10^{-18}	2.70

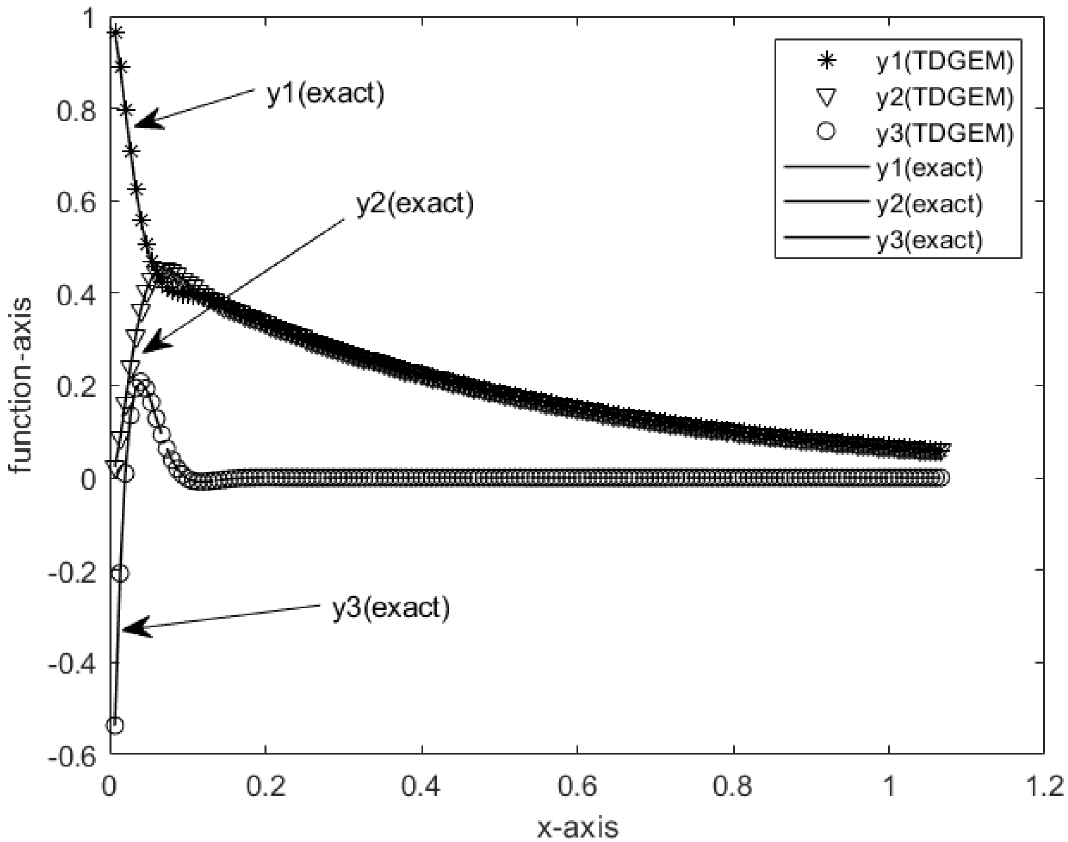

Figure 9: Numerical results for Problem 1 using the tenth order TDGEM, $N = 150$.

