Efficient Families of Multipoint Iterative Methods for Solving Nonlinear Equations

G Thangkhenpau and Sunil Panday

***Abstract*—Two new efficient families of multipoint iterative methods of fourth and eighth order convergence are constructed for finding simple roots of nonlinear equation $\Gamma(s) = 0$. Both families satisfy the optimality condition of Kung-Traub's conjecture with the family of fourth order methods requiring three function evaluations at each iteration and four evaluations of the functions at each iteration for the family of eighth order methods. Investigation on the theoritical convergence criteria of the families are carried out and fully discussed using the two main theorems which confirm their optimal convergence order. Numerical experiments on test functions are executed by comparing with existing well-known methods of similar nature to demonstrate the effectiveness and good performance of the proposed methods.**

***Index Terms*—Iterative method, Simple root, Optimal order, Nonlinear equations, Kung-Traub's conjecture.**

I. Introduction

MANY branches of science and engineering often deal with solving a number of complicated problems including real-world problems. Most of these problems can be reduced to mathematical problems as nonlinear equations of the form

$$\Gamma(s) = 0 \tag{1}$$

where $\Gamma : \mathbb{D} \subseteq \mathbb{R} \to \mathbb{R}$ is a real-valued scalar function on an open interval $\mathbb{D}$. A very effective technique for obtaining the solutions of nonlinear equation (1) is the use of iterative methods where approximate solutions are obtained with desired accuracy. Newton method [1] is perhaps the most widely used onepoint iterative method to find the simple root of (1) and is defined as

$$s_{n+1} = s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)},\ n = 0, 1, 2, ... \tag{2}$$

This method (2) has quadratic order of convergence and optimal in the sense of Kung-Traub conjecture [2]. According to this conjecture, an iterative method without memory consuming k function evaluations per iteration is optimal when the convergence order reaches the bound 2^{k-1}, this bound being called as the optimal order. Over the last few decades, numerous multipoint variants of Newton method (2) possessing higher convergence order with better efficiency have been developed using various improvement techniques ([3]- [13]). In particular, J. R. Sharma in [14] developed the well-known composite third order Newton-Steffensen method and J. Kou et al. in [15] constructed a composite fourth order Newton-type method with optimal order using the composition techniques. Weihong Bi et al. in [5], [16] and M. Salimi et al. in [11] developed the three-point optimal eight order methods using Taylor series approximation and weight function approach. The efficiency of these methods, introduced by Ostrowski [17], is measured and compared using the number of function evaluations per iteration (k) and the convergence order (p) of the iterative method by an index called the efficiency index which is given by $p^{1/k}$. Newton method (2) has the efficiency index of 1.414 for $k = 2$.

In this paper, two new families of multipoint iterative methods of optimal order of convergence, both of which are free from higher order derivatives, are presented for finding simple roots of nonlinear equations. The fourth order family uses the composition technique with three function evaluations per iteration and has the efficiency index of $4^{1/3} \approx 1.587$ while the optimal eighth order family uses the Taylor series approximation with the weight function and requires four function evaluations per iteration with improved efficiency index of $8^{1/4} \approx 1.682$.

The rest of the content of the paper is organised as follows. Section II covers the development of both the proposed families of methods. Analysis of theoritical convergence criteria of the families are also included in this section. Section III deals with the numerical test results of proposed families of methods on some nonlinear functions and the comparisons with well-known existing methods of same order to demonstrate the effectiveness and better performance of the proposed families of methods. And, section IV contains some concluding remarks.

II. Development of Methods and Convergence Analysis

This section discusses the development of the proposed families of optimal fourth and eighth order methods and the analysis of their convergence.

A. A Family of Fourth Order Methods

Recently, O. Ababneh and N. Zomot [18] developed some third order methods based on Newton method (2) as the first step. One of the methods is given below.

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ s_{n+1} &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)}.\frac{\Gamma(s_n) - 3\Gamma(y_n)}{\Gamma(s_n) - 4\Gamma(y_n)} \end{aligned} \tag{3}$$

J. R. Sharma in [14] developed a third order method by combining Newton method and Steffensen method. The

Manuscript received November 22, 2022; revised March 12, 2023.

This work was supported by the University Grants Commission (UGC), New Delhi, India.

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method is termed as Newton–Steffensen method and is written as

$$y_n = s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)}$$
$$s_{n+1} = s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} . \frac{\Gamma(s_n)}{\Gamma(s_n) - \Gamma(y_n)} \quad (4)$$

We can observe that both methods (3) and (4) are an improvement from Newton method (2) with higher efficiency index of $3^{1/3} \approx 1.442$. However, these third order methods consume three function evaluations at each iteration and so they do not satisfy the optimality conditions of Kung-Traub's conjecture. As a result, we aim to develop optimal methods of fourth order without consuming any additional cost of function evaluations per iteration thereby further increasing the efficiency index from $3^{1/3} \approx 1.442$ to $4^{1/3} \approx 1.587$.

First, let us consider a modified generalised form of Newton-Steffensen method (4), having the same third order convergence, as follows

$$y_n = s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)}$$
$$s_{n+1} = s_n - \frac{2\Gamma(s_n)^3 - \gamma\Gamma(y_n)^3}{2\Gamma'(s_n)\big(\Gamma(s_n)^2 - \Gamma(s_n)\Gamma(y_n)\big)} \quad (5)$$

where $\gamma \in \mathbb{R}$. For $\gamma = 0$, the above equation (5) becomes the Newton-Steffensen method (4).

Now, the linear combination of method (3) and method (5) produces a family of higher order methods as follows

$$\begin{aligned} y_n =& s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ s_{n+1} =& s_n - \beta \frac{\Gamma(s_n)}{\Gamma'(s_n)} \frac{\Gamma(s_n) - 3\Gamma(y_n)}{\Gamma(s_n) - 4\Gamma(y_n)} \\ &- (1-\beta) \frac{2\Gamma(s_n)^3 - \gamma\Gamma(y_n)^3}{2\Gamma'(s_n)\big(\Gamma(s_n)^2 - \Gamma(s_n)\Gamma(y_n)\big)} \end{aligned} \quad (6)$$

where $\beta, \gamma \in \mathbb{R}$. For $\beta = 0$, we get the generalised Newton-Steffensen method (5) and for $\beta = 1$, we obtain O. Ababneh and N. Zomot method (3).

The order of convergence and the conditions on β and γ for the methods (6) are discussed in the following theorem.

Theorem 1: Assume that $\Gamma : \mathbb{D} \subseteq \mathbb{R} \to \mathbb{R}$ is a sufficiently differentiable function defined on an open interval $\mathbb{D}$ with a simple root $s = \alpha \in \mathbb{D}$. For a sufficiently close initial approximation s_0 to the root α, the iterative methods given by (6) have at least third order of convergence for all $\beta, \gamma \in \mathbb{R}$. Further, for $\beta = \frac{1}{3}$ the family (6) has fourth order of convergence and satisfy the error equation

$$\varepsilon_{n+1} = \Big(\frac{1}{3}(-3+\gamma)c_2^3 - c_2c_3\Big)\varepsilon_n^4 + O(\varepsilon_n^5) \quad (7)$$

where $c_m = \frac{1}{m!}\frac{\Gamma^{(m)}(\alpha)}{\Gamma'(\alpha)}, m = 1, 2, ...$ and $\varepsilon_n = s_n - \alpha$ is the error at n^{th} iteration.

Proof: Suppose that α is a simple root of $\Gamma(s) = 0$ so that $\Gamma(\alpha) = 0$ and $\Gamma'(\alpha) \neq 0$. If s_n is sufficiently close to the root α such that $\varepsilon_n = s_n - \alpha$ is the error at n^{th} iteration, then using Taylor's expansion, we have

$$\Gamma(s_n) = \Gamma'(\alpha)\big[\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + c_4\varepsilon_n^4 + O(\varepsilon_n^5)\big] \quad (8)$$

$$\Gamma'(s_n) = \Gamma'(\alpha)\big[1 + 2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 + 5c_5\varepsilon_n^4 + O(\varepsilon_n^5)\big] \quad (9)$$

where $c_m = \frac{1}{m!}\frac{\Gamma^{(m)}(\alpha)}{\Gamma'(\alpha)}, m = 1, 2, \ldots$.

