New Lyapunov-type Inequalities for Fractional Differential Equations with Bi-ordinal Psi-Hilfer Fractional Derivative Involving Multi-point Boundary Conditions

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***Abstract*—In this work, we first propose a new fractional derivative (bi-ordinal ψ-Hilfer) and present its properties. We also study the Lyapunov-type inequalities for the fractional boundary value problem with multi-point boundary conditions in the framework of bi-ordinal ψ-Hilfer fractional derivative. Finally, we provide some corollaries for generalizing and enriching the existing literature.**

***Index Terms*—Lyapunov-type inequality, Bi-ordinal ψ-Hilfer fractional derivative, Multi-point boundary condition, ψ-Hilfer fractional derivative.**

I. Introduction

THE well-known Lyapunov result [1] states that there exists a nontrivial solution of Hill's equation with Dirichlet boundary conditions, which is expressed as follows:

Theorem 1.1 If the boundary value problem (BVP)

$$\begin{cases} x''(t)+q(t)x(t)=0, & t\in(a,b), \\ x(a)=x(b)=0, \end{cases} \tag{1}$$

has a nontrivial continuous solution, where $q(t)\in C([a,b],\mathbf{R})$, then,

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{2}$$

The Lyapunov inequality (2) and its generalizations are indispensable tools for addressing eigenvalue problems, disconjugacy, control theory, oscillation, and other fields of differential equations [2, 3].

Recently, fractional calculus has been a focus of research community due to its applicability in theory and practice [4-8]. A research for Lyapunov-type inequalities started during the study of fractional differential equations. The research was initiated by Ferreira [4] himself who considered Lyapunov-type inequalities for BVPs with Riemann-Liouville fractional derivative, which can be expressed as follows:

$$\begin{cases} ({}_aD^{\alpha}x)(t)+q(t)x(t)=0, & t\in(a,b), \\ x(a)=x(b)=0, \end{cases} \tag{3}$$

where, $q(t)\in C([a,b],\mathbf{R})$. ${}_aD^{\alpha}$ represents the Riemann-Liouville fractional derivative of order α $(1<\alpha\le 2)$. If the BVP expressed in (3) has a nontrivial solution, then,

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}. \tag{4}$$

In [5], the same author derived a Lyapunov-type inequality for Caputo fractional BVP.

Due to pioneering contribution of Ferreira, the studies regarding Lyapunov-type inequalities for fractional BVPs have been frequently considered in literature [9-22]. Few researchers have investigated the Lyapunov-type inequality of multi-point BVP for fractional differential equations for [18-20]. In 2018, Wang et al. [18] established Lyapunov-type inequalities for multi-point boundary conditions of Hilfer fractional differential equation. Zhang et al. [19] considered Lyapunov-type inequalities for the fractional BVPs involving Hilfer-Katugampola fractional derivative with multi-point boundary conditions.

Recently, few scholars have focused on the Hilfer fractional derivative of a function with respect to another function ψ. In [22], Zohra et al. derived Lyapunov-type inequalities for the fractional BVP as:

$$\begin{cases} ({}^HD^{\alpha,\beta,\psi}_{a+}x)(t)+q(t)f(x(t))=0, & a<t<b, \\ x(a)=x(b)=0, \end{cases} \tag{5}$$

where, $(a,b)\in\mathbf{R}^2$. ${}^HD^{\alpha,\beta,\psi}_{a+}$ is the ψ-Hilfer fractional derivative type of order $(1<\alpha<2,\ 0\le\beta\le 1)$, $x,\psi\in C^2([a,b],\mathbf{R})$ such that ψ is strictly increasing and $f,q:\mathbf{R}\to\mathbf{R}$. The following conclusion is obtained.

Theorem 1.2 The function $f:\mathbf{R}\to\mathbf{R}$ is continuous and sublinear, $f(u)\le\mu|u|$, for $t\in[a,b]$ and $u\in\mathbf{R},\mu>0$. If the BVP expressed in (5) has a nontrivial solution, then,

$$\int_a^b \psi'(s)|q(s)|ds \ge \frac{\Lambda}{\mu(\gamma-1)^{\gamma-1}(\psi(b)-\psi(a))^{\alpha-1}}, \tag{6}$$

where, $\Lambda=\Gamma(\alpha)(\alpha+\gamma-2)^{\alpha+\gamma-2}(\alpha-1)^{1-\alpha}$.

