An Improved Reciprocally Convex Lemma for Stability Analysis of Interval Time-Varying Delay Systems

Ze-Rong Ren, Jun-Kang Tian

Abstract—**This paper studies the stability problems of systems with an interval time-varying delay. First, an improved reciprocally convex lemma is introduced. Second, based on this reciprocally convex lemma, a less conservative stability criterion is obtained. Finally, the merits of the proposed method is shown via a numerical example.**

Index Terms—**Reciprocally convex lemma, Time-varying delay, Stability, Linear matrix inequality.**

I. Introduction

TIME -delay occurs in many practical systems, and it may cause poor performance or even instability. Therefore, the stability analysis of time-delay systems has attracted considerable attention during the past two decades [1, 2]. The Lyapunov-Krasovskii functional (LKF) method is an effective method for stability analysis of time-delay systems. There are two approaches to obtain less conservative criteria for systems with time-delay: introducing an appropriate LKF and estimating the derivative of the LKF. In constructing LKF, many types of LKFs are introduced, such as integral delay partitioning-based LKFs [3], delay partitioning-based LKFs [4], polynomial-type LKFs [5] and the augmented LKFs [6]. Sometimes in order to contain more information about the time-delay, some quadratic terms of the time-delay are introduced [7]. In [8], a new inequality is proposed for the quadratic polynomials by introducing free matrix variables. However, these free matrix variables lead to the great increase in computational complexity.

In recent years, several inequalities are introduced to estimate the integral terms in the derivative of LKFs, such as the Jensen inequality [9-10], Wirtinger inequality [11], auxiliary inequality [12], Bessel inequality [13] and free matrix inequality [14]. By using the Jensen inequality, Wirtinger inequality and Bessel inequality to estimate the integral term in the derivative of the LKF, the term $-\frac{1}{\alpha}\zeta_1^T(t)R\zeta_1(t)-\frac{1}{1-\alpha}\zeta_2^T(t)R\zeta_2(t)$ is obtained, where $\alpha\in(0,1)$, $\zeta_1(t)$ and $\zeta_2(t)$ are two real column vectors with appropriate dimensions and R is a positive symmetric matrix. This term is usually handled by a reciprocally convex combination lemma [15] and some improved reciprocally convex lemmas [16-20]. The advantage of these lemmas lie in changing the non convex terms into a single convex expression. However, it is shown that these lemmas are conservative due to still exist many zero elements in the decision matrices. This motivates the present research.

In this paper, a generalized reciprocally convex lemma is introduced which includes some existing reciprocally convex lemmas as special cases. Based on this proposed lemma and a delay-partitioning approach, a new stability criterion is obtained for time-varying delay systems. The merits of the presented criterion is demonstrated through a numerical example.

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II. Preliminary

Consider the following systems with a time-varying delay

$$\begin{cases}\dot{x}(t)=Ax(t)+Bx(t-h(t))\\ x(t)=\phi(t),\quad t\in[-h_2,0]\end{cases}\tag{1}$$

where $x(t)\in R^n$ is the state vector and $A,B\in R^{n\times n}$ are constant matrices. The time-varying delay $h(t)$ satisfies

$$0\le h_1\le h(t)\le h_2\tag{2}$$

$$h_{12}=h_2-h_1\tag{3}$$

Lemma 2.1[20] For any matrix $R\in S_+^n$, if there exist $X_1,X_2,X_3,Z_1,Z_2,Z_4\in S^n$, $Y_1,Y_2,Y_3,Y_4\in R^{n\times n}$ and $\forall\beta\in(0,1)$ such that the following inequality holds:

$$\begin{aligned}\begin{bmatrix}R & 0\\ 0 & R\end{bmatrix}&\ge\beta\begin{bmatrix}X_1 & Y_1\\ * & (1-\beta)Z_1\end{bmatrix}\\ &+(1-\beta)\begin{bmatrix}\beta X_2 & Y_2\\ * & Z_2\end{bmatrix}\\ &+\beta^2\begin{bmatrix}X_3 & Y_3\\ * & 0\end{bmatrix}+(1-\beta)^2\begin{bmatrix}0 & Y_4\\ * & Z_4\end{bmatrix}\end{aligned}\tag{4}$$