Dividing (8) by (9) gives

$$\begin{aligned} \frac{\Gamma(s_n)}{\Gamma'(s_n)} =& \varepsilon_n - c_2\varepsilon_n^2 + \big(2c_2^2 - 2c_3\big)\varepsilon_n^3 + \big(-4c_2^3 + 7c_2c_3 \\ &- 3c_4\big)\varepsilon_n^4 + O(\varepsilon_n^5)\big] \end{aligned} \quad (10)$$

Using (10), we obtain

$$\begin{aligned} y_n - \alpha =& c_2\varepsilon_n^2 + \big(-2c_2^2 + 2c_3\big)\varepsilon_n^3 + \big(4c_2^3 - 7c_2c_3 + \\ & 3c_4\big)\varepsilon_n^4 + O(\varepsilon_n^5) \end{aligned} \quad (11)$$

Using (11) to expand $\Gamma(y_n)$ near the root gives

$$\begin{aligned} \Gamma(y_n) =& \Gamma'(\alpha)\big[c_2\varepsilon_n^2 + 2(-c_2^2 + c_3)\varepsilon_n^3 + (5c_2^3 - 7c_2c_3 + \\ & 3c_4)\varepsilon_n^4 + O(\varepsilon_n^5)\big] \end{aligned} \quad (12)$$

Now, applying (8), (9), (10) and (12) in (6), we obtain the error equation as

$$\begin{aligned} \varepsilon_{n+1} =& (1-3\beta)c_2^2\varepsilon_n^3 + \Big(\frac{1}{2}\big(-6 - \beta(-12+\gamma) + \gamma\big)c_2^3 + \\ & 3(1-4\beta)c_2c_3\Big)\varepsilon_n^4 + O(\varepsilon_n^5) \end{aligned} \quad (13)$$

which shows that the family (6) has convergence order of at least three for any $\beta, \gamma \in \mathbb{R}$. Further, for $\beta = \frac{1}{3}$, the order of convergence increases from three to the optimal order four and the error equation becomes

$$\varepsilon_{n+1} = \Big(\frac{1}{3}(-3+\gamma)c_2^3 - c_2c_3\Big)\varepsilon_n^4 + O(\varepsilon_n^5) \quad (14)$$

where $\gamma \in \mathbb{R}$ is any real parameter. This completes the proof.

■

Now, substituting $\beta = \frac{1}{3}$ in (6), we get a highly efficient family of optimal fourth order methods which after simplification is expressed as follows

$$\begin{aligned} y_n =& s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ s_{n+1} =& s_n - \frac{1}{3}\frac{\Gamma(s_n)}{\Gamma'(s_n)}\Bigg[\frac{\Gamma(s_n) - 3\Gamma(y_n)}{\Gamma(s_n) - 4\Gamma(y_n)} + \\ & \frac{2\Gamma(s_n)^3 - \gamma\Gamma(y_n)^3}{\Gamma(s_n)^3 - \Gamma(s_n)^2\Gamma(y_n)}\Bigg] \end{aligned} \quad (15)$$

From (15), it can be observed that for different values of $\gamma \in \mathbb{R}$, various fourth order methods which are optimal in the sense of Kung-Traub's conjecture may be obtained.

B. A Family of Eighth Order Methods

Here, development of new highly efficient family of optimal eighth order methods are discussed along with the convergence analysis.

To begin with, by using the newly proposed fourth order family (15) as the first two steps and adding a third Newton step, we present the following three-point methods.

$$\begin{aligned} y_n =& s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ z_n =& s_n - \frac{1}{3}\frac{\Gamma(s_n)}{\Gamma'(s_n)}\Bigg[\frac{\Gamma(s_n) - 3\Gamma(y_n)}{\Gamma(s_n) - 4\Gamma(y_n)} + \\ & \frac{2\Gamma(s_n)^3 - \gamma\Gamma(y_n)^3}{\Gamma(s_n)^3 - \Gamma(s_n)^2\Gamma(y_n)}\Bigg] \\ s_{n+1} =& z_n - \frac{\Gamma(z_n)}{\Gamma'(z_n)} \end{aligned} \quad (16)$$

In order to reduce the number of function evaluations per iteration from five to four, we approximate $\Gamma'(z_n)$ using the already available informations $\Gamma(s_n), \Gamma'(s_n), \Gamma(y_n)$ and $\Gamma(z_n)$. Then, we employ the weight function technique to preserve the optimal eighth order of convergence.

By Taylor's expansion, the approximations of $\Gamma(z_n)$ and $\Gamma'(z_n)$ are written as

$$\Gamma(z_n) \approx \Gamma(y_n) + (z_n - y_n)\Gamma'(y_n) + \frac{(z_n - y_n)^2}{2}\Gamma''(y_n) \quad (17)$$

$$\Gamma'(z_n) \approx \Gamma'(y_n) + (z_n - y_n)\Gamma''(y_n) \quad (18)$$

Then, from (17), $\Gamma'(y_n)$ is approximated as

$$\Gamma'(y_n) \approx \frac{\Gamma(z_n) - \Gamma(y_n)}{z_n - y_n} - \frac{(z_n - y_n)}{2}\Gamma''(y_n) \quad (19)$$

Now, using the approximation of $\Gamma'(y_n)$ from (19) in (18), we get

$$\Gamma'(z_n) \approx \Gamma[z_n, y_n] + \frac{(z_n - y_n)}{2}\Gamma''(y_n), \quad (20)$$

where $\Gamma[z_n, y_n] = \frac{\Gamma(z_n)-\Gamma(y_n)}{z_n - y_n}$ is divided difference. Similarly, the second order derivative $\Gamma''(y_n)$ is approximated as follows

$$\Gamma''(y_n) \approx 2\Big[\frac{\Gamma[z_n, s_n] - \Gamma'(s_n)}{z_n - s_n}\Big] = 2\Gamma[z_n, s_n, s_n] \quad (21)$$

Using (21), the approximation of $\Gamma'(z_n)$ in (20) becomes

$$\Gamma'(z_n) \approx \Gamma[z_n, y_n] + (z_n - y_n)\Gamma[z_n, s_n, s_n], \quad (22)$$

Similar approximations by other authors may also be found in [5], [11], [13], [16].

Now, substituting $\Gamma'(z_n)$ from (22) in (16) and with the help of a suitable weight function, we obtain a highly efficient family of optimal eighth order methods as follows

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ z_n &= s_n - \frac{1}{3}\frac{\Gamma(s_n)}{\Gamma'(s_n)}\Big[\frac{\Gamma(s_n) - 3\Gamma(y_n)}{\Gamma(s_n) - 4\Gamma(y_n)} + \\ &\quad \frac{2\Gamma(s_n)^3 - \gamma\Gamma(y_n)^3}{\Gamma(s_n)^3 - \Gamma(s_n)^2\Gamma(y_n)}\Big] \\ s_{n+1} &= z_n - \frac{\Gamma(z_n)}{\Gamma[z_n, y_n] + (z_n - y_n)\Gamma[z_n, s_n, s_n]}\eta(\delta, \phi) \end{aligned} \quad (23)$$

where $\eta(\delta, \phi)$ is a weight function with $\delta = \frac{\Gamma(y_n)}{\Gamma(s_n)}$ and $\phi = \frac{\Gamma(z_n)}{\Gamma(s_n)}$.

Theorem 2 establishes the necessary conditions on the weight function η so that the family (23) has the optimal eighth order convergence.

Theorem 2: Assume that the function $\Gamma : \mathbb{D} \subseteq \mathbb{R} \to \mathbb{R}$ is sufficiently differentiable and has a simple root $\alpha \in \mathbb{D}$ in an open interval $\mathbb{D}$. For a sufficiently close initial approximation s_0 to the root α and a differentiable weight function η, the family of methods given by (23) has convergence order eight for all $\gamma \in \mathbb{R}$ provided η satisfies the conditions

$$\begin{aligned} &\eta(0,0) = 1;\ \eta_\delta(0,0) = 0;\ \eta_{\delta\delta}(0,0) = 0;\ \eta_\phi(0,0) = 2; \\ &\eta_{\delta\delta\delta}(0,0) = 12 - 4\gamma;\ |\eta_{\delta\phi}(0,0)| < \infty; |\eta_{\delta\delta\delta\delta}(0,0)| < \infty, \end{aligned} \quad (24)$$

where $\eta_i(\delta, \phi) = \frac{\partial \eta(\delta,\phi)}{\partial i}, i = \delta, \phi$ is the partial derivatives with respect to the index i. And, the family (23) has the following error equation

$$\begin{aligned} \varepsilon_{n+1} = &-\frac{1}{72}\Big(c_2^2\big((-3+\gamma)c_2^2 - 3c_3\big)\big(-24c_4 - 24c_2c_3(-2 + \\ &\eta_{\delta\phi}(0,0)) + c_2^3(-24(11+\gamma) + 8(-3+\gamma)\eta_{\delta\phi}(0,0) + \\ &\eta_{\delta\delta\delta\delta}(0,0))\big)\Big)\varepsilon_n^8 + O(\varepsilon_n^9) \end{aligned} \quad (25)$$

where $c_m = \frac{1}{m!}\frac{\Gamma^{(m)}(\alpha)}{\Gamma'(\alpha)}, m = 1, 2, ...$

