Adjacent Vertex Reducible Total Labeling of Corona Graph

Li Wang, Jingwen Li, Chen Song, Lijing Zhang

Abstract—**For an undirected connected graph $G(p,q)$, where p is the number of vertices and q is the number of edges, there exists a mapping f: $V(G) \cup E(G) \to \{1,2,\cdots,p+q\}$, and the sum of the labels of any two adjacent vertices of the same degree in the graph is the same, being $S(u) = f(u) + \sum_{uw \in E(G)} f(uw)$, then f is called Adjacent Vertex Reducible Total Labeling (AVRTL) of the graph G. Based on the concept of Adjacent Vertex Reducible Total Labeling, an AVRTL algorithm is designed, which finds Adjacent Vertex Reducible Total Labeling of any simple connected graph within finite vertices in an iterative, circular fashion. The labeling rules of several corona graphs were discovered through the analysis and summary of the experiment results, and the relevant theorems were further summarized. Finally, a conjecture was proposed: if the graphs G and H are AVRTL graphs, then their corona graphs $G \circ H$ or $H \circ G$ are also AVRTL graphs.**

Index Terms—**corona graphs, reducible total labeling, algorithm, graph labeling**

I. Introduction

SINCE many complex problems in computer science can be converted to graph theoretical issues and then solved using graph theoretic algorithms, graph theory has a significant theoretical and practical research value. Rosa et al. introduced the concept of graph labeling in 1966 to resolve the Graceful Tree Conjecture [1]. The presentation of the conjecture established the foundation for the ongoing development of graph labeling, even if the Graceful Tree Conjecture is still under study. The study of graph labeling has been further divided into categories by academics during the following decades, including Elegant Labeling, Harmonious Labeling, Graceful Labeling and Magic Labeling. The literature [2] is where Vertex Magic Complete Labeling initially gained traction. Many scholars have studied special graphs and certain graphs that are more readily inscribed in order to produce a wide range of research findings on Vertex Magic Total Labeling. The well-known Total Chromatic Number Conjecture was independently formulated in 1965 by M. Behzad and V.G. Vizing [3]–[4], respectively. On the basis of Vertex Distinguishable Proper Edge Coloring, literature [5] proposed the concept and some conjectures of Adjacent Strong Edge Coloring of graphs. Both domestic and international specialists have conducted extensive research [6]–[8] on this topic. The concepts and corollaries connected to Reducible Coloring were thoroughly discussed in the literature [9] in 2013.

The deployment relationships in sensor networks can be represented visually by graphs with nodes and edges. The weights of the edges reflect the amount of information transported between the nodes, while the weights of the nodes represent the energy expended by the network's computers or servers. In special circumstances, transmission paths between adjacent nodes of comparable importance are required to maintain equal energy consumption, and as many different path types as possible could be maintained. This paper builds constraints by the definition of Adjacent Vertex Reducible Total Labeling. Combined with the above real-world issues, designs a novel heuristic algorithm that can solve the problem of Adjacent Vertex Reducible Total Labeling of special graphs and their joint graphs within finite points. The labeling properties of several kinds of crown graphs were discovered in accordance with the experimental results, several theorems were summarized, and proofs were provided.

II. Preliminary Knowledge

In this paper, $G(p,q)$ is a simple connected graph with p vertices and q edges. F_m is a fan graph with central node v_0 and contains m fan vertices. A wheel graph W_n is defined as a graph with $n+1$ vertices, and the central node u_0 is adjacent to the remaining n vertices.

Definition 1: If $G(V,E)$ is a simple undirected connected graph, and there exists a bijection f: $V(G) \cup E(G) \to \{1,2,\cdots,|V|+|E|\}$, for any two adjacent vertices $uv \in E(G)$, if $d(u) = d(v)$, we have $S(u) = S(v)$, where $S(u) = f(u) + \sum_{uw \in E(G)} f(uw)$, $d(u)$ denotes the degree of vertex u, then f is said to be the Adjacent Vertex Reducible Total Labeling of the graph G, or AVRTL for short, and graph G is an AVRTL graph. If a graph does not have an AVRTL, it is said to be a NAVRTL graph.

Definition 2: The graph $I_r(G)$ denotes the r-corona graph of graph G, which is the graph created by attaching r-suspended edges on each vertex of the graph G. The 1-corona graph, also known as the crown graph, is denoted as

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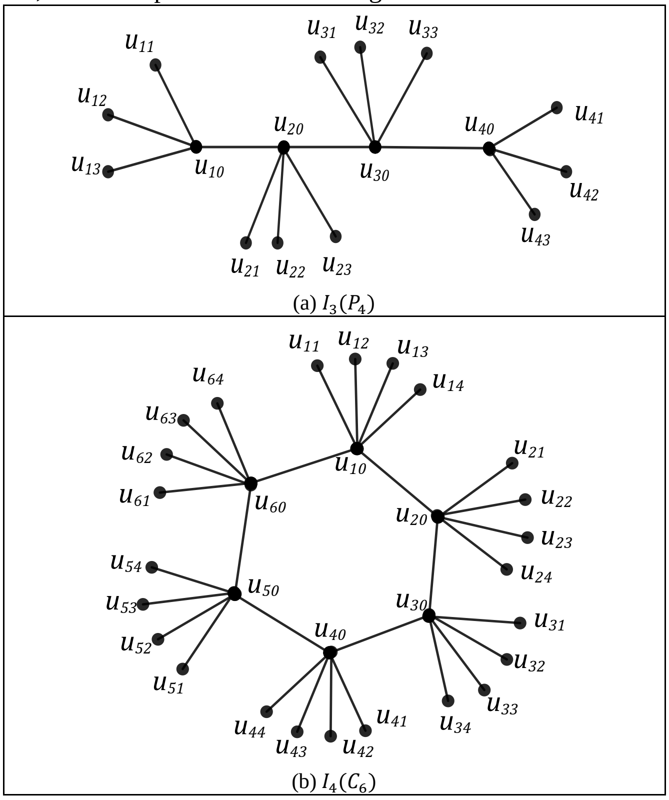
I(G). The set of endpoints of the r-hanging edges bonded at a vertex v of G is referred to as the r-hanging points, denoted as v*, two examples are shown in Figure 1.

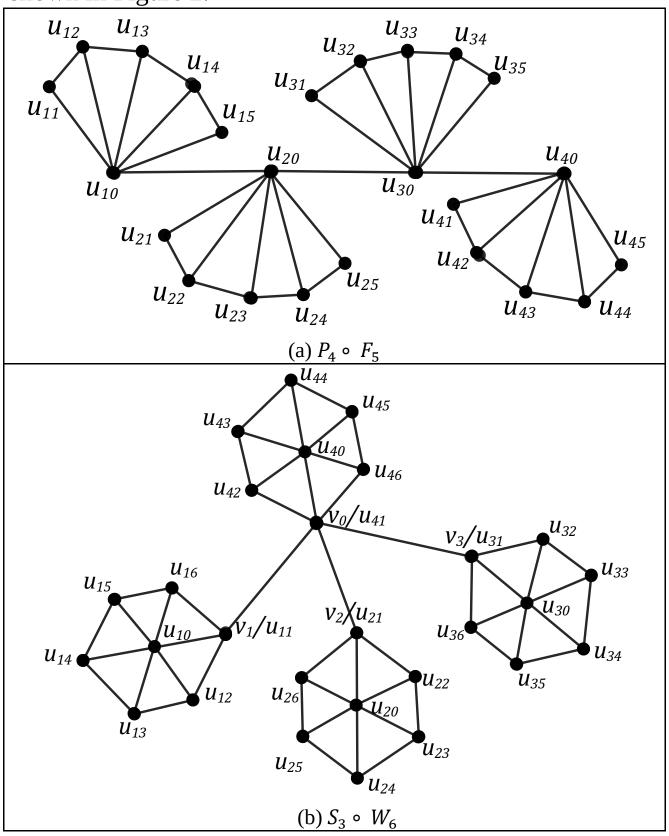


(a) $I_3(P_4)$

(b) $I_4(C_6)$

Fig 1. Examples of $I_r(G)$ graph

Definition 3: For any given graph G and H, duplicate H first based on the number of vertices in graph G, then suspend H for each vertex in graph G, where each common vertex is denoted by v_0 in H. The graph created in this way is known as the corona graph of G, noted as $G \circ H$. Two examples are shown in Figure 2.