In 2021, Karimov et al. [23] proposed bi-ordinal Hilfer fractional derivative of orders α $(n-1<\alpha\le n)$, β $(n-1<\beta\le n)$ and of type $\mu\in[0,1]$ by using the following equation:

$$D^{(\alpha,\beta)\mu}_{a+}x(t)=I^{\mu(n-\alpha)}_{0+}(\frac{d}{dt})^nI^{(1-\mu)(n-\beta)}_{0+}x(t). \tag{7}$$

Specifically, when $\mu=0$, (7) represents the Riemann-Liouville fractional derivative of order β and for $\mu=1$, the

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bi-ordinal Hilfer fractional derivative (7) denotes the Caputo fractional derivative of order α.

Based on the aforementioned literature, we study the following Lyapunov-type inequalities for BVPs of bi-ordinal ψ-Hilfer with m-point boundary conditions as:

$$\begin{cases} ({}^H D_{a+}^{(\alpha,\beta)\mu,\psi} x)(t) + q(t)x(t) = 0, \quad a < t < b, \\ x(a) = 0, \quad x(b) = \sum\limits_{i=1}^{m-2} \sigma_i x(\eta_i), \end{cases} \tag{8}$$

and

$$\begin{cases} ({}^H D_{a+}^{(\alpha,\beta)\mu,\psi} x)(t) + q(t)x(t) = 0, \quad a < t < b, \\ x(a) = 0, \quad \frac{1}{\psi'(t)} \frac{d}{dt} x(t)|_{t=b} = \sum\limits_{i=1}^{m-2} \lambda_i x(\xi_i), \end{cases} \tag{9}$$

where, $q(t) \in C([a,b], \mathbf{R})$, $\psi \in C^2[a,b]$, $\psi'(t) > 0$, ${}^H D_{a+}^{(\alpha,\beta)\mu,\psi}$ is bi-ordinal ψ-Hilfer fractional derivative of orders α $(1 < \alpha < 2)$, β $(0 \le \beta \le 1)$ and type μ $(0 \le \mu \le 1)$; $\sigma_i \ge 0$, $a < \eta_i, \xi_i < b$, $(i = 1, 2, \cdots, m-2)$, for $a < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < b$, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$. They are satisfied based on the following conditions.

(C$_1$) $(\psi(b) - \psi(a))^{\omega-1} > \sum\limits_{i=1}^{m-2} \sigma_i (\psi(\eta_i) - \psi(a))^{\omega-1}$.

(C$_2$) $(\omega-1)(\psi(b) - \psi(a))^{\omega-2} > \sum\limits_{i=1}^{m-2} \lambda_i (\psi(\xi_i) - \psi(a))^{\omega-1}$.

In this work, we propose a new definition of bi-ordinal ψ-Hilfer fractional derivative and prove its properties. We also study the Lyapunov-type inequalities for BVP expressed in (8) and (9) with m-point boundary conditions. To the best of our knowledge, only a few works have considered the Lyapunov-type inequalities for fractional BVPs involving m-point boundary conditions. This work provides new results that can extend and complement the previous literature.

The rest of this paper is summarized as follows. Section II, we briefly present the necessary definitions and lemmas related to ψ-Hilfer fractional calculus. Section III, we propose a new fractional derivative (bi-ordinal ψ-Hilfer) and prove its properties. The results are presented in Section IV. Finally, Section V concludes this work.

II. Preliminaries

In this section, we present the concepts and lemmas regarding the ψ-Hilfer fractional integral and the ψ-Hilfer fractional derivative.

Definition 2.1 ([24]) Let $(a,b)(-\infty \le a < b \le \infty)$ be a finite or infinite interval of the real line $\mathbf{R}$ and $\alpha > 0$. Also let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). The left-sided ψ-Hilfer fractional integral of a function x with respect to another function ψ on $[a,b]$ is defined by

$$I_{a+}^{\alpha,\psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} x(s) ds.$$

Definition 2.2 ([24]) Let $n-1 < \alpha \le n$ with $n \in \mathbf{N}$, $I \in [a,b]$ is the interval such that$-\infty \le a < b \le \infty$ and $x, \psi \in C^n([a,b], \mathbf{R})$ two functions such that ψ is increasing and $\psi'(t) \ne 0$, for all $t \in I$. The left-sided ψ-Hilfer fractional derivative of a function of order α and type $0 \le \beta \le 1$, is defined by

$${}^H D_{a+}^{\alpha,\beta,\psi} x(t) = I_{a+}^{\beta(n-\alpha),\psi} \mathbf{D}^n I_{a+}^{(1-\beta)(n-\alpha),\psi} x(t),$$

where, $\mathbf{D}^n = (\frac{1}{\psi'(t)} \frac{d}{dt})^n$.