then

$$\begin{bmatrix}\frac{1}{\beta}R & 0\\ 0 & \frac{1}{1-\beta}R\end{bmatrix}\ge\begin{bmatrix}R+S_1 & S_2\\ S_2^T & R+S_3\end{bmatrix}\tag{5}$$

where
$S_1=(1-\beta)X_1+(1-\beta)^2X_2+\beta(1-\beta)X_3$,
$S_2=\beta Y_1+(1-\beta)Y_2+\beta^2Y_3+(1-\beta)^2Y_4$,
$S_3=\beta^2Z_1+\beta Z_2+\beta(1-\beta)Z_4$.

Remark 2.2 Setting $X_1=X_2=X_3=Z_1=Z_2=Z_4=Y_3=Y_4=0$ and $Y_1=Y_2=S$, Lemma 2.1 reduces to Theorem 1 in [15]. Setting $X_2=X_3=Y_3=Y_4=$

$Z_1 = Z_4 = 0$ and $Y_1 = Y_2 = S$, Lemma 2.1 reduces to Theorem 1 in [17]. Setting $X_1 = \bar{X}_1+\bar{X}_2, X_2 = -\bar{X}_2, Z_1 = -\bar{Y}_2, Z_2 = \bar{Y}_1 + \bar{Y}_2, Y_1 = \bar{Z}_0 + \bar{Z}_1, Y_2 = \bar{Z}_0, Y_3 = \bar{Z}_2$ and $X_3 = Y_4 = Z_4 = 0$, Lemma 2.1 reduces to Lemma 2 in [18]. Setting $Z_1 = X_2 = 0$, Lemma 2.1 reduces to Lemma 3 in [19]. Therefore, the generalized reciprocally convex lemma proposed in Lemma 2.1 includes lemmas in [15, 17-19] as special cases.

Remark 2.3 In Lemma 2.1, the cross terms $\beta(1-\beta)Z_1$ and $\beta(1-\beta)X_2$ are introduced to exploit more information on the decision matrices. This may yield more less conservative stability results.

Lemma 2.3[21]For a matrix $R \in S_+^n$ and any continuously differentiable function $y : [a,b] \longrightarrow R^n$ the following inequality holds:

$$\int_a^b \dot{y}^T(s)R\dot{y}(s)ds \geq \frac{1}{b-a}(\Omega_0^T R\Omega_0 + 3\Omega_1^T R\Omega_1 + 5\Omega_2^T R\Omega_2), \tag{6}$$

$\Omega_0 = y(b) - y(a)$,
$\Omega_1 = y(b) + y(a) - \frac{2}{b-a}\int_a^b y(s)ds$,
$\Omega_2 = y(b) - y(a) + \frac{6}{b-a}\int_a^b y(s)ds - \frac{12}{(b-a)^2}\int_a^b\int_u^b y(s)dsdu.$

Lemma 2.4[19] Let $g(y) = a_0 + a_1 y + a_2 y^2$, where $y \in [h_1, h_2]$ and $a_0, a_1, a_2 \in R$. For a given non-negative integer N, if the following conditions hold for $i = 1, 2, \cdots, 2^N$:
(i) $g(h_1) < 0$,
(ii)$g(h_2) < 0$,
(iii) $\frac{h_{12}}{2^{N+1}}\dot{g}(\frac{i-1}{2^N}h_{12} + h_1) + g(\frac{i-1}{2^N}h_{12} + h_1) < 0, \ i = 1, 2, \cdots, 2^N$,
then $g(y) < 0$.

III. Main Results

In this section, a novel delay-dependent stability criterion is obtained as follows.