(a) $P_4 \circ F_5$

(b) $S_3 \circ W_6$

Fig 2. Examples of $G \circ H$ graph

III. AVRTL ALGORITHM

A. The basic principle of the algorithm

The adjustment principle of the algorithm is built utilizing the definition of Adjacent Vertex Reducible Total Labeling. The adjustment function breaks the current equilibrium, and then the balance is gradually restored. In this way, the final labeling matrix of the graph $G(p,q)$ is found to satisfy the requirement that the sums of the labels of the neighboring vertices of the same degree are the same, and there exists a one-to-one mapping of the merged sets of point-edge labeling values to $\{1,2,\cdots,p+q\}$.

(1) Pretreatment function:

The graph set file is read, and the adjacency matrix InitAdjust of the graph G, as well as other information, are used to figure out the number of vertices, edges, initial labeling sequence, classification set divided according to the adjacency of two points, and other statistics.

(2) Adjustment function:

Step 1: Set up the adjustment principle, one of which is to choose the current maximum number of labels to adjust, and the other is to set the adjustment span of the label value to 1.

Step 2: The current adjustment matrix is modified, and the backward function determines whether to back off. Until the condition that the sum of the labels is the same is satisfied, the intermediate matrix $MidAdjust_i$ is recorded.

Step 3: Loop iteration until the label value reaches the maximum or $MidAdjust_i = MidAdjust_{i+1}$, record the final matrix $FinalMatrix$.

(3) Backward function:

Return False if the sum of the labels between neighboring vertices in the same degree differs by more than 2 or if the collection of labels is not continuous.

(4) Output function:

Output the adjacency matrix that finally satisfies the labeling requirements.

B. Pseudocode of the algorithm

Input	The adjacency matrix of the graph $G(p,q)$
Output	The matrix satisfying the labeling requirements
1	read the adjacency matrix $initAdjust$ of the graph G
2	$Calculate\ VertexNum, EdgeNum, maxLabel, SameList$ /* $VertexNum$ is the number of vertices, $EdgeNum$ is the number of edges, $maxLabel$ is the maximum value in the label set, and $SameList$ is the set of adjacent vertices of the same degree */
3	$get\ FinalMatrix, flag = true$ /*$FinalAdjust$ is the final matrix */
4	$while(flag)$
5	$for\ i \leftarrow 0\ to\ VertexNum$
6	$ev++$
7	if (The backward function returns true)
8	$ev--$
9	$end\ if$
10	$if(currentAdjust$ satisfies the equilibrium condition) /* $currentAdjust$ is the matrix being adjusted */
11	$MidAdjust_i \leftarrow currentAdjust$
12	$end\ if$

```
if (ei == VerNum + EdgeNum ||
equalFun(MidAdjust_i, MidAdjust_{i+1})
FinalAdjust ← MidAdjust
break
end if
end for
if (FinalAdjust.maxLabel == p + q)
Output FinalAdjust
end if
end while
end
```

C. Analysis of the algorithm operation results

According to the experimental results, the following clustered bar chart shows the change in the percentage of AVRTL graphs versus NAVRTL graphs in the total number of graphs within 6-10 vertices. It can be concluded that the percentage of AVRTL graphs in the total number of graphs tends to decrease as the number of vertices increases but still accounts for the majority compared to NAVRTL graphs.

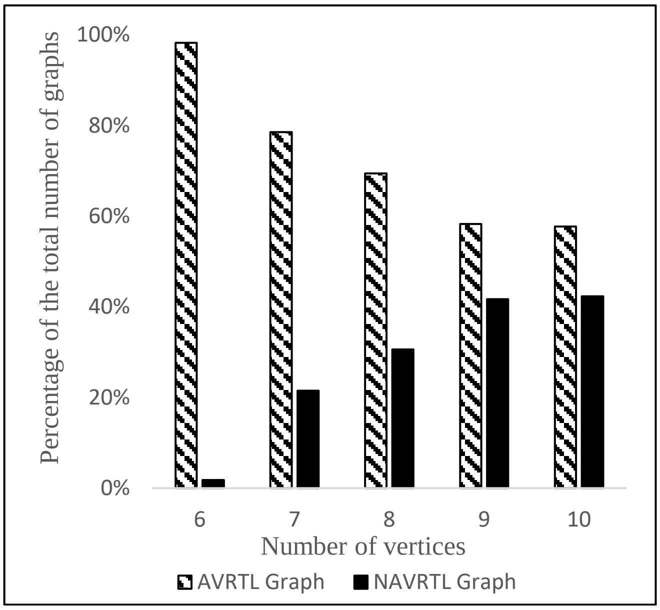
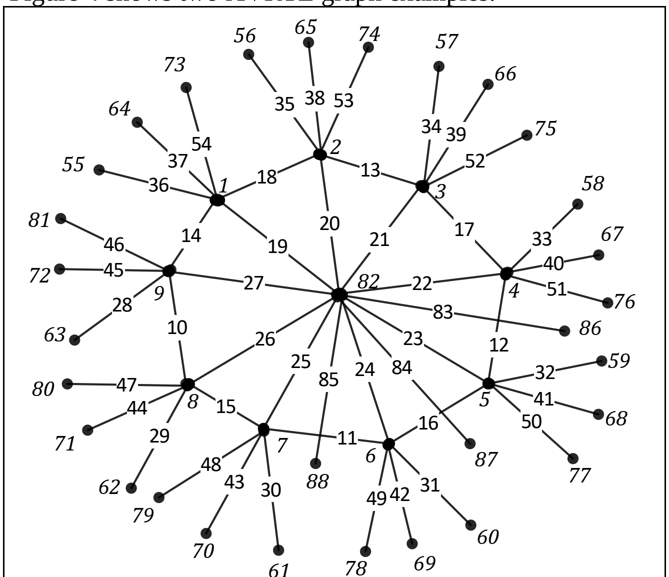
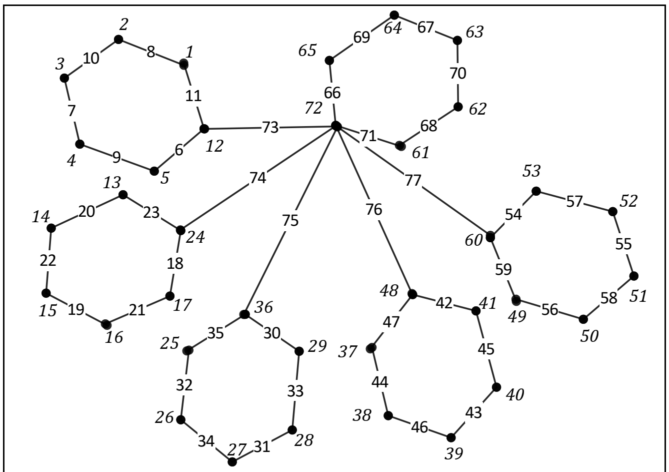


Fig 3. Variation of the percentage of AVRTL and NAVRTL in the total number of graphs within finite vertices

Figure 4 shows two AVRTL graph examples.



(a) $G(40,48)$



(b) $G(36,41)$

Fig 4. Labeling results of two random graphs

IV. THEOREM AND PROOF

Theorem 1: AVRTL exists for the road graph P_n when $n \geq 3$.

Proof: Let the set of vertices of P_n be $\{u_1, u_2, \cdots, u_n\}$, and the vertices u_1 and u_n lie at the two ends of P_n. P_n has a total of n vertices and $n-1$ edges, as shown in Figure 5(a).