Lemma 2.1 ([24]) Let $\alpha > 0$ and $\beta > 0$, then

$$I_{a+}^{\alpha,\psi} I_{a+}^{\beta,\psi} = I_{a+}^{\alpha+\beta,\psi}.$$

Lemma 2.2 ([24]) If $x \in C^n[a,b]$, $n-1 < \alpha < n$, $0 \le \beta \le 1$, and $\gamma = \alpha + \beta(n-\alpha)$, then

$$\begin{aligned} I_{a+}^{\alpha,\psi\,H} D_{a+}^{\alpha,\beta,\psi} x(t) &= x(t) - \sum_{k=1}^{n} \frac{(\psi(t) - \psi(a))^{\gamma-k}}{\Gamma(\omega - k + 1)} \\ &\times \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^{n-k} I_{a+}^{(1-\beta)(n-\alpha),\psi} x(a) \\ &= I_{a+}^{\gamma,\psi} D_{a+}^{\gamma,\psi} x(t). \end{aligned}$$

Lemma 2.3 ([24]) Let $\alpha > 0$ and $\xi > 0$, if $x(t) = (\psi(t) - \psi(a))^{\xi-1}$, then

$$I_{a+}^{\alpha,\psi} x(t) = \frac{\Gamma(\xi)}{\Gamma(\alpha+\xi)} (\psi(t) - \psi(a))^{\alpha+\xi-1},$$

$$D_{a+}^{\alpha,\psi} x(t) = \frac{\Gamma(\xi)}{\Gamma(\xi-\alpha)} (\psi(t) - \psi(a))^{\xi-\alpha-1}.$$

Lemma 2.4 ([24]) Let $x \in C^1[a,b]$, $\alpha > 0$ and $0 \le \beta \le 1$, we have

$${}^H D_{a+}^{\alpha,\beta,\psi} I_{a+}^{\alpha,\psi} x(t) = x(t).$$

III. New definition and properties of bi-ordinal ψ-Hilfer fractional derivative

In this section, we present the proposed fractional derivative (bi-ordinal ψ-Hilfer) and prove its properties.

Definition 3.1 Let $(n-1 < \alpha, \beta \le n)$ with $n \in \mathbf{N}$, $I = [a,b]$ is the interval such that $-\infty \le a < b \le \infty$ and $x, \psi \in C^n[a,b]$ two functions such that ψ is increasing and $\psi'(x) \ne 0$, for all $x \in I$. The bi-ordinal ψ-Hilfer fractional derivative (left-sided) ${}^H D_{a+}^{(\alpha,\beta)\mu,\psi} x$ of function of order α, β and type μ $(0 \le \mu \le 1)$ is defined by

$$\begin{aligned} ({}^H D_{a+}^{(\alpha,\beta)\mu,\psi} x)(t) &= (I_{a+}^{\mu(n-\alpha),\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \\ &\times I_{a+}^{(1-\mu)(n-\beta),\psi} x)(t). \end{aligned}$$

Lemma 3.1 If $x \in C^n[a,b]$, $n-1 < \alpha, \beta < n$, $0 \le \mu \le 1$, then

$$\begin{aligned} I_{a+}^{\delta,\psi\,H} D_{a+}^{(\alpha,\beta)\mu,\psi} x(t) &= (I_{a+}^{\delta,\psi} I_{a+}^{\omega-\delta,\psi} D_{a+}^{\omega,\psi} x)(t) \\ &= (I_{a+}^{\omega,\psi} D_{a+}^{\omega,\psi} x)(t) \\ &= x(t) - \sum_{k=1}^{n} \frac{(\psi(t) - \psi(a))^{\omega-k}}{\Gamma(\omega-k+1)} \\ &\times \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^{n-k} I_{a+}^{(1-\mu)(n-\beta),\psi} x(a), \end{aligned}$$

where, $\omega = \beta + \mu(n-\beta)$, $\delta = \beta + \mu(\alpha - \beta)$, and $\omega > \delta$.