Theorem 3.1 For given scalars $h_1 > 0$ and $h_2 > 0$ and non-negative integers N and m, if there exist matrices $P \in S_+^{5n}$, $Q_1 \in S_+^{3n}$, $Q_2 \in S_+^{4n}$, $Q_3 \in S_+^{mn}$, $R_1 \in S_+^n$, $R_2 \in S_+^n$, $X_1, X_2, X_3, Z_1, Z_2, Z_4 \in S^{3n}$, $Y_1, Y_2, Y_3, Y_4 \in R^{3n\times 3n}$ and $L_1, L_2 \in R^{(11+m)\times n}$ such that the following LMIs hold:

$$\psi_0 \leq 0, \tag{7}$$

$$\psi_2 + \psi_1 + \psi_0 \leq 0, \tag{8}$$

$$(\frac{1}{2^N}\bar{\rho}_j + \bar{\rho}_j^2)\psi_2 + (\frac{1}{2^{N+1}}\bar{\rho}_j + \bar{\rho}_j^2)\psi_1 + \psi_0 \leq 0, \tag{9}$$

$$h_i^2\varpi_2 + h_i\varpi_1 + \varpi_0 \leq 0, i = 1, 2, \tag{10}$$

$$(\frac{h_{12}}{2^N}\hat{\rho}_j + \hat{\rho}_j^2)\varpi_2 + (\frac{h_{12}}{2^{N+1}}\hat{\rho}_j + \hat{\rho}_j^2)\varpi_1 + \varpi_0 \leq 0, \tag{11}$$

then, system (1) is asymptotically stable.
where

$$\psi_0 = \begin{bmatrix} -\bar{R}_2 & Y_2 + Y_4 \\ * & -\bar{R}_2 + Z_2 + Z_4 \end{bmatrix},$$

$$\psi_1 = \begin{bmatrix} X_1 + X_2 & Y_1 - Y_2 + 2Y_4 \\ * & Z_1 - Z_2 - 2Z_4 \end{bmatrix},$$

$$\psi_2 = \begin{bmatrix} -X_2 + X_3 & Y_3 + 2Y_4 \\ * & Z_4 - Z_1 \end{bmatrix},$$

$$\begin{aligned}\varpi_0 =& Sym\left\{\Pi_{11}^T P\delta_1 + \Pi_{41}^T Q_2\delta_2 - L_1(h_1 e_{m+5} + e_{m+10})\right.\\ &+L_2(h_2 e_{m+6} - e_{m+11}) + \Upsilon_2^T(\frac{h_1 Y_1 - h_2 Y_2}{h_{12}})\Upsilon_3\\ &\left.+\Pi_{03}^T Q_1\Pi_{04}\right\} + \Pi_{01}^T Q_1\Pi_{01} - \Pi_{02}^T Q_1\Pi_{02}\\ &+ \Pi_{21}^T Q_2\Pi_{21} - \Pi_{22}^T Q_2\Pi_{22} + \Pi_{31}^T Q_3\Pi_{31}\\ &- \Pi_{32}^T Q_3\Pi_{32} - \Upsilon_1^T\bar{R}_1\Upsilon_1\\ &- \Upsilon_2^T(\bar{R}_2 + \frac{h_2}{h_{12}}X_1 + \frac{h_2^2}{h_{12}^2}X_2 - \frac{h_1 h_2}{h_{12}^2}X_3)\Upsilon_2\\ &- \Upsilon_3^T(\bar{R}_2 - \frac{h_1^2}{h_{12}^2}Z_1 - \frac{h_1}{h_{12}}Z_2 - \frac{h_1 h_2}{h_{12}^2}Z_4)\Upsilon_3\\ &+ e_0^T(\frac{h_1^2}{m^2}R_1 + h_{12}^2 R_2)e_0,\end{aligned}$$

$$\begin{aligned}\varpi_1 =& Sym\left\{\Upsilon_2^T(\frac{Y_2 - Y_1}{h_{12}} + \frac{2h_1 Y_3 - 2h_2 Y_4}{h_{12}^2})\Upsilon_3\right.\\ &\left.+\Pi_{12}^T P\delta_1 + \Pi_{42}^T Q_2\delta_2 + L_1 e_{m+5} - L_2 e_{m+6}\right\}\\ &+ \Upsilon_2^T(\frac{X_1}{h_{12}} + \frac{2h_2}{h_{12}^2}X_2 - \frac{h_1 + h_2}{h_{12}^2}X_3)\Upsilon_2\\ &+ \Upsilon_3^T(\frac{2h_1}{h_{12}^2}Z_1 - \frac{Z_2}{h_{12}} - \frac{h_1 + h_2}{h_{12}^2}Z_4)\Upsilon_3,\end{aligned}$$