At this point, any two elements of the vertex set $\{u_2, u_3, \cdots, u_{n-1}\}$ are adjacent and of the same degree, both of degree 2 vertices, and a mapping about f can be obtained as follows.

$$f(u_i) = \begin{cases} i, 1 \le i \le n-1 \\ 2n-1, i = n \end{cases};$$

$$f(u_i u_{i+1}) = \begin{cases} 2n - \dfrac{i}{2} - \dfrac{3}{2}, i \equiv 1 (\mathrm{mod}\, 2) \\ 2n - \left\lfloor \dfrac{n}{2} \right\rfloor - \dfrac{i}{2} - 1, i \equiv 0 (\mathrm{mod}\, 2) \end{cases} \quad 1 \le i \le n-1$$

At this time, $f(V) = \{1, 2, \cdots, n-1\} \cup \{2n-1\}$ and $f(E) = \{n, n+1, \cdots, 2n-2\}$, which gives $(V(G)) \cup f(E(G)) \to [1, 2n-1]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_i u_{i-1}) = 4n - \lfloor \frac{n}{2} \rfloor - 2, 2 \le i \le n-1$, which gives the same sum of labels of all elements in the set $\{u_2, u_3, \cdots, u_{n-1}\}$. The vertices u_1 and u_n have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

To sum up, Theorem 1 holds.

Theorem 2: AVRTL exists for the circle graph $C_n (n \geq 3)$ when $n \equiv 1 (mod\ 2)$.

Proof: Let the set of vertices of C_n be $\{u_1, u_2, \cdots, u_n\}$, with vertices u_1 adjacent to both u_n and u_2. C_n has a total of n vertices and n edges, as shown in Figure 5(b).

At this point, all vertices in the graph are adjacent and of the same degree, all of degree 2 vertices, and a mapping about f can be obtained as follows.

$$f(u_i) = i, 1 \le i \le n;$$

$$f(u_1 u_n) = 2n - \left\lfloor \frac{n}{2} \right\rfloor;$$

$$f(u_i u_{i+1}) = \begin{cases} 2n - \dfrac{i}{2} + \dfrac{1}{2}, i \equiv 1 (\mathrm{mod}\, 2) \\ 2n - \left\lfloor \dfrac{n}{2} \right\rfloor - \dfrac{i}{2}, i \equiv 0 (\mathrm{mod}\, 2) \end{cases} \quad 1 \le i \le n-1$$

At this time, $f(V)=\{1,2,\cdots,n\}$ and $f(E)=\{n+1,n+2,\cdots,2n\}$, which gives $f(V(G))\cup f(E(G))\rightarrow[1,2n]$ and $f(V(G))\cap f(E(G))=\emptyset$.

And $S(u_1)=f(u_1)+f(u_1u_2)+f(u_1u_n)=4n+1-\lfloor\frac{n}{2}\rfloor$; $S(u_n)=f(u_n)+f(u_1u_n)+f(u_{n-1}u_n)=4n+1-\lfloor\frac{n}{2}\rfloor$; $S(u_i)=f(u_i)+f(u_iu_{i+1})+f(u_iu_{i-1})=4n+1-\left\lfloor\frac{n}{2}\right\rfloor, 2\le i\le n-1$, which give the same sum of labels of all elements in the set $\{u_1,u_2,\cdots,u_n\}$.

To sum up, Theorem 2 holds.

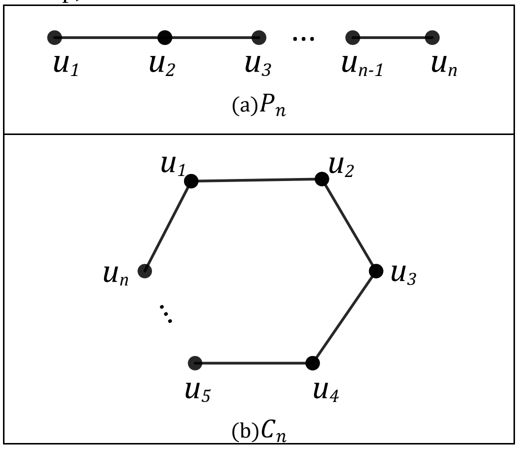


Fig 5. P_n and C_n

Theorem 3: If the graph S_m represents a star graph with $m+1(m\ge 2)$ vertices, then AVRTL exists for S_m.

Theorem 3 obviously holds according to the definition of Adjacent Vertex Reducible Total Labeling.

Theorem 4: If the graph F_n represents a fan graph with $n+1(n>3)$ vertices, then AVRTL exists for F_n.

Proof: Let the set of vertices of F_n be $\{u_0,u_1,\cdots,u_n\}$, and the center of the fan is u_0. F_n has a total of $n+1$ vertices and $2n-1$ edges, as shown in Figure 6.

At this point, any two elements of the vertex set $\{u_2,u_3,\cdots,u_{n-1}\}$ are adjacent and of the same degree, both of degree 3 vertices, and a mapping about f can be obtained as follows.

$$f(u_i)=\begin{cases}3n, i=0\\ 3n-1, i=1\\ i-1, 2\le i\le n-1\\ 3n-2, i=n\end{cases};$$

$$f(u_0u_i)=\begin{cases}3n-3, i=1\\ n+i-3, 2\le i\le n-1;\\ 3n-4, i=n\end{cases}$$

$$f(u_iu_{i+1})=3n-i-4, 1\le i\le n-1$$

At this time, $f(V)=\{1,2,\cdots,n-2\}\cup\{3n-2,3n-1,3n\}$ and $f(E)=\{n-1,n,\cdots,3n-3\}$, which gives $f(V(G))\cup f(E(G))\rightarrow[1,3n]$ and $f(V(G))\cap f(E(G))=\emptyset$.

$S(u_i)=f(u_iu_{i+1})+f(u_iu_{i-1})+f(u_iu_0)+f(u_i)=7n-11,\ 2\le i\le n-1$, which gives the same sum of labels of all elements in the set $\{u_2,u_3,\cdots,u_{n-1}\}$. The vertices u_0,u_1 and u_n have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

To sum up, Theorem 4 holds.

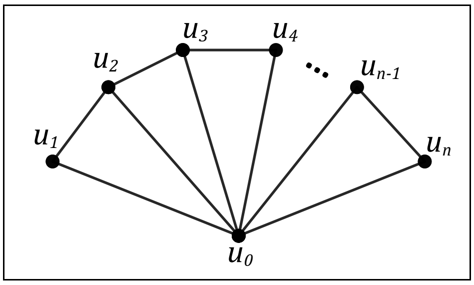


Fig 6. F_n

Theorem 5: If the graph W_n represents a wheel graph with $n+1(n>2)$ vertices, then AVRTL exists for W_n.

Proof: Let the set of vertices of W_n be $\{u_0,u_1,\cdots,u_n\}$, and the center of the wheel is u_0. The vertex u_1 is adjacent to both u_2 and u_n, W_n has a total of $n+1$ vertices and $2n$ edges, as shown in Figure 7(a).

Scenario 1: When $n=3$

At this point, all vertices in the graph are adjacent vertices of the same degree, as shown in Figure 7(b).

At this time, $f(V)=\{1,2,4,7\}$ and $f(E)=\{3,5,6,8,9,10\}$, which gives $f(V(G))\cup f(E(G))\rightarrow[1,3n+1]$ and $f(V(G))\cap f(E(G))=\emptyset$. And the sum of the labels of all adjacent vertices of the same degree is 24.

Therefore, AVRTL exists for $W_n(n=3)$.

Scenario 2: When $n\ge 4$

At this point, any two elements of the vertex set $\{u_1,u_2,\cdots,u_n\}$ are adjacent and of the same degree, both of degree 3 vertices, and a mapping about f can be obtained as follows.

$$f(u_i)=\begin{cases}3n+1, i=0\\ i, 1\le i\le n\end{cases};$$

$$f(u_0u_i)=\begin{cases}2n+i+1, 1\le i\le n-1\\ 2n+1, i=n\end{cases};$$

$$f(u_1u_n)=2n;$$

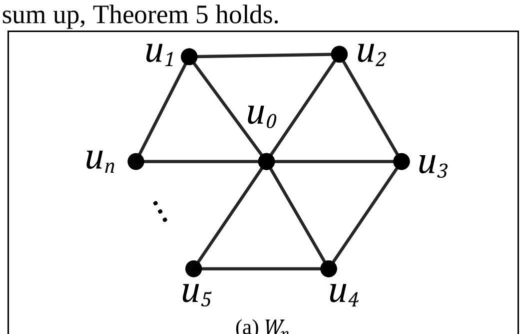
$$f(u_iu_{i+1})=2n-i, 1\le i\le n-1$$

At this time, $f(V)=\{1,2,\cdots,n\}\cup\{3n+1\}$ and $f(E)=\{n+1,n+2,\cdots,3n\}$, which gives $f(V(G))\cup f(E(G))\rightarrow[1,3n+1]$ and $f(V(G))\cap f(E(G))=\emptyset$.

