Sufficient Condition for Synchronization in Complete Networks of n Reaction-Diffusion Systems of Hindmarsh-Rose Type with Nonlinear Coupling

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***Abstract*—This work focuses on the identical synchronization in complete networks consisting of n nodes. Each node is represented by a reaction-diffusion system of Hindmarsh-Rose type connected by nonlinear coupling. Sufficient conditions on the coupling strength are identified to get the synchronization. The result shows that the complete networks are easier to synchronize if they have more nodes. This paper also shows the numerical results to verify the effectiveness of the theoretical one.**

***Index Terms*—complete network, nonlinear coupling, reaction-diffusion system of Hindmarsh-Rose, synchronization.**

I. Introduction

SYNCHRONIZATION is one of the most important dynamical properties of dynamical systems and is studied in many fields. The word *synchronization* usually means having the same behavior at the same time [1]. Therefore, the synchronization of two dynamical systems could be understood that one system copies the behavior of the other. In other words, if the behaviors of some dynamical systems are synchronized, these systems are called synchronous. In the studies of Aziz-Alaoui [1] and Corson [2], it is said that a phenomenon of synchronization may appear in a network of many weakly coupled oscillators. The phenomenon of synchronization can be seen in a lot of different applications such as increasing the power of lasers, controlling oscillations in chemical reactions, encoding electronic messages for secure communications or synchronizing the output of electric circuits [1], [3].

Synchronization has been ubiquitously studied in many domains and many natural phenomena presenting the synchronization such as the movement of birds forming the cloud, the movement of fishes in the lake, the movement of the parade, the reception and transmission of a group of neurons [1], [4], [5], [6], [7], [8]. Hence, the study of synchronization is necessary. Specifically, the neural network is investigated in this paper.

In the human brain, there are a lot of neurons, they connect together in order to form a network. A neural network is a community of neurons that are physiologically connected together. The exchange between cells is mainly based on electrochemical processes. This study focuses on the sufficient condition on coupling strength to get synchronization in complete networks of neurons. In addition, each neuron is presented by a system of reaction-diffusion equations of Hindmarsh-Rose type. This paper only considers complete networks of n neurons coupled nonlinearly.

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In 1952, Hodgkin and Huxley introduced a four-dimensional mathematical system that could approximate many properties of neural membrane potential [2], [4], [7]. Based on this system, a lot of scientists published simpler models describing the neuron voltage dynamics. In 1982, Hindmarsh J. L. and Rose R. M. introduced a new simpler model called Hindmarsh-Rose model [9]. This system was known as a simplified two-dimensional model from Hodgkin-Huxley's famous model [6]. Although this system is simpler, but it has a lot of extraordinary analytical results and retains the energizing properties and biological significance of cell. It represents the equilibrium state, activity, and bursting of the neuron voltage. The system consists of two equations in the two variables u and v. The first variable is the fast one. It is excitatory, and represents the transmembrane voltage. The second one is the slow recovery variable presenting some physical quantities, such as the electrical conductivity of ion currents across the membrane. The ordinary differential equations of Hindmarsh-Rose type are given by [2]:

$$\begin{cases} \dfrac{du}{dt} = u_t = v - u^3 + au^2 + I \\ \dfrac{dv}{dt} = v_t = 1 - bu^2 - v \end{cases} \tag{1}$$

where the parameters $a = 3, b = 5$ are constants determined by practical experience, I presents the external current.

However, the system (1) is not strong enough in order to describe the propagation of action potential. In order to solve this problem, the cable equation is investigated in this work. This mathematical system is obtained from a circuit model of the membrane and its intracellular and extracellular space to provide a quantitative description of current flow and voltage change both within and between neurons. It allows us to understand how cells function quantitatively and qualitatively. Hence, the reaction-diffusion system of Hindmarsh-Rose type (HR) is considered as follows:

$$\begin{cases} \dfrac{du}{dt} = u_t = v - u^3 + au^2 + I + d\Delta u \\ \dfrac{dv}{dt} = v_t = 1 - bu^2 - v \end{cases} \quad (2)$$