Proof: Let $\varepsilon_n = s_n - \alpha$ be the error at n^{th} iteration. Then, for a sufficiently differentiable function Γ, the expansion of $\Gamma(s_n)$ and $\Gamma'(s_n)$ about the root α using Taylor's expansion gives

$$\Gamma(s_n) = \Gamma'(\alpha)\big[\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + c_4\varepsilon_n^4 + ... + O(\varepsilon_n^9)\big] \quad (26)$$

$$\Gamma'(s_n) = \Gamma'(\alpha)\big[1 + 2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 + 5c_5\varepsilon_n^4 + ... + O(\varepsilon_n^9)\big] \quad (27)$$

where $c_m = \dfrac{1}{m!}\dfrac{\Gamma^{(m)}(\alpha)}{\Gamma'(\alpha)}$, $m = 1, 2, ...$

From (26) and (27), we obtain

$$\begin{aligned} y_n - \alpha = &c_2\varepsilon_n^2 + \big(-2c_2^2 + 2c_3\big)\varepsilon_n^3 + \big(4c_2^3 - 7c_2c_3 + 3c_4\big)\varepsilon_n^4 \\ &+ ... + O(\varepsilon_n^9) \end{aligned} \quad (28)$$

Using (28) to expand $\Gamma(y_n)$ gives

$$\begin{aligned} \Gamma(y_n) = &\Gamma'(\alpha)\big[c_2\varepsilon_n^2 + 2(-c_2^2 + c_3)\varepsilon_n^3 + \big(5c_2^3 - 7c_2c_3 + \\ &3c_4\big)\varepsilon_n^4 + ... + O(\varepsilon_n^9)\big] \end{aligned} \quad (29)$$

Applying (26), (27) and (29), we obtain

$$z_n - \alpha = \Big(\frac{1}{3}(-3+\gamma)c_2^3 - c_2c_3\Big)\varepsilon_n^4 + ... + O(\varepsilon_n^9) \quad (30)$$

Expanding $\Gamma(z_n)$ using (30) gives

$$\Gamma(z_n) = \Gamma'(\alpha)\big[(\frac{1}{3}(-3+\gamma)c_2^3 - c_2c_3)\varepsilon_n^4 + ... + O(\varepsilon_n^9)\big] \quad (31)$$

Using (28), (29), (30) and (31), we have

$$\Gamma[z_n, y_n] = \Gamma'(\alpha)\big[1 + c_2^2\varepsilon_n^2 + 2c_2(-c_2^2 + c_3)\varepsilon_n^3 + ... + O(\varepsilon_n^9)\big] \quad (32)$$

Similarly, from (26), (30) and (31) we obtain

$$\Gamma[z_n, s_n] = \Gamma'(\alpha)\big[1 + c_2\varepsilon_n + c_3\varepsilon_n^2 + c_4\varepsilon_n^3 + ... + O(\varepsilon_n^9)\big] \quad (33)$$

Again, using the equations (27), (30) and (33) give

$$\Gamma[z_n, s_n, s_n] = \Gamma'(\alpha)\big[c_2 + 2c_3\varepsilon_n + 3c_4\varepsilon_n^2 + 4c_5\varepsilon_n^3 ... + O(\varepsilon_n^9)\big] \quad (34)$$

Now, substituting (28), (30), (31), (32) and (34) in the last step of (23), the error equation is obtained as follows

$$\varepsilon_{n+1} = \rho_1\varepsilon_n^4 + \rho_2\varepsilon_n^5 + \rho_3\varepsilon_n^6 + \rho_4\varepsilon_n^7 + ... + O(\varepsilon_n^9) \quad (35)$$

where $\rho_1 = -\dfrac{1}{3}c_2\big((-3+\gamma)c_2^2 - 3c_3\big)(-1+\eta(0,0))$

$$\begin{aligned}\rho_2 =& \big((-2+3\gamma)c_2^4 - 2(-2+\gamma)c_2^2c_3 + 2c_3^2 + 2c_2c_4\big)(-1 \\ &+ \eta(0,0)) + \frac{1}{3}c_2^2\big(-(-3+\gamma)c_2^2 + 3c_3\big)\eta_\delta(0,0) \\ \rho_3 =& 7c_3c_4(-1+\eta(0,0)) + c_2^2c_4\big(-3(-2+\gamma)(-1+ \\ &\eta(0,0)) + 2\eta_\delta(0,0)\big) + c_2\Big(\big((6-4\gamma)c_3^2 + 3c_5\big)(-1 \\ &+ \eta(0,0)) + 4c_3^2\eta_\delta(0,0)\Big) + \frac{1}{6}c_2^3c_3\big(12(-3+11\gamma) \\ &(-1+\eta(0,0)) + 2(9-8\gamma)\eta_\delta(0,0) + 3\eta_{\delta\delta}(0,0)\big) \\ &+ \frac{1}{6}c_2^5\big(-4(-3+25\gamma)(-1+\eta(0,0)) + 6(-5+ \\ &4\gamma)\eta_\delta(0,0) - (-3+\gamma)\eta_{\delta\delta}(0,0)\big) \\ \rho_4 =& -\frac{2}{3}\Big((-6+4\gamma)c_3^3 - 9c_4^2 - 15c_3c_5\Big)(-1+\eta(0,0)) \\ &+ 4c_3^3\eta_\delta(0,0) + 2c_2\Big(2c_6(-1+\eta(0,0))+ \\ &c_3c_4\big(-2(-5+3\gamma)(-1+\eta(0,0)) + 7\eta_\delta(0,0)\big)\Big) \\ &+ c_2^3c_4\Big(8 - 8\eta(0,0) + 4\gamma\big(-8+8\eta(0,0)- \\ &\eta_\delta(0,0)\big) + 3\eta_\delta(0,0) + \eta_{\delta\delta}(0,0)\Big) + c_2^2\Big(c_5\big(-4(-2 \\ &+\gamma)(-1+\eta(0,0)) + 3\eta_\delta(0,0)\big) + c_3^2\big(-8+ \\ &60\gamma(-1+\eta(0,0)) + 10\eta(0,0) - \eta_\phi(0,0)- \\ &2\eta_\delta(0,0) - 8\gamma\eta_\delta(0,0) + 3\eta_{\delta\delta}(0,0)\big)\Big)+ \\ &\frac{1}{6}c_2^4c_3\Big(\gamma\big(864 - 868\eta(0,0) + 4\eta_\phi(0,0)+ \\ &224\eta_\delta(0,0) - 10\eta_{\delta\delta}(0,0)\big) + 6\big(2\eta(0,0) - 2\eta_\phi(0,0) \\ &- 24\eta_\delta(0,0) + \eta_{\delta\delta}(0,0)\big) + \eta_{\delta\delta\delta}(0,0)\Big)- \\ &\frac{1}{18}c_2^6\Big(\gamma\big(1314 - 1314\eta(0,0) + 2(-6+\gamma)\eta_\phi(0,0)+ \\ &510\eta_\delta(0,0) - 45\eta_{\delta\delta}(0,0) + \eta_{\delta\delta\delta}(0,0)\big)- \\ &3\big(12\eta(0,0) - 6(2+\eta_\phi(0,0) - 16\eta_\delta(0,0)+ \\ &4\eta_{\delta\delta}(0,0)) + \eta_{\delta\delta\delta}(0,0)\big)\Big)\end{aligned}$$

Finally, putting the conditions (24) in the above equation (35), the error equation becomes

$$\begin{aligned}\varepsilon_{n+1} =& -\frac{1}{72}\Big(c_2^2\big((-3+\gamma)c_2^2 - 3c_3\big)\big(-24c_4 - 24c_2c_3(-2+ \\ &\eta_{\delta\phi}(0,0)) + c_2^3(-24(11+\gamma) + 8(-3+\gamma)\eta_{\delta\phi}(0,0)+ \\ &\eta_{\delta\delta\delta\delta}(0,0))\big)\Big)\varepsilon_n^8 + O(\varepsilon_n^9) \qquad (36)\end{aligned}$$

which confirms the optimal eighth order convergence of the proposed family of methods (23) for any parameter $\gamma \in \mathbb{R}$. This completes the proof of the theorem. ■

Remark: As a consequence of theorem 2 and for any value of the parameter $\gamma \in \mathbb{R}$, different values satisfying the conditions (24) may be chosen for the weight function $\eta(\delta, \phi)$ so as to obtain a family of optimal eighth order methods.

Particular Case: Here, by considering a particular value of the parameter $\gamma \in \mathbb{R}$, let us analyze a particular case for the weight function $\eta(\delta, \phi)$ which satisfies the conditions (24).