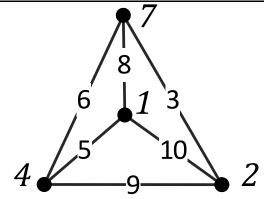
And $S(u_1)=f(u_1)+f(u_1u_2)+f(u_1u_n)+f(u_1u_0)=6n+2$; $S(u_n)=f(u_1u_n)+f(u_{n-1}u_n)+f(u_nu_0)+f(u_n)=6n+2$; $S(u_i)=f(u_iu_{i+1})+f(u_iu_0)+f(u_i)+f(u_iu_{i-1})=6n+2, 2\le i\le n-1$, which give the same sum of labels of all elements in the set $\{u_1,u_2,\cdots,u_n\}$. The vertex u_0 has no adjacent vertices of the same degree, so it is not necessary to consider the sum of its labels.

Therefore, AVRTL exists for $W_n(n\ge 4)$.

To sum up, Theorem 5 holds.



(a) W_n



(b) a labeling result of W_3

Fig 7. W_n and a labeling result of W_3

Theorem 6: AVRTL exists for the graph $I_r(F_n)$ when $n \geq 3, r \geq 2$.

Proof: Let the set of vertices of $I_r(F_n)$ be $\{u_{ij} | 1 \leq i \leq n+1, 0 \leq j \leq r\}$, and u_{i0}($1 \leq i \leq n$) is the common vertex of the graph F_n and r hanging edges. $I_r(F_n)$ has $nr + r + n + 1$ vertices and $nr + 2n + r - 1$ edges, as shown in Figure 8.

Scenario 1: When $n = 3, r \geq 2$

At this point, the vertices u_{20} and v_0 are adjacent and of the same degree, both being vertices of degree $r + 3$, and the labeling situation can be divided into two kinds.

(1) When $n = 3, r \geq 2, r \equiv 1(mod\ 2)$, a mapping about f can be obtained as follows.

$$f(u_{ij}) = \begin{cases} \begin{cases} 8, i=1 \\ 2, i=2 \\ 9, i=3 \\ 5, i=4 \end{cases} j=0 \\ (2+j)n+i+j-1, 1 \leq i \leq n+1, 1 \leq j \leq r \end{cases};$$

$$f(v_0 u_{i0}) = \begin{cases} 3, i=1 \\ 1, i=2 \\ 6, i=3 \end{cases};$$

$$f(u_{i0} u_{(i+1)0}) = \begin{cases} 4, i=1 \\ 7, i=2 \end{cases};$$

$$f(u_{i0} u_{ij}) = \begin{cases} \begin{cases} (3+r+j)n+r+i+j, j \equiv 1(\bmod 2) \\ (4+r+j)n+r-i+j+2, j \equiv 0(\bmod 2) \end{cases} 1 \leq j \leq r-1 \\ \begin{cases} (3+r)n+r+4, i=1 \\ (3+r)n+r+i, i=2,3 \\ (3+r)n+r+1, i=4 \end{cases} j=r \end{cases} 1 \leq i \leq 4$$

At this time, $f(V) = \{2,5,8,9\} \cup \{10,11, \cdots, 3n + nr + r\}$ and $f(E) = \{1,3,4,6,7\} \cup \{3n + nr + r + 1, 3n + nr + r + 2, \cdots, 2nr + 3n + 2r\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 2nr + 3n + 2r]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_{20}) = \sum_{j=1}^{r} f(u_{20} u_{2j}) + f(u_{10} u_{20}) + f(u_{20}) + f(u_{20} u_{30}) = \frac{3}{2} r^2 (n+1) + (3n + \frac{1}{2}) r - \frac{n}{2} + 15$; $S(u_{40}) = f(u_{40}) + f(u_{40} u_{10}) + f(u_{40} u_{30}) + \sum_{j=1}^{r} f(u_{40} u_{4j}) = \frac{3}{2} r^2 (n+1) + (3n + \frac{1}{2}) r - \frac{n}{2} + 15$, we can get the sum of the labels of vertices u_{20} and v_0 is the same. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $I_r(F_n)(n = 3, r \geq 2, r \equiv 1(mod\ 2))$.

(2) When $n = 3, r \geq 2, r \equiv 0(mod\ 2)$, a mapping about f can be obtained as follows.

$$f(u_{ij}) = \begin{cases} \begin{cases} 8, i=1 \\ 2, i=2 \\ 9, i=3 \\ 1, i=4 \end{cases} j=0 \\ (2+j)n+i+j-1, 1 \leq i \leq n+1, 1 \leq j \leq r \end{cases};$$

$$f(v_0 u_{i0}) = \begin{cases} 5, i=1 \\ 4, i=2 \\ 6, i=3 \end{cases};$$

$$f(u_{i0} u_{(i+1)0}) = \begin{cases} 3, i=1 \\ 7, i=2 \end{cases};$$

$$f(u_{i0} u_{ij}) = \begin{cases} (2+r+j)n+r+i+j-1, j \equiv 1(\bmod 2) \\ (3+r+j)n+r-i+j+1, j \equiv 0(\bmod 2) \end{cases} 1 \leq i \leq n+1, 1 \leq j \leq r$$

At this time, $f(V) = \{1,2,8,9\} \cup \{10,11, \cdots, 3n + nr + r\}$ and $f(E) = \{3,4,5,6,7\} \cup \{3n + nr + r + 1, 3n + nr + r + 2, \cdots, 2nr + 3n + 2r\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 2nr + 3n + 2r]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_{20}) = \sum_{j=1}^{r} f(u_{20} u_{2j}) + f(u_{10} u_{20}) + f(u_{20}) + f(u_{20} u_{30}) = \frac{3}{2} r^2 (n+1) + \left(3n + \frac{1}{2}\right) r + 16$; $S(u_{40}) = f(u_{40}) + f(u_{40} u_{10}) + f(u_{40} u_{30}) + \sum_{j=1}^{r} f(u_{40} u_{4j}) = \frac{3}{2} r^2 (n+1) + (3n + \frac{1}{2}) r + 16$, we can get the sum of the labels of vertices u_{20} and v_0 is the same. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $I_r(F_n)(n = 3, r \geq 2, r \equiv 0(mod\ 2))$.

Scenario 2: When $n \geq 4, r \geq 2$

At this point, any two elements of the vertex set $\{u_{20}, u_{30}, \cdots, u_{(n-1)0}\}$ are adjacent and of the same degree, both of degree r+3 vertices, and the labeling situation can be divided into two kinds.