where $u = u(x,t),\ v = v(x,t),\ (x,t) \in \Omega \times \mathbb{R}^+,\ d$ is a positive constant, Δu is the Laplace operator of u, $\Omega \subset \mathbb{R}^N$ is a regular bounded open set with Neumann zero flux boundary conditions, and N is a positive integer. This model allows the appearance of many patterns and relevant phenomena in physiology. This model consists of two nonlinear partial differential equations. The first one presents the action potential and the second one introduces the recovery variable describing some physical quantities, such as the electrical conductivity of ion currents across the membrane. Besides, the first equation is similar to the cable equation. It presents the distribution of the membrane potential along the axon of a single cell [6], [7]. Hereafter, system (2) is considered as a neural model, and a network of n coupled systems (2) is constructed as follows:

$$\begin{cases} u_{it} = v_i - u_i^3 + au_i^2 + I + d\Delta u_i - h(u_i, u_j) \\ v_{it} = 1 - bu_i^2 - v_i \\ i, j = 1, ..., n, i \neq j, \end{cases} \quad (3)$$

where $(u_i, v_i), i = 1, 2, ..., n$ is defined by (2).

Function h presents the coupling function describing the type of connection between cell ith and jth . Neurons connect through synapses, then it leads to two types of connections between cells such as chemical connection and electrical one. It is known that the chemical connection is more abundant than the electrical one. Hence, this study only focuses on chemical connection, then the coupling function is nonlinear [2], [10], [11] and is given by the following formula:

$$h(u_i, u_j) = (u_i - V_{syn}) g_{syn} \sum_{j=1}^{n} c_{ij} \Gamma(u_j), \quad i = 1, 2, ..., n.$$

Parameter g_{syn} represents the coupling strength. The coefficients c_{ij} are the elements of the connectivity matrix $C_n = (c_{ij})_{n \times n}$, defined by: $c_{ij} = 1$ if neuron ith and jth are coupled, $c_{ij} = 0$ if neuron ith and jth are not coupled, where $i, j = 1, 2, ..., n, i \neq j$.

Function Γ is a non-linear threshold function given by:

$$\Gamma(u_j) = \frac{1}{1 + \exp(-\lambda(u_j - \theta_{syn}))}, \quad j = 1, 2, ..., n.$$

The parameters above have the following physiological meanings:

- V_{syn} is the reversal potential and must be larger than $u_i(x,t)$, for all $i = 1, 2, ..., n$, and $x \in \Omega, t \geq 0$ since synapses are supposed excitatory.
- θ_{syn} is the threshold reached by every action potential for a neuron.
- λ is a positive number [2], [12]. The bigger λ is and the better we approach the Heaviside function.

In this paper, the rapid chemical excitatory synapse is considered, so the parameters are fixed as follows throughout this work, according to the articles [2], [12].

$$\lambda = 10,\ V_{syn} = 2,\ \theta_{syn} = -0,25.$$

Recently, there have been a lot of research papers on synchronization of the neural network, but most of them only study cells stimulated by the equations of FitzHugh-Nagumo type [10], [11] or the system of ordinary differential equations of Hindmarsh-Rose type [2]. In 2022, we have a published work about the system of reaction-diffusion equations of Hindmarsh-Rose type in complete networks with linear coupling [13]. So, there is no study related to the system of reaction-diffusion equations of Hindmarsh-Rose type in complete networks with nonlinear coupling. Thus, the research on this issue is litterally meaningful and brings a practical application value to the currently applied mathematics.

II. Synchronization of Complete Networks

In this work, the sufficient conditions on coupling strength to get synchronization in complete networks of cells are investigated. A complete network means each node connects to all others [10], [11]. For example, Fig. 1 presents the complete graphs from 3 to 10 nodes. Each node introduces a neuron modeled by a system of reaction-diffusion equations of Hindmarsh-Rose type and each edge represents a chemical connection modeled by a nonlinear coupling function. A complete network of n neurons (2) bi-directionally coupled by the chemical synapses is given as follows:

$$\begin{cases} u_{it} = v_i - u_i^3 + au_i^2 + I + d\Delta u_i \\ \qquad - \displaystyle\sum_{k=1, k\neq i}^{n} \frac{g_{syn}(u_i - V_{syn})}{1 + \exp(-\lambda(u_k - \theta_{syn}))} \\ v_{it} = 1 - bu_i^2 - v_i \\ i = 1, ..., n, \end{cases} \quad (4)$$

where g_{syn} is the coupling strength between neuron ith and jth.