Proof. Let $\omega = \beta + \mu(n-\beta)$, $\delta = \beta + \mu(\alpha - \beta)$, then by using the Definitions 2.1, 2.2 and Lemma 2.1, we get the following:

$$\begin{aligned}&({}^{H}D_{a+}^{(\alpha,\beta)\mu,\psi}x)(t) = (I_{a+}^{\mu(n-\alpha),\psi}\\ &\quad\times\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n} I_{a+}^{(1-\mu)(n-\beta),\psi}x)(t)\\ &= (I_{a+}^{\omega-\delta,\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n} I_{a+}^{(1-\mu)(n-\beta),\psi}x)(t)\\ &= (I_{a+}^{\omega-\delta,\psi}D_{a+}^{\omega,\psi}x)(t).\end{aligned}$$

Applying the integral operator $I_{a+}^{\delta,\psi}$ on the above equation, we get

$$\begin{aligned}&I_{a+}^{\delta,\psi}\,{}^{H}D_{a+}^{(\alpha,\beta)\mu,\psi}x(t) = (I_{a+}^{\delta,\psi}I_{a+}^{\omega-\delta,\psi}D_{a+}^{\omega,\psi}x)(t)\\ &= (I_{a+}^{\omega,\psi}D_{a+}^{\omega,\psi}x)(t)\\ &= x(t) - \sum_{k=1}^{n}\frac{(\psi(t)-\psi(a))^{\omega-k}}{\Gamma(\omega-k+1)}\\ &\quad\times\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n-k} I_{a+}^{(1-\mu)(n-\beta),\psi}x(a).\end{aligned}$$

Hence, Lemma 3.1 is proved.

Lemma 3.2 Let $\alpha > 0$, $n-1 < \alpha, \beta < n$, $0 \le \mu \le 1$, $m \in \mathbf{N}$ and $\mathbf{D} = \frac{1}{\psi'(t)}\frac{d}{dt}$. If the fractional derivatives $(\mathbf{D}^m x)t$ and $({}^{H}D_{a+}^{(\alpha+m,\beta+m)\mu,\psi}x)(t)$ exist, then

$$({}^{H}D_{a+}^{(\alpha,\beta)\mu,\psi}\mathbf{D}^m x)(t) = {}^{H}D_{a+}^{(\alpha+m,\beta+m)\mu,\psi}x)(t),$$

provided that

$$\mathbf{D}^j x(t)|_{t=a} = 0, \quad j = 0,1,2,\cdots,m-1.$$

Proof. Since $\mathbf{D}^j x(t)|_{t=a} = 0,\ j = 0,1,2,\cdots,m-1$, we get

$$(I_{a+}^{m,\psi}\mathbf{D}^m x)(t) = x(t),$$

consequently, the following equality is satisfied.

$$\begin{aligned}&({}^{H}D_{a+}^{(\alpha,\beta)\mu,\psi}\mathbf{D}^m x)(t) = (I_{a+}^{\mu(n-\alpha),\psi}\\ &\quad\times\mathbf{D}^n I_{a+}^{(1-\mu)(n-\beta),\psi}\mathbf{D}^m x)(t)\\ &= (I_{a+}^{\mu(n-\alpha),\psi}\mathbf{D}^{n+m} I_{a+}^{(1-\mu)(n-\beta),\psi}(I_{a+}^{m,\psi}\mathbf{D}^m x))(t)\\ &= (I_{a+}^{\mu(n-\alpha),\psi}\mathbf{D}^{n+m} I_{a+}^{(1-\mu)(n-\beta),\psi}x)(t)\\ &= ({}^{H}D_{a+}^{(\alpha+m,\beta+m)\mu,\psi}x)(t).\end{aligned}$$

Hence, Lemma 3.2 is proved.

IV. Main results

In this section, we present the Green's functions for problems presented in (8) and (9), and describe their properties.

Lemma 4.1 Let (C_1) holds. If $x(t) \in C[a,b]$ represents a solution of the BVP presented in (8), then it satisfies the following integral equation:

$$\begin{aligned}x(t) &= \int_a^b H(t,s)\psi'(s)q(s)x(s)ds\\ &+R(t)\sum_{i=1}^{m-2}\sigma_i\int_a^b H(\eta_i,s)\psi'(s)q(s)x(s)ds, \quad (10)\end{aligned}$$

where, $R(t)$ is expressed as follows:

$$R(t) = \frac{(\psi(t)-\psi(a))^{\omega-1}}{(\psi(b)-\psi(a))^{\omega-1} - \sum\limits_{i=1}^{m-2}\sigma_i(\psi(\eta_i)-\psi(a))^{\omega-1}},$$

and the Green's function $H(t,s)$ is expressed as follows:

$$H(t,s) = \frac{1}{\Gamma(\delta)(\psi(b)-\psi(a))^{\omega-1}}N(t,s),$$

$$N(t,s) = \begin{cases} d_1(t,s) & a \le s \le t \le b,\\ d_2(t,s) & a \le t \le s \le b,\end{cases}$$

with

$$\begin{aligned}d_1(t,s) &= (\psi(t)-\psi(a))^{\omega-1}(\psi(b)-\psi(s))^{\delta-1}\\ &\quad-(\psi(b)-\psi(a))^{\omega-1}(\psi(t)-\psi(s))^{\delta-1},\\ d_2(t,s) &= (\psi(t)-\psi(a))^{\omega-1}(\psi(b)-\psi(s))^{\delta-1}.\end{aligned}$$