(1) When $n \geq 4, r \geq 2, r \equiv 1(mod\ 2)$, a mapping about f can be obtained as follows.

$$f(u_{ij}) = \begin{cases} \begin{cases} i, 1 \leq i \leq n-1 \\ 2n-1, i=n \\ 2n, i=n+1 \end{cases} j=0 \\ (1+j)n+i+j-1, 1 \leq i \leq n+1, 1 \leq j \leq r \end{cases};$$

$$f(v_0 u_{i0}) = (2+r)n+r+i, 1 \leq i \leq n;$$

$$f(u_{i0} u_{(i+1)0}) = \begin{cases} 2n - \frac{i}{2} - \frac{3}{2}, i \equiv 1(\bmod 2) \\ 2n - \left\lfloor \frac{n}{2} \right\rfloor - \frac{i}{2} - 1, i \equiv 0(\bmod 2) \end{cases} 1 \leq i \leq n-1;$$

$$f(u_{i0} u_{ij}) =$$
$$\begin{cases} \begin{cases} (3+r+j)n+r+i+j, j \equiv 1(\bmod 2) \\ (4+r+j)n+r-i+j+2, j \equiv 0(\bmod 2) \end{cases} 1 \leq j \leq r-1 \\ (4+r)n+r-i+2, j=r \end{cases} 1 \leq i \leq n+1$$

At this time, $f(V) = \{1,2, \cdots, n-1\} \cup \{2n-1, 2n, \cdots, 2n + nr + r\}$ and $f(E) = \{n, n+1, \cdots, 2n-2\} \cup \{2n + nr + r + 1, 2n + nr + r + 2, \cdots, 2nr + 3n + 2r\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 2nr + 3n + 2r]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_{i0}) = f(u_{i0} u_{(i+1)0}) + f(u_{i0}) + f(u_{i0} u_{(i-1)0}) + \sum_{j=1}^{r} f(u_{i0} u_{ij}) + f(u_{i0} v_0) = \frac{3}{2} r^2 (n+1) + \frac{13}{2} n - \left\lfloor \frac{n}{2} \right\rfloor + (4n + \frac{3}{2}) r - 1, 2 \leq i \leq n-1$, which gives the same sum of labels of all elements in the set $\{u_{20}, u_{30}, \cdots, u_{(n-1)0}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $I_r(F_n)(n \geq 4, r \geq 2, r \equiv 1(mod\ 2))$.

(2) When $n \geq 4, r \geq 2, r \equiv 0(mod\ 2)$, a mapping about f can be obtained as follows.

$$f(u_{ij}) = \begin{cases} \begin{cases} 3n-1, i=1 \\ i-1, 2 \leq i \leq n-1 \\ 3n-2, i=n \\ 3n, i=n+1 \end{cases} j=0 \\ (2+r+j)n+r+i+j-1, 1 \leq i \leq n+1, 1 \leq j \leq r \end{cases};$$

$$f(v_0 u_{i0}) = \begin{cases} 3n-3, i=1 \\ n+i-3, 2 \leq i \leq n-1; \\ 3n-4, i=n \end{cases}$$

$$f(u_{i0}u_{(i+1)0}) = 3n-i-4, 1 \leq i \leq n-1;$$

$$f(u_{i0}u_{ij}) = \begin{cases} (2+j)n+i+j-1, j \equiv 1(\bmod 2) \\ (3+j)n-i+j+1, j \equiv 0(\bmod 2) \end{cases} 1 \leq i \leq n+1, 1 \leq j \leq r$$

At this time, $f(V) = \{1,2,\cdots,n-2\} \cup \{3n+nr+r+1, 3n+nr+r+2, \cdots, 2nr+3n+2r\} \cup \{3n-2, 3n-1, 3n\}$ and $f(E) = \{n-1, n, \cdots, 3n-3\} \cup \{3n+1, 3n+2, \cdots, 3n+nr+r\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 2nr+3n+2r]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_{i0}) = f(u_{i0}) + f(u_{i0}u_{(i+1)0}) + f(u_{i0}u_{(i-1)0}) + \sum_1^r f(u_{i0}u_{ij}) = \frac{n+1}{2}r^2 + (3n+\frac{1}{2})r + 7n - 11$, $2 \leq i \leq n-1$, which gives the same sum of labels of all elements in the set $\{u_{20}, u_{30}, \cdots, u_{(n-1)0}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $I_r(F_n)(n \geq 4, r \geq 2, r \equiv 0(mod\ 2))$.

To sum up, Theorem 6 holds.

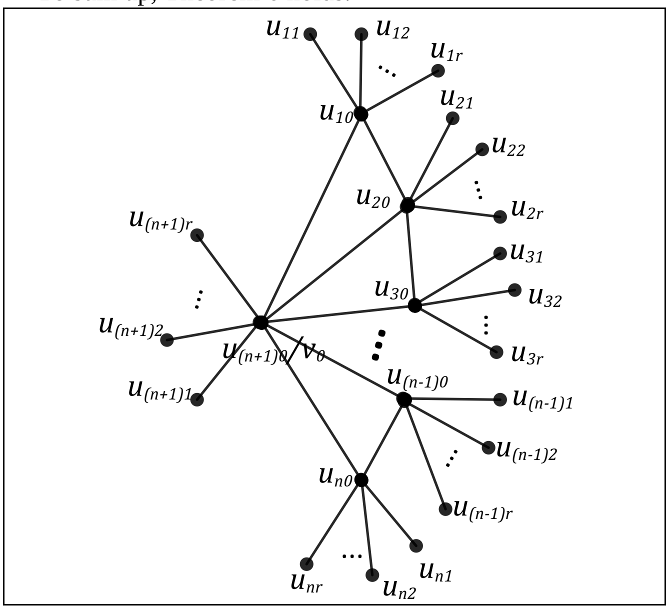


Fig 8. $I_r(F_n)$

Theorem 7: AVRTL exists for corona graph $S_m \circ C_n$ when $m \geq 2, n \geq 3$.

Proof: Let the set of vertices of $S_m \circ C_n$ be $\{u_{ij} | 1 \leq i \leq m+1, 1 \leq j \leq n-1\} \cup \{v_0, v_1, \cdots, v_m\}$, and $v_i (0 \leq i \leq m)$ is the common vertex of C_n, S_m. $S_m \circ C_n$ has a total of $mn+n$ vertices and $mn+n+m$ edges, as shown in Figure 9.

At this point, in the vertex sets $\{u_{11}, u_{12}, \cdots, u_{1(n-1)}\}$, $\{u_{21}, u_{22}, \cdots, u_{2(n-1)}\}, \cdots, \{u_{(m+1)1}, u_{(m+1)2}, \cdots, u_{(m+1)(n-1)}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 2, and a mapping about f can be obtained as follows.

$$f(u_{ri}) = \begin{cases} 2nr, i=0 \\ 2nr-2n+i, 1 \leq i \leq n-1 \end{cases} 1 \leq r \leq m+1;$$

$$f(v_0 v_i) = 2mn+2n+i, 1 \leq i \leq m;$$

$$f(u_{r0}u_{r(n-1)}) = \begin{cases} 2nr-n, i \equiv 0(\bmod 2) \\ 2nr-n+\left\lceil \frac{n}{2} \right\rceil -1, i \equiv 1(\bmod 2) \end{cases} 1 \leq r \leq m+1;$$

$$f(u_{ri}u_{r(i-1)}) = \begin{cases} 2nr-\frac{i}{2}-\frac{1}{2}, i \equiv 1(\bmod 2) \\ 2nr-\left\lceil \frac{n}{2} \right\rceil -\frac{i}{2}, i \equiv 0(\bmod 2) \end{cases} 1 \leq i < n, 1 \leq r \leq m+1$$

At this time, $f(V) = \{1,2,\cdots,n-1\} \cup \{2n+1, 2n+2, \cdots, 3n-1\} \cup \cdots \cup \{2mn+1, 2mn+2, \cdots, 2mn+n-1\} \cup \{2n, 4n, \cdots, 2mn+2n\}$ and $f(E) = \{n, n+1, \cdots, 2n-1\} \cup \{3n, 3n+1, \cdots, 4n-1\} \cup \cdots \cup \{2mn+n, 2mn+n+1, \cdots, 2mn+2n-1\} \cup \{2mn+2n+1, 2mn+2n+2, \cdots, 2mn+2n+m\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 2mn+2n+m]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And we have $S(u_{rn}) = f(u_{r(n-1)}u_{r(n-2)}) + f(u_{rn}) + f(u_{rn}u_{r(n-1)}) = 6nr - 2n - \left\lceil \frac{n}{2} \right\rceil - 1, 1 \leq r \leq m+1$; $S(u_{ri}) = f(u_{ri}u_{r(i+1)}) + f(u_{ri}u_{r(i-1)}) + f(u_{ri}) = 6nr - 2n - \left\lceil \frac{n}{2} \right\rceil - 1, 1 \leq i \leq n-1\ and\ 1 \leq r \leq m+1$, we can get that in the vertex sets $\{u_{11}, u_{12}, \cdots, u_{1(n-1)}\}, \{u_{21}, u_{22}, \cdots, u_{2(n-1)}\}, \cdots, \{u_{m1}, u_{m2}, \cdots, u_{m(n-1)}\}$, the sum of the labels of all elements in each set is the same. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

To sum up, Theorem 7 holds.