Definition 1 (see [10]). *Let $S_i = (u_i, v_i),\ i = 1, 2, ..., n$ and $S = (S_1, S_2, ..., S_n)$ be a network. We say that S is identically synchronous if*

$$\lim_{t \to +\infty} \sum_{i=1}^{n-1} \left(\|u_i - u_{i+1}\|_{L^2(\Omega)} + \|v_i - v_{i+1}\|_{L^2(\Omega)} \right) = 0.$$

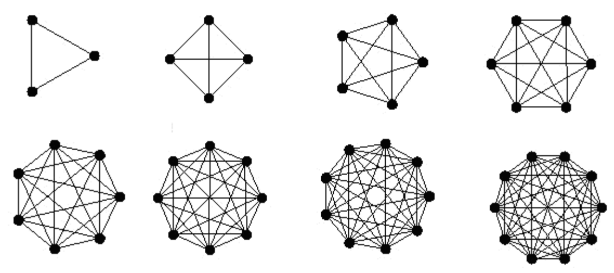

Fig. 1. Complete graphs from 3 to 10 nodes.

The system (4) can be rewritten as follows:

$$\begin{cases} u_{it} = v_i - u_i^3 + au_i^2 + I + d\Delta u_i \\ \qquad - \sum\limits_{k=1,k\neq i}^{n} \dfrac{g_{syn}(u_i - V_{syn})}{1+\exp(-\lambda(u_k - \theta_{syn}))} \\ v_{it} = 1 - bu_i^2 - v_i \\ u_{1t} = v_1 - u_1^3 + au_1^2 + I + d\Delta u_1 \\ \qquad - \sum\limits_{k=2}^{n} \dfrac{g_{syn}(u_1 - V_{syn})}{1+\exp(-\lambda(u_k - \theta_{syn}))} \\ v_{1t} = 1 - bu_1^2 - v_1 \\ i = 2, ..., n \end{cases} \tag{5}$$

Let $X = u_i - u_1, Y = v_i - v_1$ and $U = u_i + u_1, i = 2, ..., n$.

We have then the system corresponding to the variables X, Y:

$$\begin{cases} \dfrac{dX}{dt} = Y - \dfrac{1}{4}X^3 + X(aU - \dfrac{3}{4}U^2) + \Delta X \\ \qquad - \sum\limits_{k=1,k\neq i}^{n} \dfrac{g_{syn}(u_i - V_{syn})}{1+\exp(-\lambda(u_k - \theta_{syn}))} \\ \qquad + \sum\limits_{k=2}^{n} \dfrac{g_{syn}(u_1 - V_{syn})}{1+\exp(-\lambda(u_k - \theta_{syn}))} \\ \dfrac{dY}{dt} = -bXU - Y \end{cases} \tag{6}$$

Theorem 1. *Suppose that*

$$N = \inf\left\{u_i(x,t), i = 1, 2, ..., n, x \in \Omega, t \geq 0\right\},$$

and the coupling strength g_{syn} verifies the condition:

$$g_{syn} \geq \max\left\{\frac{a^2(1+\exp(-\lambda(N-\theta_{syn})))}{3(n-1)};\right.$$

$$\left.\left[\frac{3-\gamma b^2 + \gamma(b-2a)^2}{4\gamma(n-1)(3-\gamma b^2)}\right](1+\exp(-\lambda(N-\theta_{syn})))\right\},$$

where $0 < \gamma < \dfrac{3}{b^2}$, the system (5) *will synchronize in the sense of Definition 1.*

Proof: Let choose the Lyapunov function as follows:

$$E(X,Y) = \sum_{i=2}^{n} \int_{\Omega} \left(\frac{1}{2}X^2 + \frac{\gamma}{2}Y^2\right) dx,$$

where γ is a positive constant.