$$\begin{aligned}\varpi_2 =& Sym\left\{\Pi_{13}^T P\delta_1 + \Pi_{43}^T Q_2\delta_2 - \Upsilon_2^T\frac{Y_3 - Y_4}{h_{12}^2}\Upsilon_3\right\}\\ &+ \Upsilon_2^T(\frac{X_3 - X_2}{h_{12}^2})\Upsilon_2 + \Upsilon_3^T(\frac{Z_4 - Z_1}{h_{12}^2})\Upsilon_3,\end{aligned}$$

$\bar{\rho}_j = \frac{j-1}{2^N}$, $\hat{\rho}_j = \frac{j-1}{2^N}h_{12} + h_1, j = 1, 2, \cdots, 2^N$,
$\Pi_{01} = \begin{bmatrix} e_1^T & e_1^T & 0 \end{bmatrix}^T$,
$\Pi_{02} = \begin{bmatrix} e_{m+2}^T & e_1^T & h_1 e_{m+4}^T \end{bmatrix}^T$,
$\Pi_{03} = \begin{bmatrix} h_1 e_{m+4}^T & h_1 e_1^T & h_1^2 e_{m+7}^T \end{bmatrix}^T$,
$\Pi_{04} = \begin{bmatrix} 0 & \upsilon^T & e_1^T \end{bmatrix}^T$,
$\Pi_{11} = \begin{bmatrix} e_1^T & h_1 e_{m+4}^T & \bar{\Pi}_{11}^T & h_1^2 e_{m+7}^T & \upsilon^T \end{bmatrix}^T$,
$\bar{\Pi}_{11} = e_{m+10} + e_{m+11}$,
$\Pi_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & \bar{\Pi}_{12}^T \end{bmatrix}^T$,
$\bar{\Pi}_{12} = -2h_1 e_{m+8} - 2h_2 e_{m+9} - e_{m+10}$,
$\Pi_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{m+8}^T + e_{m+9}^T \end{bmatrix}^T$,
$\Pi_{21} = \begin{bmatrix} e_{m+1}^T & e_1^T & 0 & e_{m+10}^T + e_{m+11}^T \end{bmatrix}^T$,
$\Pi_{22} = \begin{bmatrix} e_{m+3}^T & e_1^T & e_{m+10}^T + e_{m+11}^T & 0 \end{bmatrix}^T$,
$\Pi_{31} = \begin{bmatrix} e_1^T & e_2^T & \cdots & e_{m-1}^T \end{bmatrix}^T$,
$\Pi_{32} = \begin{bmatrix} e_2^T & e_3^T & \cdots & e_m^T \end{bmatrix}^T$,
$\Pi_{41} = \begin{bmatrix} e_{m+10}^T + e_{m+11}^T & h_{12}e_1^T & \upsilon^T & T_1 \end{bmatrix}^T$,
$\Pi_{42} = \begin{bmatrix} 0 & 0 & \bar{\Pi}_{42}^T & T_2 \end{bmatrix}^T$,
$\bar{\Pi}_{42} = -2h_1 e_{m+8} - 2h_2 e_{m+9} - e_{m+10}$,
$\Pi_{43} = \begin{bmatrix} 0 & 0 & e_{m+8}^T + e_{m+9}^T & -e_{m+8}^T - e_{m+9}^T \end{bmatrix}^T$,

$\delta_1 = \begin{bmatrix} e_0^T & e_1^T - e_{m+1}^T & e_{m+1}^T - e_{m+3}^T & T_3 & T_4 \end{bmatrix}^T$,
$\delta_2 = \begin{bmatrix} 0 & e_0^T & e_{m+1}^T & -e_{m+3}^T \end{bmatrix}^T$,