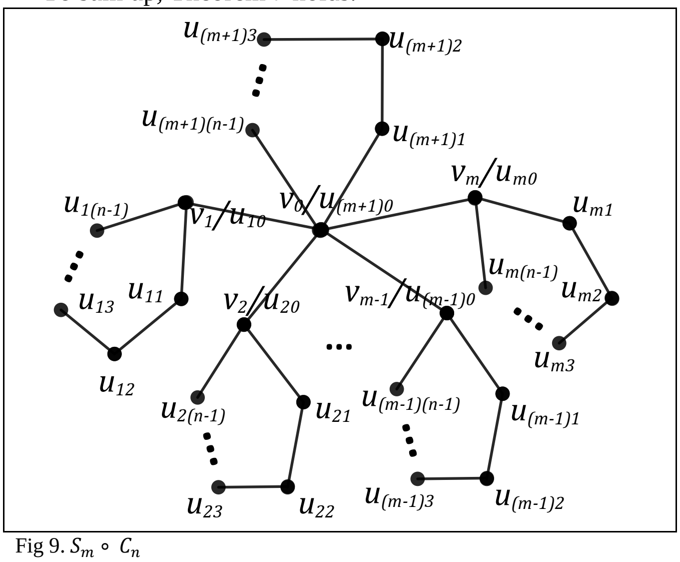


Fig 9. $S_m \circ C_n$

Theorem 8: AVRTL exists for corona graph $C_n \circ P_m$ when $n \geq 3, m \geq 2$.

Proof: Let the set of vertices of $C_n \circ P_m$ be $\{u_{ij} | 1 \leq i \leq n, 1 \leq j \leq m\}$, and $u_{i1}(1 \leq i \leq n)$ is the common vertex of C_n and P_m. $C_n \circ P_m$ has a total of mn vertices and mn edges, as shown in Figure 10.

Scenario 1: When $n \geq 3, 2 \leq m \leq 3$

At this point, any two elements of the vertex set $\{u_{11}, u_{21}, \cdots, u_{n1}\}$ are adjacent and of the same degree, both of degree 3 vertices, and a mapping about f can be obtained as follows.

$$f(u_{ij}) = \begin{cases} \begin{cases} n, i=1 \\ i-1, 2 \leq i \leq n \end{cases} j=1 \\ (m+j-1)n+i, 2 \leq j \leq m, 1 \leq i \leq n \end{cases};$$

$$f(u_{11}u_{n1}) = n+1;$$

$$f(u_{i1}u_{(i+1)1}) = 2n-i+1, 1 \leq i \leq n-1;$$

$$f(u_{ij}u_{i(j+1)}) = (j+1)n+i, 1 \leq i \leq n, 1 \leq j \leq m-1$$

At this time, $f(V) = \{1,2,\cdots,n\} \cup \{mn+n+1, mn+n+2, \cdots, 2mn\}$ and $f(E) = \{n+1, n+2, \cdots, mn+n\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 2mn]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_{11}) = f(u_{11}u_{21}) + f(u_{11}u_{n1}) + f(u_{11}u_{12}) + f(u_{11}) = 6n+2$; $S(u_{i1}) = f(u_{i1}u_{(i+1)1}) + f(u_{i1}u_{i2}) + f(u_{i1}u_{(i-1)1}) + f(u_{i1}) = 6n+2, 2 \leq i \leq n-1$; $S(u_{n1}) = f(u_{n1}) + f(u_{n1}u_{11}) + f(u_{n1}u_{n2}) + f(u_{n1}u_{(n-1)1}) = 6n+2$, which give the same sum of labels of all elements in the set $\{u_{11}, u_{21}, \cdots, u_{n1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $C_n \circ P_m (n \geq 3, 2 \leq m \leq 3)$.

Scenario 2: When $n \geq 3, m \geq 4$

At this point, in the vertex sets $\{u_{12}, u_{13}, \cdots, u_{1(m-1)}\}, \{u_{22}, u_{23}, \cdots, u_{2(m-1)}\}, \cdots, \{u_{n2}, u_{n3}, \cdots, u_{n(m-1)}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 2; any two elements of the vertex set $\{u_{11}, u_{21}, \cdots, u_{n1}\}$ are adjacent and of the same degree, both of degree 3 vertices, and a mapping about f can be obtained as follows.

$$f(u_{ij}) = \begin{cases} \begin{cases} n, i=1 \\ i-1, 2 \leq i \leq n \end{cases} j=1 \\ \begin{cases} 2mn-2n+i, j=2 \\ jn+i, 3 \leq j \leq m-1 \quad 1 \leq i \leq n \\ 2mn-n+i, j=m \end{cases} \end{cases};$$

$$f(u_{11}u_{n1}) = n+1;$$

$$f(u_{i1}u_{(i+1)1}) = 2n-i+1, 1 \leq i \leq n-1;$$

$$f(u_{ij}u_{i(j+1)}) = \begin{cases} 2n+i, j=1 \\ \begin{cases} (2m-2-\left\lfloor \frac{j}{2} \right\rfloor)n+i, j \equiv 1 (\mathrm{mod}\, 2) \\ (2m-\frac{j}{2}-1-\left\lfloor \frac{m}{2} \right\rfloor)n+i, j \equiv 0 (\mathrm{mod}\, 2) \end{cases} 1 < j \leq m \end{cases} 1 \leq i \leq n$$

At this time, $f(V) = \{1,2,\cdots,n\} \cup \{3n+1, 3n+2, \cdots, mn\} \cup \{2mn-2n+1, 2mn-2n+2, \cdots, 2mn\}$ and $f(E) = \{n+1, n+2, \cdots, 3n\} \cup \{mn+1, mn+2, \cdots, 2mn-2n\}$, which gives $f(V(G)) \cup f(E(G)) \to [1, 2mn]$ and $f(V(G)) \cap f(E(G)) = \emptyset$.

And $S(u_{ij}) = f(u_{ij}u_{i(j-1)}) + f(u_{ij}u_{i(j+1)}) + f(u_{ij}) = 4mn - 2n - \left\lfloor \frac{m}{2} \right\rfloor n + 3i, 1 \leq i \leq n, 2 \leq j \leq m-1$, we can get that in the vertex sets $\{u_{12}, u_{13}, \cdots, u_{1(m-1)}\}, \{u_{22}, u_{23}, \cdots, u_{2(m-1)}\}, \cdots, \{u_{n2}, u_{n3}, \cdots, u_{n(m-1)}\}$, the sum of the labels of all elements in each set is the same. $S(u_{i1}) = f(u_{i1}) + f(u_{i1}u_{(i+1)1}) + f(u_{i1}u_{(i-1)1}) + f(u_{i1}u_{i2}) = 6n+2, 2 \leq i \leq n-1$; $S(u_{11}) = f(u_{11}u_{12}) + f(u_{11}u_{21}) + f(u_{11}u_{n1}) + f(u_{11}) = 6n+2$; $S(u_{n1}) = f(u_{n1}u_{(n-1)1}) + f(u_{n1}u_{11}) + f(u_{n1}u_{n2}) + f(u_{n1}) = 6n+2$, which give the same sum of labels of all elements in the set $\{u_{11}, u_{21}, \cdots, u_{n1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $C_n \circ P_m (n \geq 3, m \geq 4)$.

To sum up, Theorem 8 holds.