Proof. We use Lemma 3.1 to reduce (8) into an equivalent integral equation as:

$$\begin{aligned}x(t) &= -I_{a+}^{\delta,\psi}q(t)x(t) + c_1(\psi(t)-\psi(a))^{\omega-1}\\ &+c_2(\psi(t)-\psi(a))^{\omega-2}, \quad (11)\end{aligned}$$

where, $c_1,\ c_2 \in \mathbf{R}$. Since $x(a) = 0,\ x(b) = \sum\limits_{i=1}^{m-2}\sigma_i x(\eta_i)$, we have $c_2 = 0$, and

$$c_1 = \frac{1}{(\psi(b)-\psi(a))^{\omega-1}}\Big[\sum_{i=1}^{m-2}\sigma_i x(\eta_i) + I_{a+}^{\delta,\psi}q(t)x(t)|_{t=b}\Big].$$

Substituting the results of $c_1,\ c_2$ into (11), we obtain the following:

$$\begin{aligned}x(t) &= -I_{a+}^{\delta,\psi}q(t)x(t) + \left(\frac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}\right)^{\omega-1}\\ &\quad\times\Big[\sum_{i=1}^{m-2}\sigma_i x(\eta_i) + I_{a+}^{\delta,\psi}q(t)x(t)|_{t=b}\Big]\\ &= \int_a^b H(t,s)\psi'(s)q(s)x(s)ds\\ &\quad+\left(\frac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}\right)^{\omega-1}\sum_{i=1}^{m-2}\sigma_i x(\eta_i). \quad (12)\end{aligned}$$

Now,

$$\begin{aligned}\sum_{i=1}^{m-2}\sigma_i x(\eta_i) &= \sum_{i=1}^{m-2}\sigma_i\int_a^b H(\eta_i,s)\psi'(s)q(s)x(s)ds\\ &+\sum_{i=1}^{m-2}\sigma_i\left(\frac{\psi(\eta_i)-\psi(a)}{\psi(b)-\psi(a)}\right)^{\omega-1}\sum_{i=1}^{m-2}\sigma_i x(\eta_i),\end{aligned}$$

which can be further solved to obtain:

$$\sum_{i=1}^{m-2}\sigma_i x(\eta_i) = \frac{A_1\sum\limits_{i=1}^{m-2}\sigma_i\int_a^b H(\eta_i,s)\psi'(s)q(s)x(s)ds}{A_1 - \sum\limits_{i=1}^{m-2}\sigma_i(\psi(\eta_i)-\psi(a))^{\omega-1}}, \quad (13)$$

where, $A_1 = (\psi(b)-\psi(a))^{\omega-1}$.

Using (12) and (13), we obtain the desired result expressed in (10). Hence, Lemma 4.1 is proved.

Lemma 4.2 Let us assume that (C_2) holds. If $x(t) \in C[a,b]$

is the solution of the BVP expressed in (9), then it satisfies the following integral equation:

$$x(t) = \int_a^b G(t,s)\psi'(s)q(s)x(s)ds$$
$$+Q(t)\sum_{i=1}^{m-2}\lambda_i \int_a^b G(\xi_i,s)\psi'(s)q(s)x(s)ds, \quad (14)$$

where, $Q(t)$ and A_2 are defined as follows:

$$Q(t) = \frac{(\psi(t)-\psi(a))^{\omega-1}}{A_2}, \ \ t\in[a,b],$$

$$A_2 = (\omega-1)(\psi(b)-\psi(a))^{\omega-2} - \sum_{i=1}^{m-2}\lambda_i(\psi(\xi_i)-\psi(a))^{\omega-1},$$

$G(t,s)$ denotes the Green's function, which is defined as follows:

$$G(t,s) = \frac{(\psi(b)-\psi(s))^{\delta-2}}{\Gamma(\delta)(\omega-1)}\begin{cases} g_1(t,s), & a\le s\le t\le b,\\ g_2(t,s), & a\le t\le s\le b,\end{cases}$$

and

$$g_1(s,t) = (\delta-1)(\psi(b)-\psi(a))^{2-\omega}(\psi(t)-\psi(a))^{\omega-1}$$
$$-(\omega-1)\frac{(\psi(t)-\psi(s))^{\delta-1}}{(\psi(b)-\psi(s))^{\delta-2}},$$
$$g_2(s,t) = (\delta-1)(\psi(b)-\psi(a))^{2-\omega}(\psi(t)-\psi(a))^{\omega-1}.$$