$\upsilon = h_1^2 e_{m+8} + h_2^2 e_{m+9} + h_2 e_{m+10}$,

$\Upsilon_1 = \begin{bmatrix} e_1^T - e_2^T & e_1^T + e_2^T - 2e_{m+4}^T & T_5 \end{bmatrix}^T$,

$\Upsilon_2 = \begin{bmatrix} e_{m+1}^T - e_{m+2}^T & \bar{\Upsilon}_2^T & T_6 \end{bmatrix}^T$,
$\bar{\Upsilon}_2 = e_{m+1} + e_{m+2} - 2e_{m+5}$,
$\Upsilon_3 = \begin{bmatrix} e_{m+2}^T - e_{m+3}^T & \bar{\Upsilon}_3^T & T_7 \end{bmatrix}^T$,
$\bar{\Upsilon}_3 = e_{m+2} + e_{m+3} - 2e_{m+6}$,
$T_1 = h_{12}(e_{m+10}^T + e_{m+11}^T) - \upsilon^T$,
$T_2 = 2h_1 e_{m+8}^T + 2h_2 e_{m+9}^T + e_{m+10}^T$,
$T_3 = h_1(e_1^T - e_{m+4}^T)$,
$T_4 = h_{12}e_{m+1}^T - e_{m+10}^T - e_{m+11}^T$,
$T_5 = e_1^T - e_2^T + 6e_{m+4}^T - 12e_{m+7}^T$,
$T_6 = e_{m+1}^T - e_{m+2}^T + 6e_{m+5}^T - 12e_{m+8}^T$,
$T_7 = e_{m+2}^T - e_{m+3}^T + 6e_{m+6}^T - 12e_{m+9}^T$,
$e_0 = Ae_1 + Be_{m+2}$,
$\bar{R}_i = diag(R_i, 3R_i, 5R_i), i = 1, 2$,
$e_i = \begin{bmatrix} 0_{n\times(i-1)n} & I_n & 0_{n\times((11+m)-i)n} \end{bmatrix}$,
for $i = 1, 2, \cdots, (11+m)$.

Proof. Let an integer $m > 0$, $[0, h_1]$ is divided into m segments equally, i.e., $[0, h_1] = \bigcup_{i=1}^{m}[\frac{i-1}{m}h_1, \frac{i}{m}h_1]$. Then, we introduce a LKF candidate as follows:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t), \tag{12}$$

where

$$V_1(x_t) = \eta_1^T(t)P\eta_1(t), \tag{13}$$

$$\begin{aligned} V_2(x_t) = &\int_{t-h_1}^{t} \eta_2^T(t,s)Q_1\eta_2(t,s)ds \\ &+ \int_{t-h_2}^{t-h_1} \eta_3^T(t,s)Q_2\eta_3(t,s)ds \\ &+ \int_{t-\frac{h_1}{m}}^{t} \eta_4^T(s)Q_3\eta_2(s)ds, \end{aligned} \tag{14}$$

$$\begin{aligned} V_3(x_t) = &\frac{h_1}{m}\int_{t-\frac{h_1}{m}}^{t}\int_{u}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsdu \\ &+ h_{12}\int_{t-h_2}^{t-h_1}\int_{u}^{t} \dot{x}^T(s)R_2\dot{x}(s)dsdu, \end{aligned} \tag{15}$$

$\eta_1(t) = \begin{bmatrix} x^T(t) & \mu_1^T(t) & \bar{\eta}_1(t)^T & \mu_4^T(t) & \mu_5^T(t) \end{bmatrix}^T$

$\bar{\eta}_1(t) = \mu_2(t) + \mu_3(t)$

$\eta_2(t,s) = \begin{bmatrix} x^T(s) & x^T(t) & \int_s^t x^T(\beta)d\beta \end{bmatrix}^T$

$\eta_3^T(t,s) = \begin{bmatrix} x^T(s) & x^T(t) & \int_s^{t-h_1} x^T(\beta)d\beta & \bar{\eta}_3(t,s)^T \end{bmatrix}$