By taking derivative this Lyapunov function according to t, we have:

$$\begin{aligned} \tfrac{dE(X,Y)}{dt} = \sum_{i=2}^{n} \int_{\Omega} \Big[& -\tfrac{X^4}{4} + X\Delta X - \big(X^2\big(\tfrac{3}{4}U^2 - aU \\ & + \sum_{k=1,k\neq i}^{n} \tfrac{g_{syn}}{1+\exp(-\lambda(u_k-\theta_{syn}))}\Big) \\ & -(\gamma bU - 1)XY + \gamma Y^2\big) \\ & + g_{syn}(u_1 - V_{syn})(u_i - u_1)\Big(\sum_{l=2}^{n} \tfrac{1}{1+\exp(-\lambda(u_l-\theta_{syn}))} \\ & - \sum_{k=1,k\neq i}^{n} \tfrac{1}{1+\exp(-\lambda(u_k-\theta_{syn}))}\Big)\Big] dx. \end{aligned}$$

Hence,

$$\begin{aligned} \tfrac{dE(X,Y)}{dt} \leq \sum_{i=2}^{n} \int_{\Omega} \Big[& -\tfrac{X^4}{4} - \big(X^2\big(\tfrac{3}{4}U^2 - aU \\ & + \sum_{k=1,k\neq i}^{n} \tfrac{g_{syn}}{1+\exp(-\lambda(u_k-\theta_{syn}))}\Big) \\ & -(\gamma bU - 1)XY + \gamma Y^2\big) \\ & + g_{syn}(u_1 - V_{syn})(u_i - u_1)\Big(\tfrac{1}{1+\exp(-\lambda(u_i-\theta_{syn}))} \\ & - \tfrac{1}{1+\exp(-\lambda(u_1-\theta_{syn}))}\Big)\Big] dx. \end{aligned} \tag{7}$$

Since we are interested in the rapid chemical excitatory synapses, so

$$u_1 < V_{syn}, \forall x \in \Omega, t \geq 0 \Rightarrow u_1 - V_{syn} < 0, \forall x \in \Omega, t \geq 0.$$

Note that:

If $u_i > u_1$, then

$$u_i - u_1 > 0 \Rightarrow g_n(u_i - u_1)(u_1 - V_{syn}) < 0,$$

and

$$\frac{1}{1+\exp(-\lambda(u_i - \theta_{syn}))} > \frac{1}{1+\exp(-\lambda(u_1 - \theta_{syn}))}.$$

Thus

$$g_n(u_i - u_1)(u_1 - V_{syn})\Big(\tfrac{1}{1+\exp(-\lambda(u_i-\theta_{syn}))} - \tfrac{1}{1+\exp(-\lambda(u_1-\theta_{syn}))}\Big) < 0.$$

If $u_i < u_1$, then

$$u_i - u_1 < 0 \Rightarrow g_n(u_i - u_1)(u_1 - V_{syn}) > 0,$$

and

$$\frac{1}{1+\exp(-\lambda(u_i - \theta_{syn}))} < \frac{1}{1+\exp(-\lambda(u_1 - \theta_{syn}))}.$$

Thus

$$g_n(u_i - u_1)(u_1 - V_{syn})\Big(\tfrac{1}{1+\exp(-\lambda(u_i-\theta_{syn}))} - \tfrac{1}{1+\exp(-\lambda(u_1-\theta_{syn}))}\Big) < 0.$$

It means in any cases, there is always the inequality:

$$g_n(u_i - u_1)(u_1 - V_{syn})\Big(\tfrac{1}{1+\exp(-\lambda(u_i-\theta_{syn}))} - \tfrac{1}{1+\exp(-\lambda(u_1-\theta_{syn}))}\Big) < 0.$$

Therefore, the system (7) becomes:

$$\begin{aligned} \tfrac{dE(X,Y)}{dt} \leq \sum_{i=2}^{n} \int_{\Omega} \Big[& -\tfrac{X^4}{4} - \big(X^2\big(\tfrac{3}{4}U^2 - aU \\ & + \sum_{k=1,k\neq i}^{n} \tfrac{g_{syn}}{1+\exp(-\lambda(u_k-\theta_{syn}))}\Big) \\ & -(\gamma bU - 1)XY + \gamma Y^2\big)\Big] dx. \end{aligned}$$