Proof. By applying the integral operator $I_{a+}^{\delta,\psi}$ on (9) and using Lemma 3.1, we obtain the following equation:

$$x(t) = -I_{a+}^{\delta,\psi}q(t)x(t) + c_1(\psi(t)-\psi(a))^{\omega-1}$$
$$+c_2(\psi(t)-\psi(a))^{\omega-2},$$

where, $c_1,\ c_2 \in \mathbf{R}$. The boundary condition $x(a) = 0$ implies that $c_2 = 0$. Hence,

$$x(t) = -I_{a+}^{\delta,\psi}q(t)x(t) + c_1(\psi(t)-\psi(a))^{\omega-1}. \quad (15)$$

Applying derivative $\mathbf{D} = \frac{1}{\psi'(t)}\frac{d}{dt}$ on the both sides of the (15) with respect to t, we get the following equation:

$$\frac{1}{\psi'(t)}\frac{d}{dt}x(t) = -I_{a+}^{\delta-1,\psi}q(t)x(t)$$
$$+ c_1(\omega-1)(\psi(t)-\psi(a))^{\omega-2}.$$

Since $\frac{1}{\psi'(t)}\frac{d}{dt}x(t)|_{t=b} = \sum_{i=1}^{m-2}\lambda_i x(\xi_i)$, we get:

$$c_1 = \frac{1}{(\omega-1)(\psi(b)-\psi(a))^{\omega-2}}$$
$$\times\Big[\sum_{i=1}^{m-2}\lambda_i x(\xi_i) + I_{a+}^{\delta-1,\psi}q(t)x(t)|_{t=b}\Big].$$

Now, the unique solution of the problem presented in (9) is expressed as follows:

$$x(t) = -I_{a+}^{\delta,\psi}q(t)x(t) + \frac{(\psi(t)-\psi(a))^{\omega-1}}{(\omega-1)(\psi(b)-\psi(a))^{\omega-2}}$$
$$\times\Big[\sum_{i=1}^{m-2}\lambda_i x(\xi_i) + I_{a+}^{\delta-1,\psi}q(t)x(t)|_{t=b}\Big],$$
$$= \int_a^b G(t,s)\psi'(s)q(s)x(s)ds$$
$$+\frac{(\psi(t)-\psi(a))^{\omega-1}}{(\omega-1)(\psi(b)-\psi(a))^{\omega-2}}\sum_{i=1}^{m-2}\lambda_i x(\xi_i). (16)$$

We obtain:

$$\sum_{i=1}^{m-2}\lambda_i x(\xi_i) = \sum_{i=1}^{m-2}\lambda_i\int_a^b G(\xi_i,s)\psi'(s)q(s)x(s)ds$$
$$+\frac{\sum_{i=1}^{m-2}\lambda_i(\psi(\xi_i)-\psi(a))^{\omega-1}}{(\omega-1)(\psi(b)-\psi(a))^{\omega-2}}$$
$$\times\sum_{i=1}^{m-2}\lambda_i x(\xi_i),$$

which implies,

$$\sum_{i=1}^{m-2}\lambda_i x(\xi_i) = \frac{(\omega-1)\sum_{i=1}^{m-2}\lambda_i\int_a^b G(\xi_i,s)\psi'(s)q(s)x(s)ds}{(\psi(b)-\psi(a))^{2-\omega}A_2}. \quad (17)$$

Using (17) in (16), we obtain the solution of the problem presented in (14). Hence, Lemma 4.2 is proved.

Lemma 4.3 ([5]) If $1<\nu<2$, then,

$$\frac{2-\nu}{(\nu-1)^{\frac{\nu-1}{\nu-2}}} \le \frac{(\nu-1)^{(\nu-1)}}{\nu^\nu}.$$

Lemma 4.4 The Green's functions $H(t,s)$ and $G(t,s)$ defined in (10) and (14) respectively, satisfy the following properties:

(i) $H(t,s)$ and $G(t,s)$ are continuous functions in $[a,b]\times[a,b]$.