Now, let us take $\gamma = 3$ so that the condition $\eta_{\delta\delta\delta}(0,0) = 12 - 4\gamma$ in (24) reduces to zero, i.e., $\eta_{\delta\delta\delta}(0,0) = 0$. Then, a particular case for the weight function $\eta(\delta, \phi)$ satisfying the conditions (24) for which $\gamma = 3$ is given below.

$$\eta(\delta, \phi) = 1 + \lambda\delta^4 + 2\phi(1+\delta) \qquad (37)$$

where $\lambda \in \mathbb{R}$, $\delta = \frac{\Gamma(y_n)}{\Gamma(s_n)}$ and $\phi = \frac{\Gamma(z_n)}{\Gamma(s_n)}$. Thus, the family (23) becomes

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ z_n &= s_n - \frac{1}{3}\frac{\Gamma(s_n)}{\Gamma'(s_n)}\Big[\frac{\Gamma(s_n) - 3\Gamma(y_n)}{\Gamma(s_n) - 4\Gamma(y_n)} + \\ &\quad \frac{2\Gamma(s_n)^3 - 3\Gamma(y_n)^3}{\Gamma(s_n)^3 - \Gamma(s_n)^2\Gamma(y_n)}\Big] \qquad (38) \\ s_{n+1} &= z_n - \frac{\Gamma(z_n)}{\Gamma[z_n, y_n] + (z_n - y_n)\Gamma[z_n, s_n, s_n]} \times \\ &\quad \Big(1 + \lambda\delta^4 + 2\phi(1+\delta)\Big) \end{aligned}$$

And, it has the error equation

$$\varepsilon_{n+1} = c_2^2c_3\Big((-14+\lambda)c_2^3 - c_4\Big)\varepsilon_n^8 + O(\varepsilon_n^9) \qquad (39)$$

III. Numerical Results

In this section, we perform numerical experiments on the two proposed families of methods in order to analyze their effectiveness and computational efficiencies using some nonlinear functions as test functions. Then, we compare the results with some well-known existing methods of the same order available in literature. In particular, the following optimal methods are considered for the comparison.

Kou's method of fourth order (KM) [15].

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ s_{n+1} &= y_n - \frac{\Gamma(s_n) + \Gamma(y_n)}{\Gamma(s_n) - \Gamma(y_n)}\frac{\Gamma(y_n)}{\Gamma'(s_n)} \qquad (40)\end{aligned}$$

The well-known optimal fourth order Ostrowski's method (OM) [17].

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ s_{n+1} &= y_n - \Big(\frac{\Gamma(y_n)}{\Gamma(s_n) - 2\Gamma(y_n)}\Big)\frac{\Gamma(s_n)}{\Gamma'(s_n)} \qquad (41)\end{aligned}$$

Jarratt's method of fourth order (JM) [19], which is defined as

$$\begin{aligned} y_n &= s_n - \frac{2}{3}\frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ s_{n+1} &= s_n - \left(\frac{3\Gamma'(y_n) + \Gamma'(s_n)}{6\Gamma'(y_n) - 2\Gamma'(s_n)}\right)\frac{\Gamma(s_n)}{\Gamma'(s_n)} \qquad (42)\end{aligned}$$

One of the variants of King's fourth-order family of methods developed by Chun (CM) [20],

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ s_{n+1} &= y_n - \frac{\Gamma(s_n)^2}{\big(\Gamma(s_n) - \Gamma(y_n)\big)^2}\frac{\Gamma(y_n)}{\Gamma'(s_n)} \qquad (43)\end{aligned}$$

The new Potra–Pták-type optimal fourth order method developed by Prem B. Chand et al. (PBM) in [21],

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ s_{n+1} &= s_n - \Big[1 + 2\Big(\frac{\Gamma(y_n)}{\Gamma(s_n)}\Big)^2\Big]\frac{\Gamma(s_n) + \Gamma(y_n)}{\Gamma'(s_n)} \qquad (44)\end{aligned}$$

The eighth order method by Kung-Traub in [2] (KTM),

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ z_n &= y_n - \frac{\Gamma(s_n)\Gamma(y_n)}{\big(\Gamma(s_n)-\Gamma(y_n)\big)^2}\frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ s_{n+1} &= z_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)}\frac{\Gamma(s_n)\Gamma(y_n)\Gamma(z_n)}{\big(\Gamma(s_n)-\Gamma(y_n)\big)^2}\times \\ &\quad \frac{\Gamma(s_n)^2+\Gamma(y_n)\big(\Gamma(y_n)-\Gamma(z_n)\big)}{\big(\Gamma(s_n)-\Gamma(z_n)\big)^2\big(\Gamma(y_n)-\Gamma(z_n)\big)} \end{aligned} \tag{45}$$

The eighth order method developed by Liu and Wang in [3] (LWM).

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ z_n &= y_n - \frac{\Gamma(y_n)}{\Gamma'(s_n)}\frac{\Gamma(s_n)}{\Gamma(s_n)-2\Gamma(y_n)} \\ s_{n+1} &= z_n - \frac{\Gamma(z_n)}{\Gamma'(s_n)}\Big[\Big(\frac{\Gamma(s_n)-\Gamma(y_n)}{\Gamma(s_n)-2\Gamma(y_n)}\Big)^2+\frac{\Gamma(z_n)}{\Gamma(y_n)-5\Gamma(z_n)} \\ &\quad +\frac{4\Gamma(z_n)}{\Gamma(s_n)-7\Gamma(z_n)}\Big] \end{aligned} \tag{46}$$

Alicia Cordero et al. in [7] (ACM) developed two optimal classes of eighth order methods, one of them is given below.

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ z_n &= y_n - \frac{\Gamma(s_n)^2}{\big(\Gamma(s_n)-\Gamma(y_n)\big)^2}\frac{\Gamma(y_n)}{\Gamma'(s_n)} \\ s_{n+1} &= z_n - \Big(1+2t+4t^2+6t^3+u+4tu\Big)\frac{\Gamma(z_n)}{\Gamma'(s_n)} \end{aligned} \tag{47}$$

where $t=\frac{\Gamma(y_n)}{\Gamma(s_n)}, u=\frac{\Gamma(z_n)}{\Gamma(y_n)}$.

Weihong Bi, Qingbiao Wu and Hongmin Ren in [16] designed two-parameter eighth-order family of methods, one of them has the expression

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ z_n &= y_n - \Big[1+2\frac{\Gamma(y_n)}{\Gamma(s_n)}+5\frac{\Gamma^2(y_n)}{\Gamma^2(s_n)}+\lambda\frac{\Gamma^3(y_n)}{\Gamma^3(s_n)}\Big]\frac{\Gamma(y_n)}{\Gamma'(s_n)} \\ s_{n+1} &= z_n - \frac{\Gamma(s_n)+(\gamma+2)\Gamma(z_n)}{\Gamma(s_n)+\gamma\Gamma(z_n)}\times \\ &\quad \frac{\Gamma(z_n)}{\Gamma[z_n,y_n]+\Gamma[z_n,s_n,s_n](z_n-y_n)} \end{aligned} \tag{48}$$

where $\lambda=\gamma=1$. We denote this method by BWRM.

The efficient family of optimal eighth-order methods by A. Singh and J. P. Jaiswal in [22] (AJM), which is written as follows

$$\begin{aligned} y_n &= s_n - \big(1+\lambda t_1^3\big)\frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ z_n &= y_n - \big(1+t_2+\beta t_2^2\big)\frac{\Gamma(y_n)}{\Gamma[s_n,y_n]} \\ s_{n+1} &= z_n - \Big(\big(t_2^2+2(\beta-1)t_2^3\big)+\big(\gamma t_3^2\big)+\big(1+2t_4+\delta t_4^2\big)\Big)\times \\ &\quad \frac{\Gamma(z_n)}{\Gamma[y_n,z_n]} \end{aligned} \tag{49}$$

where $\lambda=1,\beta=-1,\gamma=\delta=1, t_1=\frac{\Gamma(s_n)}{\Gamma'(s_n)}, t_2=\frac{\Gamma(y_n)}{\Gamma(s_n)}$, $t_3=\frac{\Gamma(z_n)}{\Gamma(y_n)}$ and $t_4=\frac{\Gamma(z_n)}{\Gamma(s_n)}$.

The family of three-point optimal methods developed by Thukral R. in 2010 [23] (TM):

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ z_n &= y_n - \frac{\Gamma(y_n)}{\Gamma'(s_n)}\frac{\Gamma(s_n)+b\Gamma(y_n)}{\Gamma(s_n)+(b-2)\Gamma(y_n)} \\ s_{n+1} &= z_n - \frac{\Gamma(z_n)}{\Gamma'(s_n)}\Big(\varphi(t)+\frac{\Gamma(z_n)}{\Gamma(y_n)-a\Gamma(z_n)}+4\frac{\Gamma(z_n)}{\Gamma(s_n)}\Big) \end{aligned} \tag{50}$$

where $a=b=0$ and $\varphi(t)=12t^3+5t^2+2t+1$, $t=\frac{\Gamma(y_n)}{\Gamma(s_n)}$.