If $N = \inf \{u_i(x,t), i = 1,2,...,n, x \in \Omega, t \geq 0\}$, then

$$\frac{dE(X,Y)}{dt} \leq \sum_{i=2}^{n} \int_{\Omega} \Big[-\frac{X^4}{4} - \Big(X^2 \big(\tfrac{3}{4}U^2 - aU + \tfrac{(n-1)g_{syn}}{1+\exp(-\lambda(N-\theta_{syn}))} \big) - (\gamma bU - 1)XY + \gamma Y^2 \Big) \Big] dx$$

$$\leq \sum_{i=2}^{n} \int_{\Omega} \Big[-\frac{X^4}{4} - \big(AX^2 - BXY + \gamma Y^2 \big) \Big] dx,$$

where $A = \dfrac{3}{4}U^2 - aU + C$, with $C = \dfrac{(n-1)g_{syn}}{1+\exp(-\lambda(N-\theta_{syn}))}$, $B = \gamma bU - 1$.

It can be seen that $AX^2 - BXY + \gamma Y^2 > 0$ if the following two conditions are verified:

- Since $A = \dfrac{3}{4}U^2 - aU + ng_{syn}$, the solutions of the equation $A = 0$ are $U_{1,2} = \dfrac{2\left(a \pm \sqrt{a^2 - 3C}\right)}{3}$ if $C \leq \dfrac{a^2}{3}$.
  Therefore, $A > 0$ if $C > \dfrac{a^2}{3}$.
  It means
  $$g_{syn} > \frac{a^2(1+\exp(-\lambda(N-\theta_{syn})))}{3(n-1)}.$$
- Moreover,
  $$\gamma A - \frac{B^2}{4} > 0 \Leftrightarrow (3-\gamma b^2)U^2 - 2(a-2b)U + 4C - \frac{1}{\gamma} > 0.$$
  This condition is satisfied if $C > \dfrac{1}{4\gamma} + \dfrac{(b-2a)^2}{4(3-\gamma b^2)}$ and $\gamma < \dfrac{3}{b^2}$.
  It means
  $$g_{syn} > \left[\frac{3 - \gamma b^2 + \gamma (b-2a)^2}{4\gamma(n-1)(3-\gamma b^2)} \right] (1+\exp(-\lambda(N-\theta_{syn}))),$$
  and
  $$\gamma < \frac{3}{b^2}.$$
  Then, if the coupling strength g_{syn} verifies the condition:
  $$g_{syn} \geq \max \left\{ \frac{a^2(1+\exp(-\lambda(N-\theta_{syn})))}{3(n-1)}; \left[\frac{3 - \gamma b^2 + \gamma (b-2a)^2}{4\gamma(n-1)(3-\gamma b^2)} \right] (1+\exp(-\lambda(N-\theta_{syn}))) \right\},$$
  with $0 < \gamma < \dfrac{3}{b^2}$, we have $AX^2 - BXY + \gamma Y^2 > 0$.
  It implies that $\dfrac{dE(X,Y)}{dt} < 0$, for all X, Y. It means the origin is globally asymptotically stable for $E(X,Y)$ (see [14]). Hence,the network (5) synchronizes globally asymptotically. Theorem has been proven.

■

As the result of Theorem 1 is proven, we can easily see that the coupling strength needs to reach certain threshold values to synchronize the complete networks. Moreover, if the number of nodes in the network is higher, it synchronizes more easily.

III. Numerical results and discussion

This section focuses on finding numerically the minimal values of coupling strength g_{syn} to observe the synchronization between n subsystems modeling the function of neuron networks. In the following, the paper shows the numerical results obtained by integrating the system (5) with

$$a = 3, b = 5, I = 0, d = 1,$$
$$[0;T] \times \Omega = [0;200] \times [0;100] \times [0;100].$$

The integration of the system is realized by using C++ and the results are represented by Gnuplot.

Fig. 2 below illustrates the synchronization of the complete network of 3 systems of reaction-diffusion equations of Hindmarsh-Rose type with nonlinear coupling. It means we integrate the system (5) for $n = 3$. The simulations show that the system synchronizes from the value $g_{syn} = 1.11$.