$\bar{\eta}_3(t,s) = \int_{t-h_2}^{s} x(\beta)d\beta$

$\eta_4(t) = \begin{bmatrix} x^T(s) & x^T(s-\frac{h_1}{m}) & \cdots & x^T(s-\frac{m-1}{m}h_1) \end{bmatrix}^T$

$\mu_1(t) = \int_{t-\frac{h_1}{m}}^{t} x(\theta)d\theta$

$\mu_2(t) = \int_{t-h(t)}^{t-h_1} x(\theta)d\theta$

$\mu_3(t) = \int_{t-h_2}^{t-h(t)} x(\theta)d\theta$

$\mu_4(t) = \int_{t-\frac{h_1}{m}}^{t}\int_{\theta}^{t} x(s)dsd\theta$

$\mu_5(t) = \int_{t-h_2}^{t-h_1}\int_{\theta}^{t-h_1} x(s)dsd\theta$

$\mu_6(t) = \int_{t-h(t)}^{t-h_1}\int_{\theta}^{t-h_1} x(s)dsd\theta$

$\mu_7(t) = \int_{t-h_2}^{t-h(t)}\int_{\theta}^{t-h(t)} x(s)dsd\theta$

$\zeta(t) = \begin{bmatrix} \zeta_1^T(t) & \zeta_2^T(t) & \zeta_3^T(t) \end{bmatrix}^T$

$\zeta_0(t) = \begin{bmatrix} x^T(t) & x^T(t-\frac{1}{m}h_1) & \cdots & x^T(t-\frac{m-1}{m}h_1) \end{bmatrix}^T$

$\zeta_1(t) = \begin{bmatrix} \zeta_0^T(t) & x^T(t_1) & x^T(t-h(t)) & x^T(t_2) \end{bmatrix}^T$

$t_1 = t - h_1, t_2 = t - h_2$

$\zeta_2(t) = \begin{bmatrix} \frac{m}{h_1}\mu_1^T(t) & \frac{1}{\hat{h}_1}\mu_2^T(t) & \frac{1}{\hat{h}_2}\mu_3^T(t) & \frac{m^2}{h_1^2}\mu_4^T(t) \end{bmatrix}^T$

$\zeta_3(t) = \begin{bmatrix} \frac{1}{(\hat{h}_1)^2}\mu_6^T(t) & \frac{1}{(\hat{h}_2)^2}\mu_7^T(t) & \mu_2^T(t) & \mu_3^T(t) \end{bmatrix}^T$

$\hat{h}_1 = h(t) - h_1, \hat{h}_2 = h_2 - h(t)$

Calculating the derivative of $V(x_t)$ along the system (1) yields:

$$\dot{V}_1(x_t) = 2\eta_1^T(t)P\dot{\eta}_1(t), \tag{16}$$

$$\begin{aligned} \dot{V}_2(x_t) = &\eta_2^T(t,t)Q_1\eta_2(t,t) \\ &- \eta_2^T(t,t-h_1)Q_1\eta_2(t,t-h_1) \\ &+ 2\int_{t-h_1}^{t} \eta_2^T(t,s)Q_1\frac{\partial\eta_2(t,s)}{\partial t}ds \\ &+ \eta_3^T(t,t-h_1)Q_2\eta_3(t,t-h_1) \\ &- \eta_3^T(t,t-h_2)Q_2\eta_3(t,t-h_2) \\ &+ \eta_4^T(t)Q_3\eta_3(t) \\ &- \eta_4^T(t-\frac{h_1}{m})Q_3\eta_4(t-\frac{h_1}{m}) \\ &+ 2\int_{t-h_2}^{t-h_1} \eta_3^T(t,s)Q_2\frac{\partial\eta_3(t,s)}{\partial t}ds \\ = &\zeta^T(t)(h^2(t)\varpi_{21} + h(t)\varpi_{11} + \varpi_{01})\zeta(t), \end{aligned} \tag{17}$$

where

$$\begin{aligned} \varpi_{01} = &Sym\left\{\Pi_{41}^T Q_2\delta_2 + \Pi_{03}^T Q_1\Pi_{04}\right\} \\ &+ \Pi_{01}^T Q_1\Pi_{01} - \Pi_{02}^T Q_1\Pi_{02} + \Pi_{21}^T Q_2\Pi_{21} \\ &- \Pi_{22}^T Q_2\Pi_{22} + \Pi_{31}^T Q_3\Pi_{31} - \Pi_{32}^T Q_3\Pi_{32}, \end{aligned}$$