The optimal eighth order method by Petković et al. in [24] (PM).

$$\begin{aligned} y_n &= s_n - \frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ z_n &= s_n - \Big[\Big(\frac{\Gamma(y_n)}{\Gamma(s_n)}\Big)^2-\frac{\Gamma(s_n)}{\Gamma(y_n)-\Gamma(s_n)}\Big]\frac{\Gamma(s_n)}{\Gamma'(s_n)} \\ s_{n+1} &= z_n - \frac{\Gamma(z_n)}{\Gamma'(s_n)}\Big[\varphi(t)+\frac{\Gamma(z_n)}{\Gamma(y_n)-\Gamma(z_n)}+\frac{4\Gamma(z_n)}{\Gamma(s_n)}\Big] \end{aligned} \tag{51}$$

where $\varphi(t)=1+2t+2t^2-t^3$ with $t=\frac{\Gamma(y_n)}{\Gamma(s_n)}$.

As for the new proposed families of methods (PFM), we represent the fourth order family (15) and eighth order family (38) by PFM4 and PFM8 respectively. In both cases, the two real parameters γ and λ are taken as $\gamma=3$ and $\lambda=14$ throughout the whole computations. Some numerical test functions $\Gamma(s)$ and initial guesses (s_0) are displayed in Table I.

TABLE I
NUMERICAL TEST FUNCTIONS AND INITIAL GUESSES.

Test function, $\Gamma(s)$	Initial guess (s_0)
$\Gamma_1(s)=s^3-s^2+3s\cos s-1$	0.1
$\Gamma_2(s)=\log_e[s^2+s+2]-s+1$	3.9
$\Gamma_3(s)=\sin^2 s-s^2+1$	1.6
$\Gamma_4(s)=\sin^2 s+s$	0.2
$\Gamma_5(s)=\pi-2s\sin(\frac{\pi}{s})$	1.8
$\Gamma_6(s)=\cos s-s$	0.9

We further assess the real-world feasibility of the introduced methods PFM4 and PFM8 by examining their application to a range of engineering problems.

1) Environmental engineering problem [25]: In environmental engineering, the equation below can be utilized to determine the downstream oxygen level C (mg/L) in a river following a sewage discharge.

$$C=10-20(e^{-0.15s}-e^{-0.5s}) \tag{52}$$

where s (in kilometres) is the distance downstream to be calculated. When the oxygen level C falls to a reading of 5 mg/L, the above equation simplifies to:

$$\Gamma_7(s)=20(e^{-0.15s}-e^{-0.5s})-5 \tag{53}$$

Taking $s_0=1.2$ as the initial guess, the above equation (53) converges to the solution (simple root) $\alpha\approx 0.97622986351687693$. The comparison results are presented in Table IV.

TABLE II
COMPARISONS OF FOURTH ORDER METHODS UNDER THE SAME TNFE=12.

Method	$\Gamma(s)$	$\|\Gamma(s_n)\|$	$\|s_n - s_{n-1}\|$	s_n	COC	CPU Time	ρ
OM	$\Gamma_1(s)$	9.8365×10^{-184}	1.7232×10^{-46}	0.39532362298631519	4.0000	0.021021	0.56
JM	$\Gamma_1(s)$	1.3067×10^{-184}	1.0466×10^{-46}	0.39532362298631519	4.0000	0.024144	0.55
CM	$\Gamma_1(s)$	4.4609×10^{-169}	6.8753×10^{-43}	0.39532362298631519	4.0000	0.023024	2.33
PBM	$\Gamma_1(s)$	3.1008×10^{-159}	1.8119×10^{-40}	0.39532362298631519	4.0000	0.014662	1.45
KM	$\Gamma_1(s)$	2.9939×10^{-160}	1.0100×10^{-40}	0.39532362298631519	4.0000	0.022723	1.45
PFM4	$\Gamma_1(s)$	6.5944×10^{-280}	2.3019×10^{-70}	0.39532362298631519	4.0000	0.007981	0.12
OM	$\Gamma_2(s)$	6.3415×10^{-422}	3.6133×10^{-105}	4.1525907367571583	4.0000	0.013568	6.18×10^{-4}
JM	$\Gamma_2(s)$	5.7633×10^{-418}	3.4281×10^{-104}	4.1525907367571583	4.0000	0.014027	6.93×10^{-4}
CM	$\Gamma_2(s)$	5.2189×10^{-410}	3.1893×10^{-102}	4.1525907367571583	4.0000	0.014888	1.50×10^{-3}
PBM	$\Gamma_2(s)$	4.9959×10^{-401}	5.2924×10^{-100}	4.1525907367571583	4.0000	0.017613	1.06×10^{-3}
KM	$\Gamma_2(s)$	8.2084×10^{-401}	5.9919×10^{-100}	4.1525907367571583	4.0000	0.012023	1.06×10^{-3}
PFM4	$\Gamma_2(s)$	3.6625×10^{-437}	6.2525×10^{-109}	4.1525907367571583	4.0000	0.005731	3.98×10^{-4}
OM	$\Gamma_3(s)$	3.2552×10^{-226}	4.2228×10^{-57}	1.4044916482153412	4.0000	0.014661	0.41
JM	$\Gamma_3(s)$	1.1307×10^{-227}	1.8389×10^{-57}	1.4044916482153412	4.0000	0.021859	0.40
CM	$\Gamma_3(s)$	1.0032×10^{-202}	2.5933×10^{-51}	1.4044916482153412	4.0000	0.018541	2.34
PBM	$\Gamma_3(s)$	4.4411×10^{-189}	6.0063×10^{-48}	1.4044916482153412	4.0000	0.014729	1.37
KM	$\Gamma_3(s)$	2.0486×10^{-190}	2.7836×10^{-48}	1.4044916482153412	4.0000	0.020871	1.37
PFM4	$\Gamma_3(s)$	9.4025×10^{-245}	1.5323×10^{-61}	1.4044916482153412	4.0000	0.009142	0.07
OM	$\Gamma_4(s)$	1.0858×10^{-197}	5.7403×10^{-50}	0	4.0000	0.011792	1.00
JM	$\Gamma_4(s)$	2.2790×10^{-199}	2.2056×10^{-50}	0	4.0000	0.018880	0.96
CM	$\Gamma_4(s)$	1.1374×10^{-178}	2.7461×10^{-45}	0	4.0000	0.013016	5.00
PBM	$\Gamma_4(s)$	6.9441×10^{-167}	2.1934×10^{-42}	0	4.0000	0.010878	3.00
KM	$\Gamma_4(s)$	3.9161×10^{-168}	1.0689×10^{-42}	0	4.0000	0.014614	3.00
PFM4	$\Gamma_4(s)$	2.9974×10^{-392}	3.4278×10^{-79}	0	5.0000	0.005899	8.88×10^{-16}
OM	$\Gamma_5(s)$	2.7357×10^{-311}	3.2222×10^{-78}	1.6574002402580061	4.0000	0.022340	0.08
JM	$\Gamma_5(s)$	8.9135×10^{-307}	4.2367×10^{-77}	1.6574002402580061	4.0000	0.022040	0.09
CM	$\Gamma_5(s)$	1.2499×10^{-250}	3.2265×10^{-63}	1.6574002402580061	4.0000	0.016760	1.24
PBM	$\Gamma_5(s)$	8.5666×10^{-230}	4.5197×10^{-58}	1.6574002402580061	4.0000	0.024507	0.66
KM	$\Gamma_5(s)$	3.3751×10^{-226}	3.5808×10^{-57}	1.6574002402580061	4.0000	0.017252	0.66
PFM4	$\Gamma_5(s)$	3.1976×10^{-352}	1.4917×10^{-88}	1.6574002402580061	4.0000	0.007239	0.21
OM	$\Gamma_6(s)$	1.9400×10^{-345}	1.4591×10^{-86}	0.73908513321516064	4.0000	0.015242	0.03
JM	$\Gamma_6(s)$	1.0255×10^{-348}	2.2589×10^{-87}	0.73908513321516064	4.0000	0.016281	0.02
CM	$\Gamma_6(s)$	1.1769×10^{-334}	6.6324×10^{-84}	0.73908513321516064	4.0000	0.015627	0.07
PBM	$\Gamma_6(s)$	2.2887×10^{-326}	7.3402×10^{-82}	0.73908513321516064	4.0000	0.017111	0.05
KM	$\Gamma_6(s)$	1.1866×10^{-326}	6.2286×10^{-82}	0.73908513321516064	4.0000	0.015800	0.05
PFM4	$\Gamma_6(s)$	1.0661×10^{-364}	2.5608×10^{-91}	0.73908513321516064	4.0000	0.006798	0.01

2) Aerospace engineering problem (The Kepler's Equation) [26]: Let us examine the Kepler's equation in astronomy, as expressed below.

$$M = E - e\sin(E),\ M \in [0, 2\pi),\ e \in [0, 1] \tag{54}$$

The mean anomaly M and eccentricity e play a crucial role in the Kepler's equation. The eccentric anomaly E can be used to calculate the position of a point moving in a Keplerian orbit. For a specific case of $M = 0.6$ and $e = 0.9$, the equation simplifies to:

$$\Gamma_8(s) = s - 0.9\sin(s) - 0.6 \tag{55}$$

where the variable s represents the eccentric anomaly E to be determined. Taking $s_0 = 1.8$ as the initial guess, the above equation (55) converges to the solution (simple root) $\alpha \approx 1.4975894133904085$. The comparison results are presented in Table V.