Fig. 2(a), 2(b), 2(f), 2(g), 2(k), 2(l), 2(p), 2(q) represent the synchronization errors of the coupled solutions $(u_1(x_1,x_2,t), u_2(x_1,x_2,t))$ and $(u_2(x_1,x_2,t), u_3(x_1,x_2,t))$, where $t \in [0;T]$ and for all $(x_1,x_2) \in \Omega$. In Fig. 2(p) and 2(q) with $g_{syn} = 1.11$, the simulation shows that the synchronization errors reach zero, it means:

$$u_1(x_1,x_2,t) \approx u_2(x_1,x_2,t)$$

and

$$u_2(x_1,x_2,t) \approx u_3(x_1,x_2,t)$$

for all $(x_1,x_2) \in \Omega$.

Fig. 2(c), 2(d), 2(e), 2(h), 2(i), 2(j), 2(m), 2(n), 2(o), 2(r), 2(s), 2(t) represent the solutions $u_i(x_1,x_2,190), i = 1,2,3$, of the network from the moment when no synchronization has occurred until they have the same shape, i.e, the synchronization is performed.

Before synchronization with $g_{syn} = 0.1$, Fig. 2(a) represents the synchronization error between u_1 and u_2, for all $(x_1,x_2) \in \Omega$; Fig. 2(b) represents the synchronization error between u_3 and u_2; Fig. 2(c) represents a solution $u_1(x_1,x_2,190)$; similarly, Fig. 2(d) and 2(e) represent the solutions $u_2(x_1,x_2,190)$ and $u_3(x_1,x_2,190)$ when they are coupled together; the results are similarly done for $g_{syn} = 0.5$ (Fig. 2(f), 2(g), 2(h), 2(i), 2(j)), $g_{syn} = 1.0$ (Fig. 2(k), 2(l), 2(m), 2(n), 2(o)) and $g_{syn} = 1.11$ (Fig. 2(p), 2(q), 2(r), 2(s), 2(t)). For $g_{syn} = 1.11$, the synchronization occurs. Since it is easy to see that the synchronization errors in Fig. 2(r), 2(r) reach zero, and all patterns in Fig. 2(r), 2(s), 2(t) are the same.

In other words, the coupling strength is over or equal to $g_{syn} = 1.11$, three nonlinearly coupled neurons have synchronous properties. By doing similarly for the complete networks of nonlinearly identical coupled neurons, the values of coupling strength with respect to the number of neurons n are reported in Table I. In Table I, for each value of n, we seek one necessary value of coupling strength to get the synchronization in complete networks with respect to n from 3 to 20.

Based on these numerical experiments, it is clear to see that the coupling strength required to obtain the synchronization of n neurons depends on the number of neurons. Indeed, the points in Fig. 3 represent the coupling strength