$$\varpi_{11} = Sym\left\{\Pi_{42}^T Q_2\delta_2\right\},$$

$$\varpi_{21} = Sym\left\{\Pi_{43}^T Q_2\delta_2\right\},$$

$$\begin{aligned} \dot{V}_3(x_t) = &\frac{h_1^2}{m^2}\dot{x}^T(t)R_1\dot{x}(t) + h_{12}^2\dot{x}^T(t)R_2\dot{x}(t) \\ &- \omega_1 - \omega_2, \end{aligned} \tag{18}$$

where

$$\omega_1 = \frac{h_1}{m}\int_{t-\frac{h_1}{m}}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds,$$

$$\begin{aligned} \omega_2 = &h_{12}\int_{t-h(t)}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)ds \\ &+ h_{12}\int_{t-h_2}^{t-h(t)} \dot{x}^T(s)R_2\dot{x}(s)ds. \end{aligned}$$

Based on Lemma 2.3, the following inequalities hold:

$$\omega_1 \geq \zeta^T(t)\Upsilon_1^T\bar{R}_1\Upsilon_1\zeta(t), \tag{19}$$

$$\omega_2 \geq \zeta^T(t)(\frac{1}{\beta}\Upsilon_2^T\bar{R}_2\Upsilon_2 + \frac{1}{1-\beta}\Upsilon_3^T\bar{R}_2\Upsilon_3)\zeta(t), \tag{20}$$

where
$\beta = \frac{h(t)-h_1}{h_{12}}$.
Then, based on Lemma 2.2, we have:

$$\omega_2 \geq \zeta^T(t)(\Upsilon_2^T\bar{S}_1\Upsilon_2 + 2\Upsilon_2^T\bar{S}_2\Upsilon_3 + \Upsilon_3^T\bar{S}_3\Upsilon_3)\zeta(t), \quad (21)$$

where
$\bar{S}_1 = \bar{R}_2 + (1-\beta)X_1 + (1-\beta)^2X_2 + \beta(1-\beta)X_3$,
$\bar{S}_2 = \beta Y_1 + (1-\beta)Y_2 + \beta^2Y_3 + (1-\beta)^2Y_4$,
$\bar{S}_3 = \bar{R}_2 + \beta^2Z_1 + \beta Z_2 + \beta(1-\beta)Z_4$.
According to (18)-(21), we can obtain:

$$\dot{V}_3(x_t) \leq \zeta^T(t)(h^2(t)\varpi_{22} + h(t)\varpi_{12} + \varpi_{02})\zeta(t), \quad (22)$$

where

$$\begin{aligned}\varpi_{02} =& Sym\left\{\Upsilon_2^T(\frac{h_1Y_1 - h_2Y_2}{h_{12}})\Upsilon_3\right\} - \Upsilon_1^T\bar{R}_1\Upsilon_1\\ &- \Upsilon_2^T(\bar{R}_2 + \frac{h_2}{h_{12}}X_1 + \frac{h_2^2}{h_{12}^2}X_2 - \frac{h_1h_2}{h_{12}^2}X_3)\Upsilon_2\\ &- \Upsilon_3^T(\bar{R}_2 - \frac{h_1^2}{h_{12}^2}Z_1 - \frac{h_1}{h_{12}}Z_2 - \frac{h_1h_2}{h_{12}^2}Z_4)\Upsilon_3\\ &+ e_0^T(\frac{h_1^2}{m^2}R_1 + h_{12}^2R_2)e_0,\end{aligned}$$