3) Ocean engineering problem [25]: In ocean engineering, the equation below represents the height of a reflected standing wave in a harbour, represented by the variable h.

$$h = h_0\left[\sin\left(\frac{2\pi s}{\Lambda}\right)\cos\left(\frac{2\pi t v}{\Lambda}\right) + e^{-s}\right] \tag{56}$$

where s denotes the distance from the source of the wave, t denotes the time elapsed since the wave was created, v denotes the velocity of the wave, h_0 denotes the height of the wave at the source and Λ denotes the wavelength of the wave. For particular values of $\Lambda = 16$, $t = 12$, $v = 48$ and $h = 0.4h_0$, the above equation reduces to the following nonlinear equation.

$$\Gamma_9(s) = e^{-s} + \sin\left(\frac{\pi s}{8}\right)\cos(72\pi) - 0.4 = 0 \tag{57}$$

Taking $s_0 = 7.4$ as the initial guess, the above equation (57) converges to the solution (simple root) $\alpha \approx 6.9547312898815048$. The comparison results are presented in Table VI.

All numerical computations have been performed using the programming software Mathematica 12.2 with 2000 significant digits so as to avoid loss of significant digits and also to obtain pinpoint accuracy. From Table II to Table

TABLE III
COMPARISONS OF EIGHTH ORDER METHODS UNDER THE SAME TNFE=12.

Method	$\Gamma(s)$	$\lvert\Gamma(s_n)\rvert$	$\lvert s_n - s_{n-1}\rvert$	s_n	COC	CPU Time	ρ
BWRM	$\Gamma_1(s)$	7.1111×10^{-332}	3.6968×10^{-42}	0.39532362298631519	8.0000	0.022253	1.02
KTM	$\Gamma_1(s)$	1.3824×10^{-318}	1.5532×10^{-40}	0.39532362298631519	8.0000	0.017727	2.05
AJM	$\Gamma_1(s)$	8.1866×10^{-325}	2.2411×10^{-41}	0.39532362298631519	8.0000	0.019444	6.46
TM	$\Gamma_1(s)$	7.1022×10^{-287}	1.1969×10^{-36}	0.39532362298631519	8.0000	0.007850	8.47
ACM	$\Gamma_1(s)$	4.1061×10^{-278}	1.4290×10^{-35}	0.39532362298631519	8.0000	0.017768	11.86
LWM	$\Gamma_1(s)$	6.5457×10^{-344}	1.0892×10^{-43}	0.39532362298631519	8.0000	0.020690	1.66
PM	$\Gamma_1(s)$	7.8834×10^{-274}	4.7165×10^{-35}	0.39532362298631519	8.0000	0.022821	16.17
PFM8	$\Gamma_1(s)$	4.1947×10^{-446}	3.4394×10^{-56}	0.39532362298631519	8.0000	0.006564	0.01
BWRM	$\Gamma_2(s)$	3.4098×10^{-804}	2.9384×10^{-100}	4.1525907367571583	8.0000	0.013557	1.02×10^{-7}
KTM	$\Gamma_2(s)$	3.0978×10^{-786}	4.7750×10^{-98}	4.1525907367571583	8.0000	0.020219	1.90×10^{-7}
AJM	$\Gamma_2(s)$	3.9686×10^{-627}	1.7429×10^{-78}	4.1525907367571583	8.0000	0.017345	7.74×10^{-5}
TM	$\Gamma_2(s)$	2.4669×10^{-747}	3.0084×10^{-93}	4.1525907367571583	8.0000	0.009351	6.10×10^{-7}
ACM	$\Gamma_2(s)$	2.0783×10^{-741}	1.6174×10^{-92}	4.1525907367571583	8.0000	0.014735	7.37×10^{-7}
LWM	$\Gamma_2(s)$	2.8312×10^{-820}	3.0834×10^{-102}	4.1525907367571583	8.0000	0.013618	5.75×10^{-8}
PM	$\Gamma_2(s)$	1.8515×10^{-740}	2.1218×10^{-92}	4.1525907367571583	8.0000	0.013184	7.48×10^{-7}
PFM8	$\Gamma_2(s)$	1.4555×10^{-892}	3.3230×10^{-111}	4.1525907367571583	8.0000	0.005806	1.63×10^{-8}
BWRM	$\Gamma_3(s)$	4.0070×10^{-377}	8.3430×10^{-48}	1.4044916482153412	8.0000	0.020364	0.69
KTM	$\Gamma_3(s)$	6.4883×10^{-389}	2.5744×10^{-49}	1.4044916482153412	8.0000	0.014461	1.35
AJM	$\Gamma_3(s)$	1.2858×10^{-351}	9.3394×10^{-45}	1.4044916482153412	8.0000	0.013797	8.95
TM	$\Gamma_3(s)$	2.5200×10^{-352}	8.0079×10^{-45}	1.4044916482153412	8.0000	0.007792	6.00
ACM	$\Gamma_3(s)$	1.0647×10^{-338}	3.8134×10^{-43}	1.4044916482153412	8.0000	0.014344	9.59
LWM	$\Gamma_3(s)$	9.1257×10^{-398}	2.0727×10^{-50}	1.4044916482153412	8.0000	0.021201	1.08
PM	$\Gamma_3(s)$	2.9068×10^{-328}	7.2760×10^{-42}	1.4044916482153412	8.0000	0.017362	14.91
PFM8	$\Gamma_3(s)$	1.1012×10^{-427}	7.1047×10^{-54}	1.4044916482153412	8.0000	0.006650	6.83×10^{-3}
BWRM	$\Gamma_4(s)$	4.8241×10^{-407}	3.7127×10^{-46}	0	8.0000	0.016371	1.68×10^{-3}
KTM	$\Gamma_4(s)$	1.3632×10^{-339}	3.3156×10^{-43}	0	8.0000	0.012342	9.33
AJM	$\Gamma_4(s)$	6.1592×10^{-321}	5.7816×10^{-41}	0	8.0000	0.015796	49.34
TM	$\Gamma_4(s)$	1.8734×10^{-305}	5.1042×10^{-39}	0	8.0000	0.006548	40.67
ACM	$\Gamma_4(s)$	2.6968×10^{-294}	1.2034×10^{-37}	0	8.0000	0.013679	61.32
LWM	$\Gamma_4(s)$	3.5019×10^{-352}	9.0670×10^{-45}	0	8.0000	0.016047	7.67
PM	$\Gamma_4(s)$	7.7595×10^{-287}	9.8073×10^{-37}	0	8.0000	0.012231	90.64
PFM8	$\Gamma_4(s)$	2.8452×10^{-482}	2.8764×10^{-54}	0	9.0000	0.005253	2.19×10^{-6}
BWRM	$\Gamma_5(s)$	5.0814×10^{-382}	1.8721×10^{-48}	1.6574002402580061	8.0000	0.021472	1.08
KTM	$\Gamma_5(s)$	1.1528×10^{-495}	1.5410×10^{-62}	1.6574002402580061	8.0000	0.014317	0.12
AJM	$\Gamma_5(s)$	7.8185×10^{-390}	2.0210×10^{-49}	1.6574002402580061	8.0000	0.016269	0.90
TM	$\Gamma_5(s)$	1.3718×10^{-443}	4.1677×10^{-56}	1.6574002402580061	8.0000	0.007254	0.49
ACM	$\Gamma_5(s)$	1.0341×10^{-406}	1.4805×10^{-51}	1.6574002402580061	8.0000	0.021146	1.44
LWM	$\Gamma_5(s)$	2.8923×10^{-534}	2.6325×10^{-67}	1.6574002402580061	8.0000	0.018224	0.04
PM	$\Gamma_5(s)$	2.0678×10^{-384}	8.3109×10^{-49}	1.6574002402580061	8.0000	0.018698	2.93
PFM8	$\Gamma_5(s)$	1.4192×10^{-558}	2.9005×10^{-70}	1.6574002402580061	8.0000	0.006793	9.12×10^{-3}
BWRM	$\Gamma_6(s)$	4.1015×10^{-652}	8.5247×10^{-82}	0.73908513321516064	8.0000	0.018632	8.79×10^{-4}
KTM	$\Gamma_6(s)$	1.7351×10^{-648}	2.4210×10^{-81}	0.73908513321516064	8.0000	0.011694	8.79×10^{-4}
AJM	$\Gamma_6(s)$	9.9005×10^{-544}	2.1016×10^{-68}	0.73908513321516064	8.0000	0.020361	0.02
TM	$\Gamma_6(s)$	3.8555×10^{-605}	5.2896×10^{-76}	0.73908513321516064	8.0000	0.008007	3.76×10^{-3}
ACM	$\Gamma_6(s)$	3.0675×10^{-599}	2.8190×10^{-75}	0.73908513321516064	8.0000	0.021936	4.60×10^{-3}
LWM	$\Gamma_6(s)$	1.5104×10^{-784}	2.6788×10^{-98}	0.73908513321516064	8.0000	0.022728	3.40×10^{-4}
PM	$\Gamma_6(s)$	2.3233×10^{-600}	2.0296×10^{-75}	0.73908513321516064	8.0000	0.013253	4.82×10^{-3}
PFM8	$\Gamma_6(s)$	4.4174×10^{-803}	1.6042×10^{-100}	0.73908513321516064	8.0000	0.005786	6.02×10^{-5}