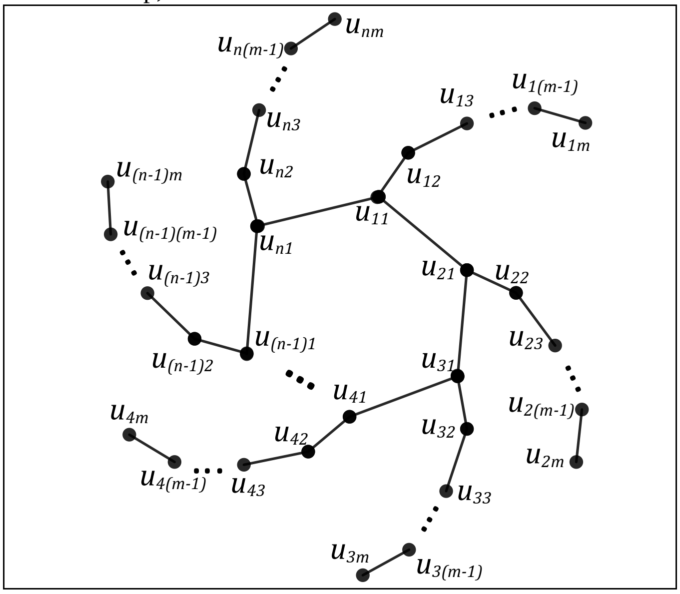


Fig 10. $C_n \circ P_m$

Theorem 9: AVRTL exists for corona graph $W_n \circ C_m$ when $n \geq 4, m \geq 3, m \equiv 1 (mod\ 2)$.

Proof: Let the set of vertices of $W_n \circ C_m$ be $\{u_{ij} | 1 \leq i \leq n+1, 1 \leq j \leq m\}$, and $v_i (0 \leq i \leq n)$ is the common vertex of W_n and C_m. $W_n \circ C_m$ has a total of $mn+m$ vertices and $mn+m+2n$ edges, as shown in Figure 11.

At this point, in the vertex sets $\{u_{12}, u_{13}, \cdots, u_{1m}\}, \{u_{22}, u_{23}, \cdots, u_{2m}\}, \cdots, \{u_{(n+1)2}, u_{(n+1)3}, \cdots, u_{(n+1)m}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 2; any two elements of the vertex set $\{u_{11}, u_{21}, \cdots, u_{n1}\}$ are adjacent and of the same degree, both of degree 5 vertices, and a mapping about f can be obtained as follows.

$$f(u_{ij}u_{i(j+1)}) = \begin{cases} 4n+i+2, j=1 \\ \begin{cases} (m+3-j)n+m+i-j+1, j \equiv 1 (\mathrm{mod}\, 2) \\ (2m+3-j)n+2m+i-j+1, j \equiv 0 (\mathrm{mod}\, 2) \end{cases} 1 < j \leq m \end{cases} 1 \leq i \leq n+1;$$

$$f(u_{ij})=\begin{cases}(2+m)n+m+i,\ j=2\\ 2(n+1)j+i-2, 3\le j\le m\end{cases};$$

$$f(v_i)=\begin{cases}3n+1, i=0\\ n, i=1\\ i-1, 2\le i\le n\end{cases};$$

$$f(v_1v_n)=n+1;$$

$$f(v_iv_{i+1})=2n-i+1, 1\le i\le n-1;$$

$$f(v_0v_i)=3n-i+1, 1\le i\le n;$$

$$f(u_{i1}u_{im})=3n+i+1, 1\le i\le n+1$$

At this time, $f(V)=\{1,2,\cdots,n\}\cup\{6n+5,6n+6,\cdots,7n+5\}\cup\{8n+7,8n+8,\cdots,9n+7\}\cup\cdots\cup\{2mn+2m+1,2mn+2m+2,\cdots,2mn+2m+n-1\}\cup\{3n+1\}$ and $f(E)=\{n+1,n+2,\cdots,3n\}\cup\{3n+2,3n+3,\cdots,6n+4\}\cup\{7n+6,7n+7,\cdots,8n+6\}\cup\cdots\cup\{2mn+2m+n, 2mn+2m+n+1,\cdots,2mn+2m+2n\}$, which gives $f(V(G))\cup f\big(E(G)\big)\to[1,2mn+2m+2n]$ and $f\big(V(G)\big)\cap f\big(E(G)\big)=\emptyset$.

And $S(u_{ij})=f\big(u_{ij}u_{i(j-1)}\big)+f\big(u_{ij}u_{i(j+1)}\big)+f\big(u_{ij}\big)=3mn+3m+7n+3i+1, 1\le i\le n+1, 2\le j\le m,$ we can get that in the vertex sets $\{u_{12},u_{13},\cdots,u_{1m}\}$, $\{u_{22},u_{23},\cdots,u_{2m}\}$, $\cdots$, $\{u_{(n+1)2},u_{(n+1)3},\cdots,u_{(n+1)m}\}$, the sum of the labels of all elements in each set is the same. $S(u_{11})=f(u_{11}u_{21})+f(u_{11}u_{n1})+f(u_{11}v_0)+f(u_{11})+f(u_{11}u_{1m})+f(u_{11}u_{12})=14n+6; S(u_{n1})=f(u_{n1}v_0)+f(u_{11}u_{n1})+f\big(u_{n1}u_{(n-1)1}\big)+f(u_{n1}u_{nm})+f(u_{n1}u_{n2})+f(u_{n1})=14n+6, S(u_{i1})=f\big(u_{i1}u_{(i+1)1}\big)+f(u_{i1}u_{im})+f(u_{i1}v_0)+f\big(u_{i1}u_{(i-1)1}\big)+f(u_{i1}u_{i2})+f(u_{i1})=14n+6, 2\le i\le n-1$, which give the same sum of labels of all elements in the set $\{u_{11},u_{21},\cdots,u_{n1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

To sum up, Theorem 9 holds.

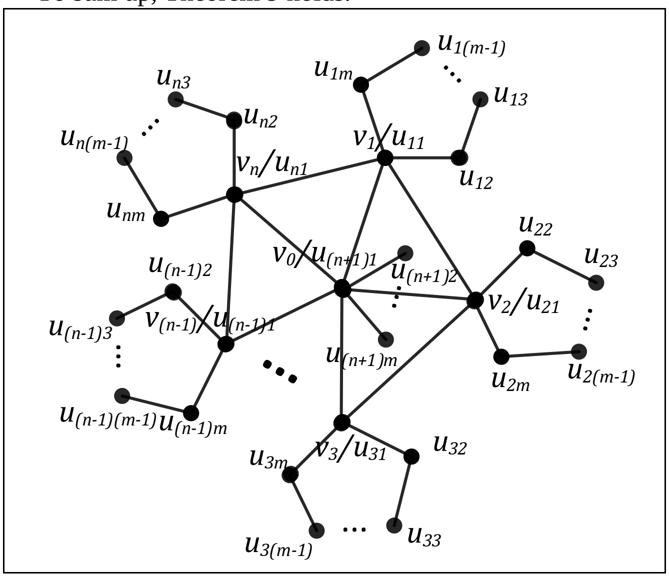


Fig 11. $W_n\circ C_m$

Theorem 10: AVRTL exists for corona graph $F_n\circ P_m$ when $n\ge 3, m\ge 4$.

Proof: Let the set of vertices of $F_n\circ P_m$ be $\{u_{ij}|1\le i\le n+1,1\le j\le m\}$, and $v_i(0\le i\le n)$ is the common vertex of P_n and F_m. $F_n\circ P_m$ has a total of $mn+m$ vertices and $mn+m+n-2$ edges, as shown in Figure 12.

Scenario 1: When $n=3, m\ge 4$

At this point, the vertices v_0 and v_2 are adjacent and of the same degree, both being vertices of degree 4; any two elements of the vertex set $\{u_{i2},u_{i3},\cdots,u_{i(m-1)}\}(1\le i\le 4)$ are adjacent and of the same degree, both of degree 2 vertices, and a mapping about f can be obtained as follows.

$$f(v_i)=i, 1\le i\le 4;$$

$$f(u_{ij})=\begin{cases}8m+i-7,\ j=2\\ 4j+i+1, 3\le j\le m-1\\ 8m+i-3,\ j=m\end{cases}\ 1\le i\le 4;$$

$$f(v_0v_i)=\begin{cases}6, i=1\\ 5, i=2\\ 7, i=3\end{cases};$$

$$f(v_iv_{i+1})=i+7, 1\le i\le 2;$$

$$f(u_{ij}u_{i(j+1)})=$$

$$\begin{cases}3n+i,\ j=1\\ \begin{cases}8m+i-4\left\lfloor\dfrac{j}{2}\right\rfloor-7,\ j\equiv 1(\mathrm{mod}\,2)\\ 8m+i-4\left\lfloor\dfrac{m}{2}\right\rfloor-2j-3,\ j\equiv 0(\mathrm{mod}\,2)\end{cases} 1<j\le m\end{cases} 1\le i\le 4$$

At this time, $f(V)=\{1,2,3,4\}\cup\{14,15,\cdots,4m+1\}\cup\{8m-6,8m-5,\cdots,8m+1\}$ and $f(E)=\{5,6,\cdots,13\}\cup\{4m+2,4m+3,\cdots,8m-7\}$, which gives $f(V(G))\cup f\big(E(G)\big)\to[1,2mn+2m+n-2]$ and $f\big(V(G)\big)\cap f\big(E(G)\big)=\emptyset$.