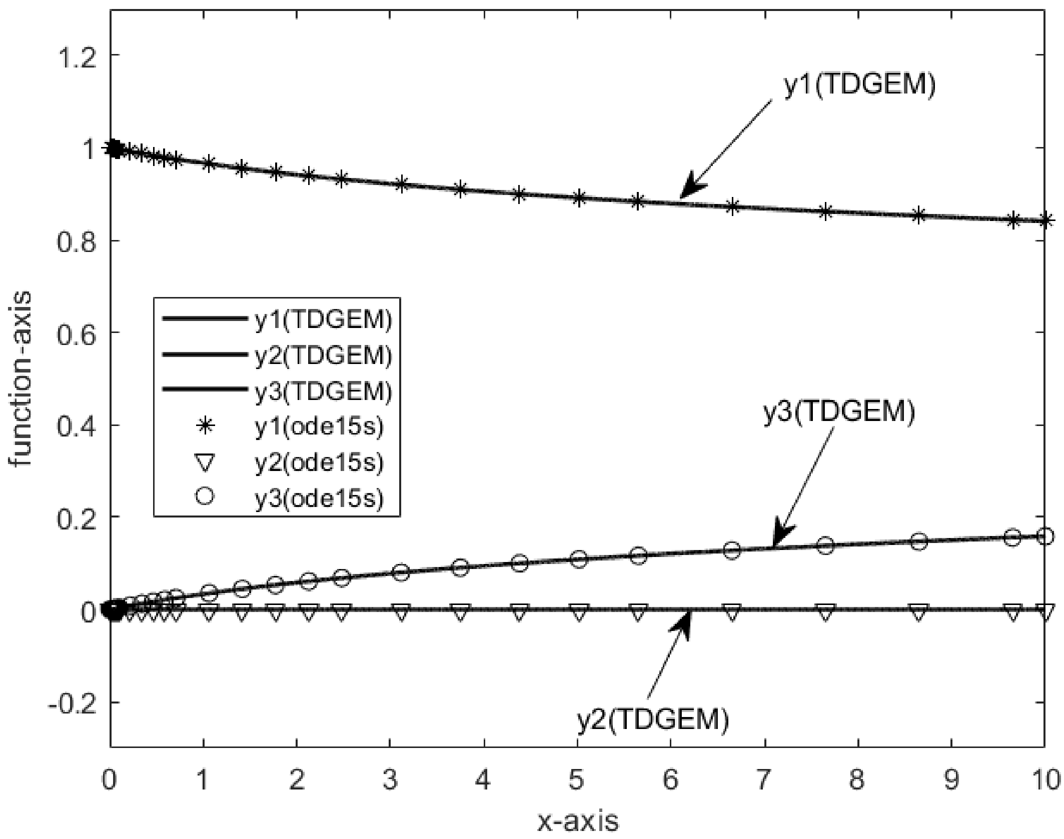

Figure 10: Numerical results for Robertson's Equation using the sixth order TDGEM.

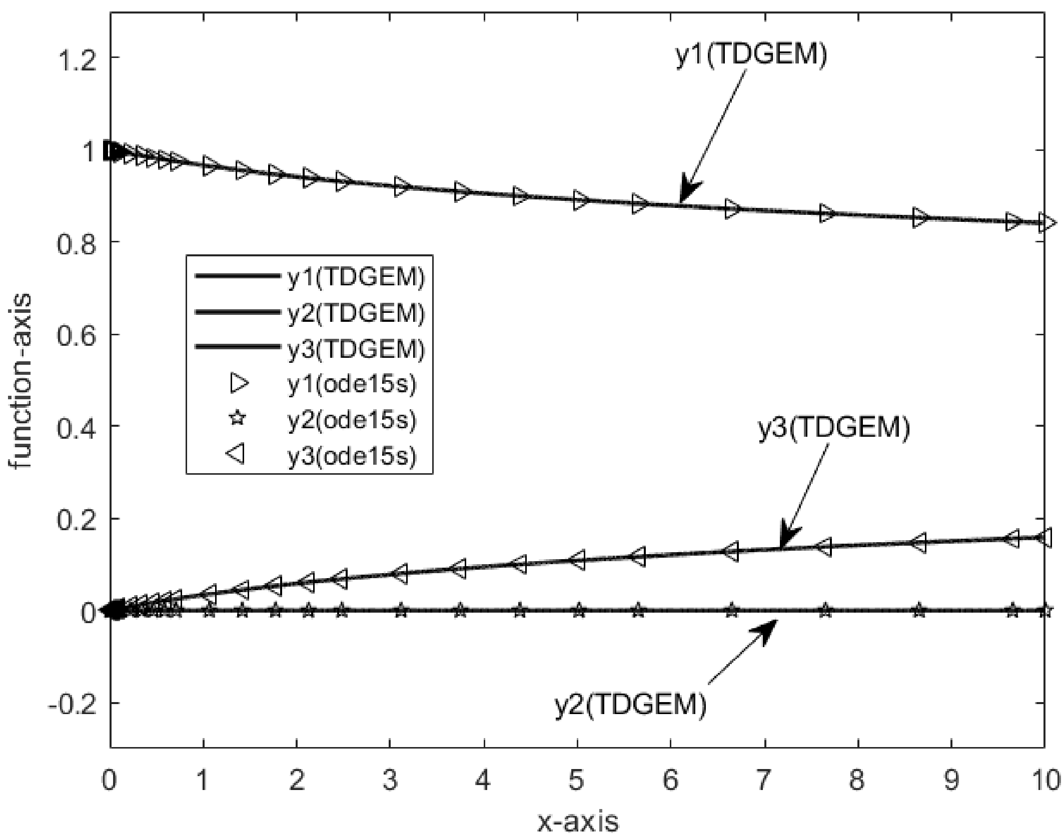

Figure 11: Numerical results for Robertson's Equation using the eighth order TDGEM.