VI, we have presented the absolute residual error $|\Gamma(s_n)|$ for each test function, approximated roots (s_n), error in the consecutive iterations $|s_n - s_{n-1}|$ and the computational order of convergence (COC) for all the compared methods after twelve function evaluations are completed, i.e., the total number of function evaluations (TNFE) for each test function is 12. The computational order of convergence (COC) is calculated using the following expression [27]:

$$COC = \frac{\log|\Gamma(s_n)/\Gamma(s_{n-1})|}{\log|\Gamma(s_{n-1})/\Gamma(s_{n-2})|} \quad (58)$$

The calculations of the estimated values of the asymptotic error constant (ρ) are provided in the last column of the Tables. It has the following expression [28].

$$\rho \approx \lim_{n\to\infty} \left| \frac{s_n - s_{n-1}}{(s_{n-1} - s_{n-2})^p} \right| \quad (59)$$

TABLE IV
COMPARISONS FOR ENVIRONMENTAL ENGINEERING PROBLEM UNDER THE SAME TNFE=12.

Method	$\Gamma(s)$	$\lvert\Gamma(s_n)\rvert$	$\lvert s_n - s_{n-1}\rvert$	s_n	COC	CPU Time	ρ
OM	$\Gamma_7(s)$	1.0173×10^{-294}	5.6706×10^{-74}	0.97622986351687693	4.0000	0.022972	0.03
JM	$\Gamma_7(s)$	1.7996×10^{-293}	1.1528×10^{-73}	0.97622986351687693	4.0000	0.028964	0.03
CM	$\Gamma_7(s)$	5.6684×10^{-207}	2.8462×10^{-52}	0.97622986351687693	4.0000	0.019691	0.24
PBM	$\Gamma_7(s)$	1.8450×10^{-234}	4.4253×10^{-59}	0.97622986351687693	4.0000	0.025502	0.14
KM	$\Gamma_7(s)$	1.6870×10^{-231}	2.4335×10^{-58}	0.97622986351687693	4.0000	0.024749	0.14
PFM4	$\Gamma_7(s)$	8.7660×10^{-316}	3.1162×10^{-79}	0.97622986351687693	4.0000	0.010461	0.03
BWRM	$\Gamma_7(s)$	1.1571×10^{-405}	3.5049×10^{-51}	0.97622986351687693	8.0000	0.019307	0.01
KTM	$\Gamma_7(s)$	1.3542×10^{-490}	9.7814×10^{-62}	0.97622986351687693	8.0000	0.019994	4.56×10^{-3}
AJM	$\Gamma_7(s)$	1.1633×10^{-376}	1.3165×10^{-47}	0.97622986351687693	8.0000	0.009433	0.04
TM	$\Gamma_7(s)$	3.4495×10^{-436}	5.1620×10^{-55}	0.97622986351687693	8.0000	0.014236	0.02
ACM	$\Gamma_7(s)$	5.3778×10^{-414}	2.8069×10^{-52}	0.97622986351687693	8.0000	0.010414	0.04
LWM	$\Gamma_7(s)$	1.3773×10^{-507}	7.8439×10^{-64}	0.97622986351687693	8.0000	0.015214	2.71×10^{-3}
PM	$\Gamma_7(s)$	6.0767×10^{-396}	4.7098×10^{-50}	0.97622986351687693	8.0000	0.016534	0.07
PFM8	$\Gamma_7(s)$	1.1803×10^{-530}	1.5742×10^{-66}	0.97622986351687693	8.0000	0.007329	8.83×10^{-5}

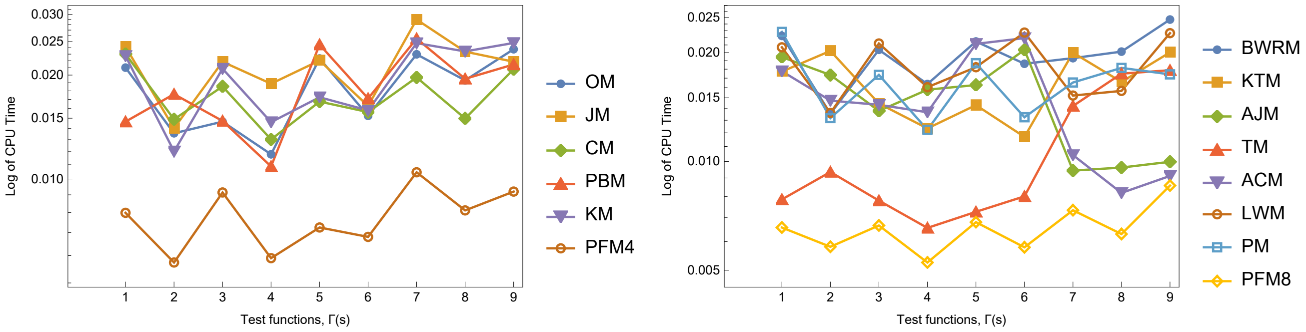
TABLE V
COMPARISONS FOR THE KEPLER'S EQUATION PROBLEM UNDER THE SAME TNFE=12.

Method	$\Gamma(s)$	$\lvert\Gamma(s_n)\rvert$	$\lvert s_n - s_{n-1}\rvert$	s_n	COC	CPU Time	ρ
OM	$\Gamma_8(s)$	4.8616×10^{-229}	1.4912×10^{-57}	1.4975894133904085	4.0000	0.019308	0.11
JM	$\Gamma_8(s)$	7.8446×10^{-231}	5.3725×10^{-58}	1.4975894133904085	4.0000	0.023384	0.10
CM	$\Gamma_8(s)$	8.4733×10^{-181}	1.1339×10^{-45}	1.4975894133904085	4.0000	0.014972	0.55
PBM	$\Gamma_8(s)$	1.3056×10^{-194}	4.5468×10^{-49}	1.4975894133904085	4.0000	0.019512	0.33
KM	$\Gamma_8(s)$	7.9546×10^{-196}	2.2590×10^{-49}	1.4975894133904085	4.0000	0.023423	0.33
PFM4	$\Gamma_8(s)$	9.7987×10^{-270}	2.0765×10^{-67}	1.4975894133904085	4.0000	0.008112	5.64×10^{-3}
BWRM	$\Gamma_8(s)$	3.6608×10^{-393}	1.6277×10^{-49}	1.4975894133904085	8.0000	0.020129	7.96×10^{-3}
KTM	$\Gamma_8(s)$	1.0787×10^{-397}	3.5069×10^{-50}	1.4975894133904085	8.0000	0.016597	0.05
AJM	$\Gamma_8(s)$	2.9913×10^{-336}	1.2708×10^{-42}	1.4975894133904085	8.0000	0.009618	0.47
TM	$\Gamma_8(s)$	1.2357×10^{-360}	1.2489×10^{-45}	1.4975894133904085	8.0000	0.017494	0.22
ACM	$\Gamma_8(s)$	2.5683×10^{-348}	4.1014×10^{-44}	1.4975894133904085	8.0000	0.008183	0.34
LWM	$\Gamma_8(s)$	1.0616×10^{-406}	2.6909×10^{-51}	1.4975894133904085	8.0000	0.015674	0.04
PM	$\Gamma_8(s)$	4.9737×10^{-339}	5.6452×10^{-43}	1.4975894133904085	8.0000	0.018146	0.52
PFM8	$\Gamma_8(s)$	6.9822×10^{-456}	4.0249×10^{-57}	1.4975894133904085	8.0000	0.006296	1.09×10^{-4}

TABLE VI
COMPARISONS FOR OCEAN ENGINEERING PROBLEM UNDER THE SAME TNFE=12.