And $S(v_0)=f(v_0)+f(v_0v_1)+f(v_0v_3)+f(v_0v_2)+f(v_0u_{42})=35; S(v_2)=f(v_2)+f(v_2v_1)+f(v_2v_3)+f(v_0v_2)+f(v_2u_{22})=35$, we can get the sum of the labels of vertices v_0 and v_2 is the same. $S(u_{ij})=f(u_{ij})+f\big(u_{ij}u_{i(j+1)}\big)+f\big(u_{ij}u_{i(j-1)}\big)=16m-4\left\lfloor\frac{m}{2}\right\rfloor+3i-5, 1\le i\le n+1, 2\le j\le m-1$, which give the same sum of labels of all elements in the set $\{u_{ij}|1\le i\le 4, 2\le j\le m-1\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $F_n\circ P_m(n=3, m\ge 4)$.

Scenario 2: When $n\ge 4, m\ge 4$

At this point, in the vertex sets $\{u_{12},u_{13},\cdots,u_{1(m-1)}\},\{u_{22},u_{23},\cdots,u_{2(m-1)}\},\cdots,\{u_{(n+1)2},u_{(n+1)3},\cdots,u_{(n+1)(m-1)}\}$, any two elements in each set are adjacent and of the same degree, and all are points of degree 2; any two elements of the vertex set $\{u_{21},u_{31},\cdots,u_{(n-1)1}\}$ are adjacent and of the same degree, both of degree 4 vertices, and a mapping about f can be obtained as follows.

$$f(u_{ij}u_{i(j+1)})=$$

$$\begin{cases}3n+i,\ j=1\\ \begin{cases}2(n+1)m+i-n-(n+1)\left\lfloor\dfrac{j}{2}\right\rfloor-4,\ j\equiv 1(\mathrm{mod}\,2)\\ 2(n+1)m+i-(n+1)(\dfrac{j}{2}+\left\lfloor\dfrac{m}{2}\right\rfloor)-3,\ j\equiv 0(\mathrm{mod}\,2)\end{cases} 1<j\le m\end{cases} 1\le i\le n+1;$$

$$f(v_i)=\begin{cases} i, 1\le i\le n-1 \\ 2n-1, i=n \\ 3n, i=n+1 \end{cases};$$

$$f(u_{ij})=\begin{cases} 2(n+1)m-n+i-4,\ j=2 \\ (j+1)n+i+j-2, 3\le j\le m-1 \\ 2(n+1)m+i-3,\ j=m \end{cases}\ 1\le i\le n+1;$$

$$f(v_0v_i)=3n-i, 1\le i\le n;$$

$$f(v_iv_{i+1})=\begin{cases} 2n-\frac{i}{2}-\frac{3}{2}, i\equiv 1(\mathrm{mod}\ 2) \\ 2n-\left\lfloor\frac{n}{2}\right\rfloor-\frac{i}{2}-1, i\equiv 0(\mathrm{mod}\ 2) \end{cases}\ 1\le i\le n-1$$

At this time, $f(V)=\{1,2,\cdots,n-1\}\cup\{2n-1,3n\}\cup\{4n+2,4n+3,\cdots,nm+m+n-2\}\cup\{2nm+2m-2, 2nm+2m-1,\cdots,2nm+2m+n-2\}$ and $f(E)=\{n,n+1,\cdots,2n-2\}\cup\{3n+1,3n+2,\cdots,4n+1\}\cup\{2n,2n+1,\cdots,3n-1\}\cup\{nm+m+n-1,nm+m+n,\cdots,2mn+2m-3\}$, which gives $f(V(G))\cup f(E(G))\to[1,2mn+2m+n-2]$ and $f(V(G))\cap f(E(G))=\emptyset$.

And $S(u_{ij})=f(u_{ij}u_{i(j-1)})+f(u_{ij}u_{i(j+1)})+f(u_{ij})=4mn+4m+n-(n+1)\left\lfloor\frac{m}{2}\right\rfloor+3i-8, 1\le i\le n+1, 2\le j\le m-1$, we can get that in the vertex sets $\{u_{12},u_{13},\cdots,u_{1(m-1)}\}$, $\{u_{22},u_{23},\cdots,u_{2(m-1)}\}$, $\cdots$, $\{u_{(n+1)2},u_{(n+1)3},\cdots,u_{(n+1)(m-1)}\}$, the sum of the labels of all elements in each set is the same. $S(u_{i1})=f(u_{i1}u_{(i+1)1})+f(u_{i1}u_{(i-1)1})+f(u_{i1}u_{i2})+f(v_0v_i)+f(u_{i1})=10n-\left\lfloor\frac{n}{2}\right\rfloor-2, 2\le i\le n-1$, which gives the same sum of labels of all elements in the set $\{u_{21},u_{31},\cdots,u_{(n-1)1}\}$. The rest of the vertices have no adjacent vertices of the same degree, so it is not necessary to consider the sum of their labels.

Therefore, AVRTL exists for $F_n\circ P_m(n\ge 4, m\ge 4)$.

To sum up, Theorem 10 holds.

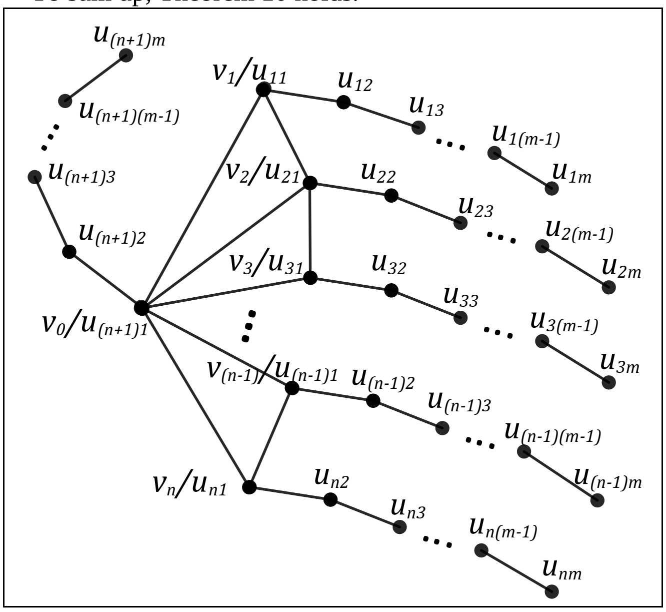


Fig 12. $F_n\circ P_m$

Conjecture 1: If graphs G and H are AVRTL graphs, then their corona graphs $G\circ H$ or $H\circ G$ are also AVRTL graphs.

V. Conclusion

This paper designs a novel AVRTL algorithm based on the ideas of traditional intelligent algorithms to address the practical problem that special scenarios exist in sensor networks that the Adjacent Vertex Reducible Total Labeling model can describe. The algorithm labels the points and edges in the graph with the help of preprocessing, adjustment and backward functions in a circular, iterative merit-seeking manner. By analyzing the results of labeling from the result set, the labeling rules of several corona graphs were found, and several theorems were summarized to enrich the research results of the reducible series. Finally, a conjecture was given based on the experimental results and the summarized theorems: if graphs G and H are AVRTL graphs, then their corona graphs $G\circ H$ or $H\circ G$ are also AVRTL graphs.

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