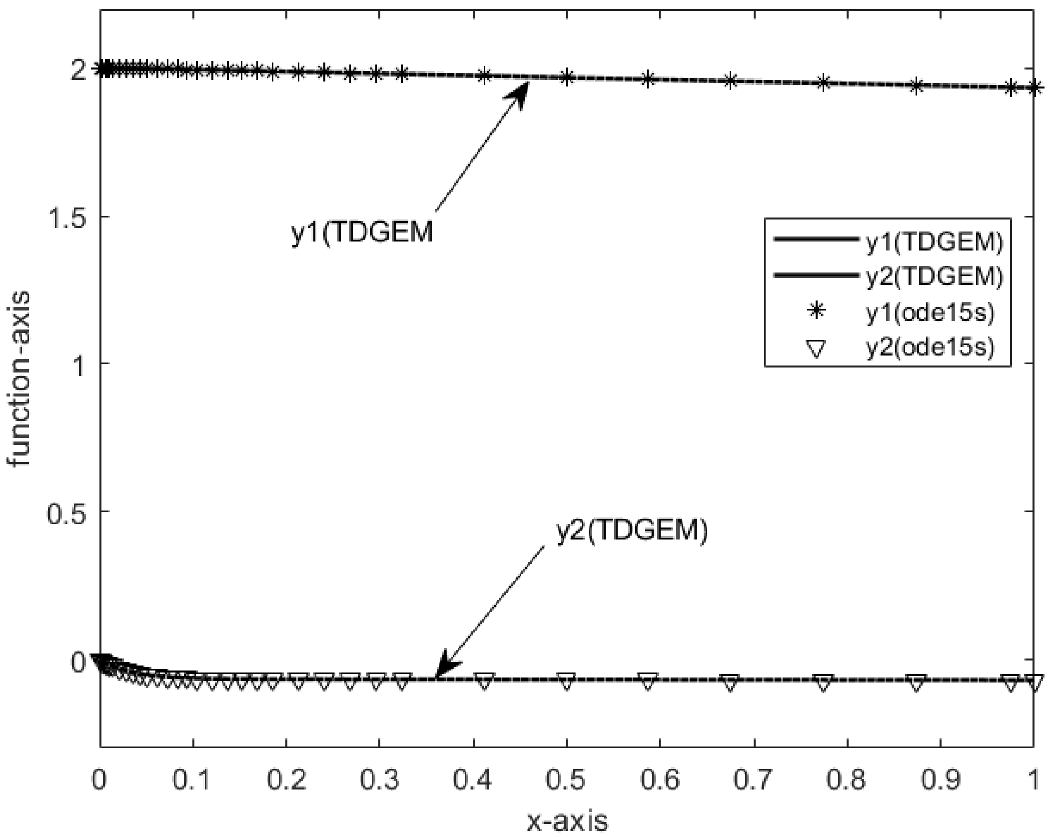

Figure 12: Numerical results for Van der Pol equation using the sixth order TDGEM.

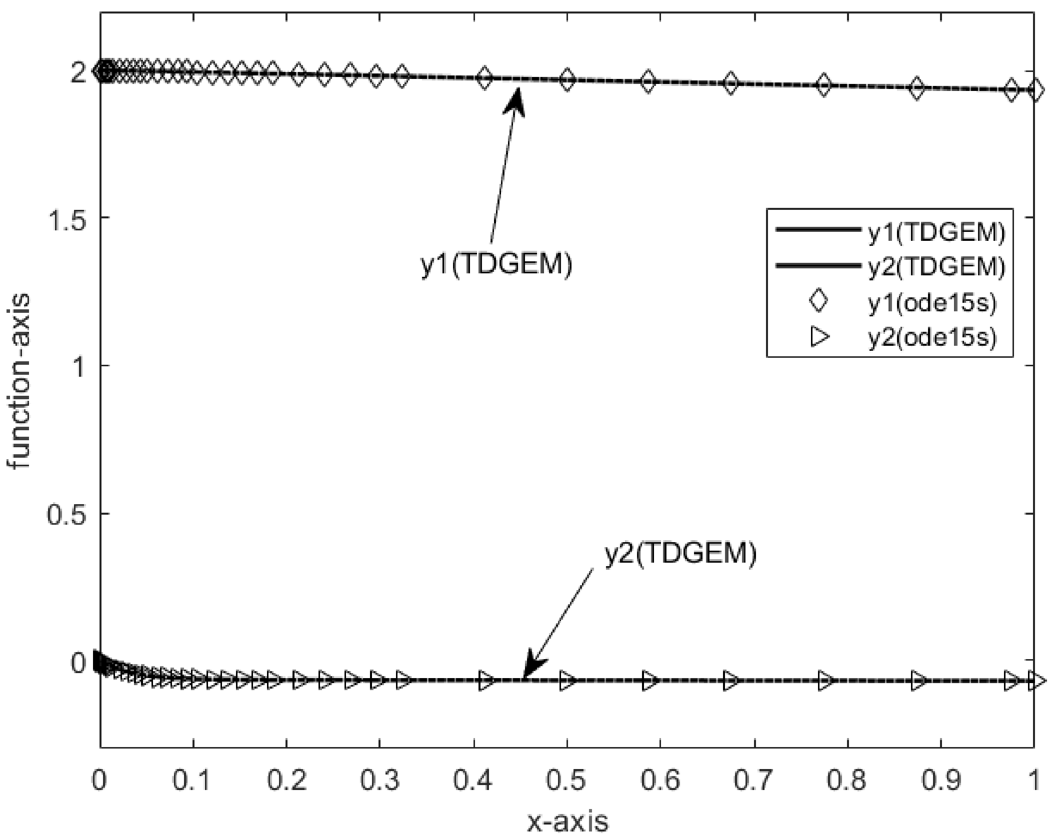

Figure 13: Numerical results for Van der Pol equation using the eighth order TDGEM.

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