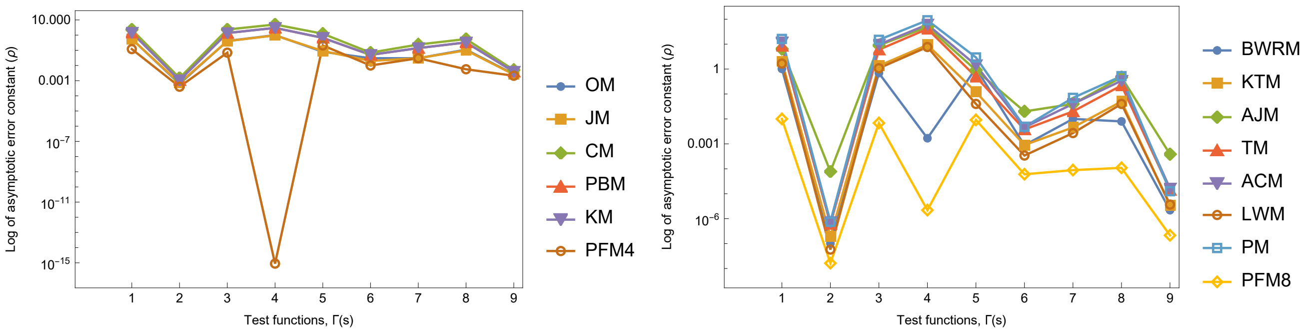
Method	$\Gamma(s)$	$\lvert\Gamma(s_n)\rvert$	$\lvert s_n - s_{n-1}\rvert$	s_n	COC	CPU Time	ρ
OM	$\Gamma_9(s)$	1.7938×10^{-322}	2.0703×10^{-80}	6.9547312898815048	4.0000	0.023765	2.70×10^{-3}
JM	$\Gamma_9(s)$	1.3431×10^{-324}	6.1678×10^{-81}	6.9547312898815048	4.0000	0.021864	2.57×10^{-3}
CM	$\Gamma_9(s)$	5.0861×10^{-307}	1.2913×10^{-76}	6.9547312898815048	4.0000	0.020798	5.07×10^{-3}
PBM	$\Gamma_9(s)$	1.2116×10^{-313}	3.0484×10^{-78}	6.9547312898815048	4.0000	0.021468	3.89×10^{-3}
KM	$\Gamma_9(s)$	9.2845×10^{-314}	2.8522×10^{-78}	6.9547312898815048	4.0000	0.024779	3.89×10^{-3}
PFM4	$\Gamma_9(s)$	8.5285×10^{-329}	5.7818×10^{-82}	6.9547312898815048	4.0000	0.009196	2.11×10^{-3}
BWRM	$\Gamma_9(s)$	1.6050×10^{-682}	3.4553×10^{-85}	6.9547312898815048	8.0000	0.024691	2.19×10^{-6}
KTM	$\Gamma_9(s)$	2.2106×10^{-615}	8.0695×10^{-77}	6.9547312898815048	8.0000	0.020034	3.41×10^{-6}
AJM	$\Gamma_9(s)$	1.0567×10^{-359}	4.0755×10^{-45}	6.9547312898815048	8.0000	0.009977	3.85×10^{-4}
TM	$\Gamma_9(s)$	1.0027×10^{-569}	3.4409×10^{-71}	6.9547312898815048	8.0000	0.017877	1.41×10^{-5}
ACM	$\Gamma_9(s)$	2.4519×10^{-566}	8.9722×10^{-71}	6.9547312898815048	8.0000	0.009118	1.62×10^{-5}
LWM	$\Gamma_9(s)$	4.1202×10^{-579}	2.7327×10^{-72}	6.9547312898815048	8.0000	0.022643	3.67×10^{-6}
PM	$\Gamma_9(s)$	2.3455×10^{-581}	1.2277×10^{-72}	6.9547312898815048	8.0000	0.017413	1.26×10^{-5}
PFM8	$\Gamma_9(s)$	4.9145×10^{-699}	3.9856×10^{-87}	6.9547312898815048	8.0000	0.008575	2.14×10^{-7}

where p is determined as either 4 or 8, depending on the order of convergence of each method being compared. It has to be noted that a reduced asymptotic error constant indicates that the associated method is faster in convergence compared to other methods. However, there are instances where the method might have smaller residual errors, smaller error differences in consecutive iterations, yet a greater asymptotic error. Also, in Table II to Table VI we have provided the CPU time (in seconds) which is the average of three CPU time consumed by each method after three executions for each test function. The CPU time is computed by taking $|\Gamma(s_n)| \leq 10^{-1000}$ as the stopping criterion using



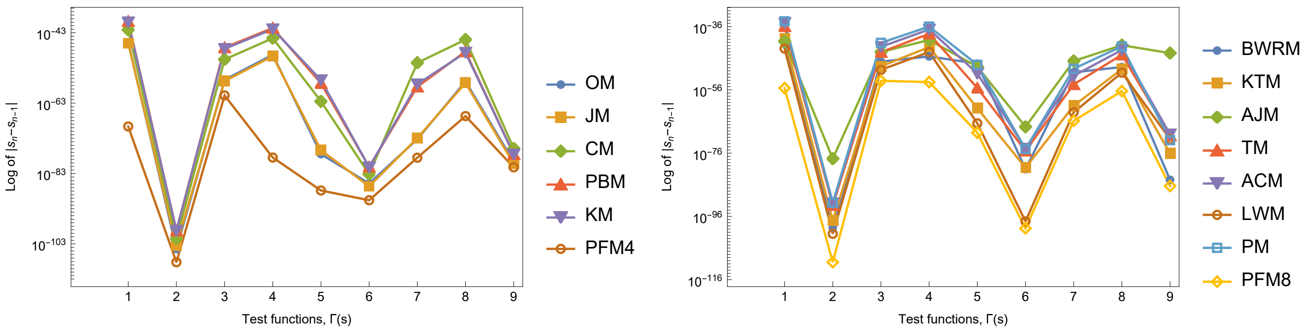
(a) Fourth order methods (b) Eighth order methods

Fig. 1. Graphical comparison between the methods based on CPU Time for each test function, $\Gamma(s)$.



(a) Fourth order methods (b) Eighth order methods

Fig. 2. Graphical comparison between the methods based on the estimated values of the asymptotic error constant (ρ) for each test function, $\Gamma(s)$.



(a) Fourth order methods (b) Eighth order methods

Fig. 3. Graphical comparison between the methods based on the error in consecutive iteration, $|s_n - s_{n-1}|$ for each test function, $\Gamma(s)$.

Mathematica 12.2 software on a system running Windows 11 with Intel(R) Core(TM) i5-10210U CPU @ 1.60GHz 2.11 GHz and 8GB of RAM. To perform graphical comparisons, we have utilized the same Mathematica 12.2 software and the figures depicting the graphical behaviour of the methods are presented in Fig. 1 to Fig. 3.

The numerical results presented in Tables II to VI highlight the competitiveness and accuracy of both the PFM4 and PFM8 families of methods, as they converge faster to the root with lower CPU time and asymptotic error constant values compared to existing methods. These results are further supported by visual illustrations through graphical representations provided in Fig. 1 to Fig. 3. Moreover, the results affirm that the computational order of convergence aligns with the theoretical order of convergence for both proposed families of methods.

Additionally, it is evident from Tables IV to VI that the practical use of the newly introduced families PFM4 and PFM8 on real-world problems shows its usefulness and applicability. Additionally, the families PFM4 and PFM8 demonstrated superior performance when compared to other existing methods of similar nature.

IV. Concluding Remarks

In this paper, we have introduced two new families of iterative methods of optimal fourth and eighth order convergence for solving nonlinear equations. By using the composition technique along with a modified generalised form of Newton-Steffensen method and the weight function approach, we have achieved the idea of developing new families of iterative methods with optimal order convergence and improved efficiency. Analysis of the numerical results have illustrated the efficiency and better performance of our new proposed families of methods in terms of minimal absolute residual error, minimal error in consecutive iterations. The proposed methods, PFM4 and PFM8, exhibit faster convergence with smaller asymptotic error constant values, resulting in reduced CPU time compared to other existing methods in comparison. Moreover, the overall performance of our proposed work is quite good with fast convergence speed and can be a great alternative for solving nonlinear equations.

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