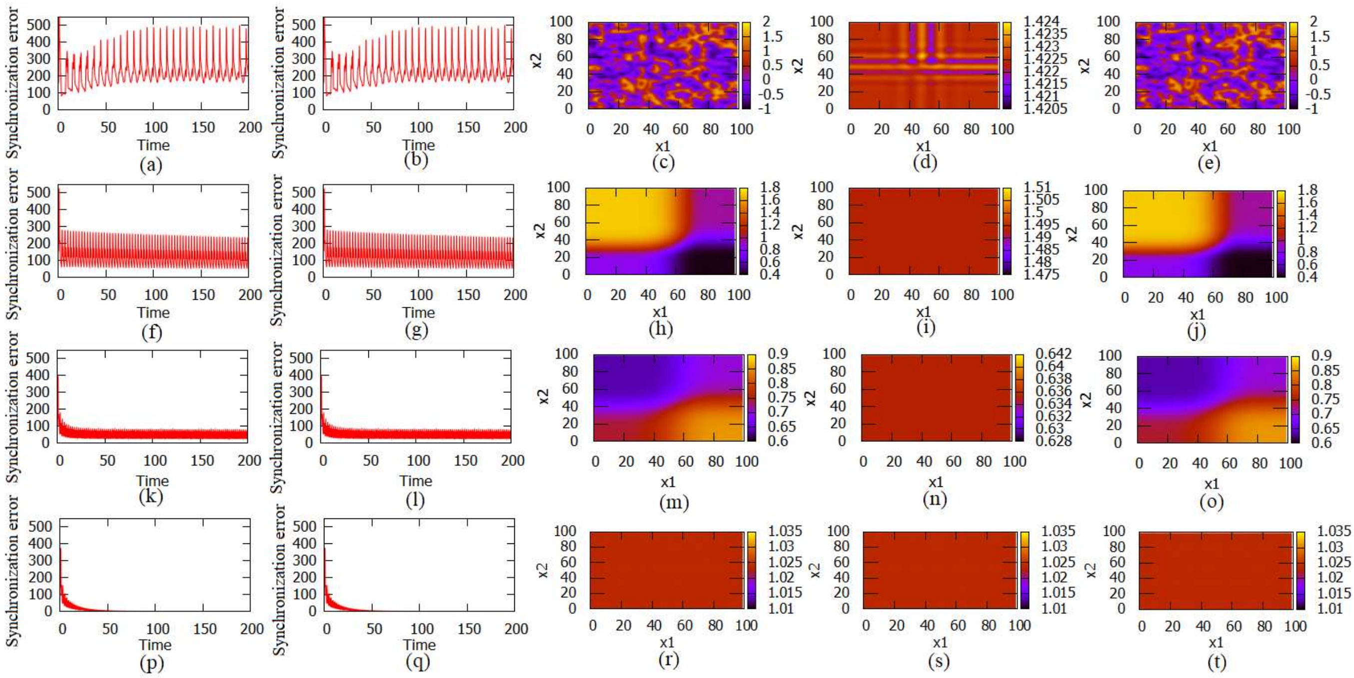

Fig. 2. Synchronization in complete network of 3 nonlinearly connected cells.

TABLE I
MINIMAL COUPLING STRENGTH NECESSARY TO OBSERVE THE SYNCHRONIZATION

n		3	4	5	6
g_{syn}		1.11	0.75	0.58	0.48
n	7	8	9	10	11
g_{syn}	0.4	0.35	0.32	0.29	0.25
n		12	13	14	15
g_{syn}		0.24	0.23	0.21	0.2
n	16	17	18	19	20
g_{syn}	0.19	0.18	0.17	0.165	0.16

of synchronization with respect to the number of neurons in complete networks from Table I, and we would like to find a relationship between the number of neurons n and the coupling strength reported in Table I. This relationship is presented by the following function:

$$g_{syn} = \frac{2.1}{n-1} + 0.05. \tag{8}$$

In Fig. 3, the function (8) is represented by a curve where the points corresponding to the coupling strengths are almost on. It means the coupling strength necessary to obtain the synchronization in complete networks follows the law given by (8). These simulations show that the bigger the number of neurons is, the smaller the coupling strength is. It means that synchronization is easier when the number of neurons in complete networks is bigger.

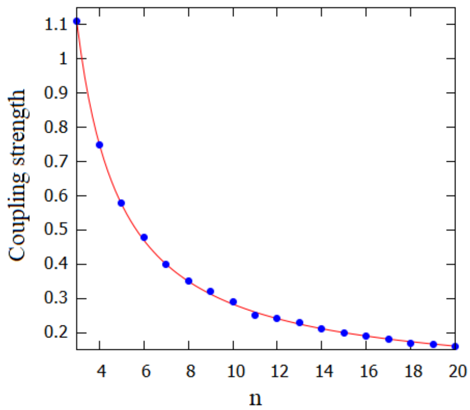

Fig. 3. The evolution of the coupling strength with respect to the number of neurons.

IV. CONCLUSION

This study provided a sufficient condition on the coupling strength to obtain the synchronization in the complete networks of n non-linearly coupled systems of reaction-diffusion equations of Hindmarsh-Rose type. Theorem 1 shows that the bigger the value of n is, the smaller g_{syn} is. Numerically, it shows that the synchronization is stable when the coupling strength exceeded a certain threshold and depends on the number of neurons in network. The bigger the number of cells is, the easier the ynchronization will be obtained. In other words, the numerical results meet the theoretical one. In addition, we will study the different synchronization regimes in free networks coupled with chemical connection in the future works.

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