(ii) For any $(t,s)\in[a,b]\times[a,b]$, we have,

$$|H(t,s)| \le \frac{(\delta-1)^{\delta-1}(\omega-1)^{\omega-1}(\psi(b)-\psi(a))^{\delta-1}}{\Gamma(\delta)(\omega+\delta-2)^{\omega+\delta-2}}.$$

(iii) For any $(t,s)\in[a,b]\times[a,b]$, we have,

$$|G(t,s)| \le \frac{(\psi(b)-\psi(s))^{\delta-2}(\psi(b)-\psi(a))}{(\omega-1)\Gamma(\delta)}$$
$$\times\max\{\omega-\delta,\delta-1\}.$$

Proof. It is evident that (i) is satisfied. In order to prove that (ii) holds, for all $(t,s)\in[a,b]\times[a,b]$, we start with the function $d_2(t,s)$, which is easier to achieve,

$$0\le d_2(t,s)\le d_2(s,s).$$

Now, we start by deriving the function $d_1(t,s)$ with respect to s, as follows:

$$\frac{\partial d_1(t,s)}{\partial s} = (\delta-1)\psi'(s)(\psi(b)-\psi(a))^{\omega-1}$$
$$\times(\psi(t)-\psi(s))^{\delta-2}$$
$$\times\Big[1-\Big(\frac{\psi(t)-\psi(s)}{\psi(b)-\psi(s)}\Big)^{2-\delta}$$
$$\times\Big(\frac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}\Big)^{\omega-1}\Big]\ge 0.$$

This shows that $d_1(t,s)$ is an increasing function with respect to $s\in[a,t]$. Therefore, we obtain the following:

$$d_1(t,a)\le d_1(t,s)\le d_1(t,t).$$

Considering,

$$d_1(t,a) = (\psi(t)-\psi(a))^{\omega-1}(\psi(b)-\psi(a))^{\delta-1}$$
$$\times\Big[1-\Big(\frac{\psi(b)-\psi(a)}{\psi(t)-\psi(a)}\Big)^{\omega-\delta}\Big]\le 0,$$

We obtain the following:

$$|d_1(t,s)| \le \max\left\{\max_{t\in[a,b]} d_1(t,t), \max_{t\in[a,b]}(-d_1(t,a))\right\}.$$

For convenience, we define two functions $y_1(t)$ and $y_2(t)$ as follows:

$$y_1(t) = d_1(t,t) = (\psi(t)-\psi(a))^{\omega-1}(\psi(b)-\psi(t))^{\delta-1},$$

and

$$\begin{aligned}
y_2(t) &= -d_1(t,a)\\
&= (\psi(b)-\psi(a))^{\omega-1}(\psi(t)-\psi(a))^{\delta-1}\\
&\quad -(\psi(t)-\psi(a))^{\omega-1}(\psi(b)-\psi(a))^{\delta-1}\\
&= (\psi(b)-\psi(a))^{\omega-1}(\psi(t)-\psi(a))^{\delta-1}\\
&\quad \times\Big[1-\Big(\frac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}\Big)^{\omega-\delta}\Big].
\end{aligned}$$

Then, differentiating $y_1(t)$ on (a,b), following expression is obtained:

$$\begin{aligned}
{y_1}'(t) &= \psi'(t)(\psi(t)-\psi(a))^{\omega-2}(\psi(b)-\psi(t))^{\delta-2}\\
&\times\Big[(\omega-1)(\psi(b)-\psi(t))-(\delta-1)(\psi(t)-\psi(a))\Big].
\end{aligned}$$

Please note that ${y_1}'(t_1)=0$ if and only if

$$\psi(t_1) = \frac{(\omega-1)\psi(b)+(\delta-1)\psi(a)}{\delta+\omega-2},$$

this follows $\psi(a) < \psi(t_1) < \psi(b)$, or $a = \psi^{-1}(\psi(a)) < t_1 < \psi^{-1}(\psi(b)) = b$. Then, $y_1(a) = y_1(b) = 0$ and $y_1(t) > 0$ on (a,b). According to Rolle's theorem, we deduce that $y_1(t)$ is maximum at $t = t_1$

$$\begin{aligned}
&\max_{t\in[a,b]} y_1(t) = y(t_1)\\
&= \Big(\frac{(\omega-1)(\psi(b)-\psi(a))}{\delta+\omega-2}\Big)^{\omega-1}\\
&\quad\times\Big(\frac{(\delta-1)(\psi(b)-\psi(a))}{\delta+\omega-2}\Big)^{\delta-1}\\
&= \frac{(\omega-1)^{\omega-1}(\delta-1)^{\delta-1}(\psi(b)-\psi(a))^{\delta+\omega-2}}{(\delta+\omega-2)^{\delta+\omega-2}}.
\end{aligned}$$