$$\begin{aligned}\varpi_{12} =& Sym\left\{\Upsilon_2^T(\frac{Y_2 - Y_1}{h_{12}} + \frac{2h_1Y_3 - 2h_2Y_4}{h_{12}^2})\Upsilon_3\right\}\\ &+ \Upsilon_2^T(\frac{X_1}{h_{12}} + \frac{2h_2}{h_{12}^2}X_2 - \frac{h_1 + h_2}{h_{12}^2}X_3)\Upsilon_2\\ &+ \Upsilon_3^T(\frac{2h_1}{h_{12}^2}Z_1 - \frac{Z_2}{h_{12}} - \frac{h_1 + h_2}{h_{12}^2}Z_4)\Upsilon_3,\end{aligned}$$

$$\begin{aligned}\varpi_{22} =& Sym\left\{-\Upsilon_2^T\frac{Y_3 - Y_4}{h_{12}^2}\Upsilon_3\right\} + \Upsilon_2^T(\frac{X_3 - X_2}{h_{12}^2})\Upsilon_2\\ &+ \Upsilon_3^T(\frac{Z_4 - Z_1}{h_{12}^2})\Upsilon_3.\end{aligned}$$

For any matrices $L_1, L_2 \in R^{(11+m)\times n}$, we have:

$$\begin{aligned}&2\zeta^T(t)\left[L_1((h(t) - h_1)e_{m+5} - e_{m+10})\right.\\ &\quad\left.+L_2((h_2 - h(t))e_{m+6} - e_{m+11})\right]\zeta(t) = 0.\end{aligned} \quad (23)$$

According to (16), (17), (22) and (23), we can obtain

$$\dot{V}(x_t) \leq \zeta^T(t)(h^2(t)\varpi_2 + h(t)\varpi_1 + \varpi_0)\zeta(t), \quad (24)$$

where ϖ_2, ϖ_1 and ϖ_0 are defined in Theorem 3.1. For $h(t) \in [h_1, h_2]$, based on Lemma 2.1, if (7)-(11) hold, then we have $h^2(t)\varpi_2 + h(t)\varpi_1 + \varpi_0 < 0$, $i.e.,\dot{V}(x_t) < 0$. Therefore, system (1) is asymptotically stable. This completes the proof.

Remark 3.2 To reduce the conservativeness, the delay interval $[0, h_1]$ is divided into m segments equally. The LKF includes the term $\int_{t-\frac{h_1}{m}}^{t}\eta_4^T(s)Q_3\eta_4(s)ds$, so the relationship among some state vectors $x^T(t), x^T(t - \frac{1}{m}h_1), \cdots, x^T(t - \frac{m-1}{m}h_1)$ and $x^T(t - h_1)$ are considered sufficiently, which may yield less conservative results.

IV. Numerical examples

In this section, a numerical example is given to demonstrate the advantages of the proposed criterion.

Example 4.1 Consider system (1) with

TABLE I
UPPER BOUND OF h_2 FOR DIFFERENT h_1

h_1	0.0	0.4	0.7	1.0
[11]	1.59	2.01	2.41	2.62
[12]	1.64	2.13	2.70	2.96
[16]	1.86	2.28	2.69	2.89
[13]	2.39	2.76	3.15	3.41
[19]	2.54	2.90	3.23	3.44
Theorem 3.1($m = 1$)	2.59	2.94	3.26	3.46
Theorem 3.1($m = 2$)	2.63	2.97	3.28	3.48
Theorem 3.1($m = 3$)	2.66	2.99	3.30	3.49

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}.$$

For different h_1, the upper bounds of h_2 calculated by Theorem 3.1 in this paper are listed in table 1, along with others reported in [11-13, 16, 19]. Comparing with recently existing results, it is obvious that the stability criterion presented in this paper is less conservative than those in [11-13, 16, 19]. For different m and $h_1 = 0.0$, the upper bounds of h_2 calculated by Theorem 3.1 in this paper are 2.59 ($m = 1$), 2.63 ($m = 2$) and 2.66 ($m = 3$). Therefore, the conservativeness of obtained results will be reduced with the increase of m.

V. Conclusion

In this paper, an improved reciprocally convex lemma was introduced, which yielded a less conservative stability criterion. It was observed that the generalized reciprocally convex lemma proposed in this paper included lemmas in [15, 17-19] as special cases. Finally, a numerical example was provided to show the effectiveness of the presented method.

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