Now, we assume that $\max_{t\in[a,b]} y_2(t) \le \max_{t\in[a,b]} y_1(t)$. If $\omega=\delta$, it is obviously. If $\omega\ne\delta$, differentiating $y_2(t)$ on (a,b), we get the following:

$$\begin{aligned}
{y_2}'(t) &= \psi'(t)(\psi(b)-\psi(a))^{\delta-1}(\psi(t)-\psi(a))^{\delta-2}\\
&\times[(\delta-1)(\psi(b)-\psi(a))^{\omega-\delta}\\
&-(\omega-1)(\psi(t)-\psi(a))^{\omega-\delta}].
\end{aligned}$$

Therefore, by calculating ${y_2}'(t_2)=0$ if and only if

$$\psi(t_2) = \left(\psi(a)+\Big(\frac{\delta-1}{(\omega-1)}\Big)^{\frac{1}{\omega-\delta}}(\psi(b)-\psi(a))\right),$$

where, $t_2\in(a,b)$, as, $\psi(a)<\psi(t_2)<\psi(b)$. In fact, we concludes $y_2(a)=y_2(b)=0$ and $y_2(t)>0$ on (a,b), such that $y_2(t)$ is maximum at $t=t_2$, then

$$\begin{aligned}
&\max_{t\in[a,b]} y_2(t) = y_2(t_2)\\
&\qquad= \frac{\omega-\delta}{\omega-1}\Big(\frac{\delta-1}{\omega-1}\Big)^{\frac{\delta-1}{\omega-\delta}}(\psi(b)-\psi(a))^{\delta+\omega-2}.
\end{aligned}$$

Now, we demonstrate that $y_2(t_2)\le y_1(t_1)$. Considering $\nu = \frac{\delta+\omega-2}{\omega-1}$, and by using Lemma 4.3, we get the following:

$$\begin{aligned}
y_2(t_2) &= \frac{\omega-\delta}{\omega-1}\Big(\frac{\delta-1}{\omega-1}\Big)^{\frac{\delta-1}{\omega-\delta}}(\psi(b)-\psi(a))^{\delta+\omega-2}\\
&\le \left(\frac{(\delta-1)^{\delta-1}(\omega-1)^{\omega-1}}{(\delta+\omega-2)^{\delta+\omega-2}}\right)^{\frac{1}{\omega-1}}(\psi(b)-\psi(a))^{\delta+\omega-2}\\
&\le \frac{(\omega-1)^{\omega-1}(\delta-1)^{\delta-1}(\psi(b)-\psi(a))^{\delta+\omega-2}}{(\delta+\omega-2)^{\delta+\omega-2}}\\
&= y_1(t_1).
\end{aligned}$$

This proves the second property. Hence

$$\begin{aligned}
&|d_1(t,s)| \le \max\left\{\max_{t\in[a,b]} y_1(t), \max_{t\in[a,b]} y_2(t))\right\}\\
&= \max_{t\in[a,b]} y_1(t)\\
&= \frac{(\omega-1)^{\omega-1}(\delta-1)^{\delta-1}(\psi(b)-\psi(a))^{\delta+\omega-2}}{(\delta+\omega-2)^{\delta+\omega-2}}.
\end{aligned}$$

Therefore, we conclude the following:

$$|H(t,s)| \le \frac{(\delta-1)^{\delta-1}(\omega-1)^{\omega-1}(\psi(b)-\psi(a))^{\delta-1}}{\Gamma(\delta)(\omega+\delta-2)^{\omega+\delta-2}}.$$

Hence, (ii) is now satisfied. Now, we prove that (iii). It is noteworthy that, for any $(t,s)\in[a,b]\times[a,b]$, it is easy to show that:

$$0 \le g_2(t,s) \le g_2(s,s) = g_1(s,s).$$

Differentiating $g_1(t,s)$ with respect to t, we obtain the following:

$$\begin{aligned}
\frac{\partial g_1(t,s)}{\partial t} &= (\delta-1)(\omega-1))\psi'(t)\Big[\Big(\frac{\psi(b)-\psi(a)}{\psi(t)-\psi(a)}\Big)^{2-\omega}\\
&\quad-\Big(\frac{\psi(b)-\psi(s)}{\psi(t)-\psi(s)}\Big)^{2-\delta}\Big]\\
&\le 0.
\end{aligned}$$

Please note that for fixed $s\in[a,b]$, $g_1(t,s)$ is a decreasing function of $t\in[s,b]$. Therefore,

$$g_1(b,s) \le g_1(t,s) \le g_1(s,s) = g_2(s,s).$$

Hence,

$$|g_1(t,s)| \le \max\left\{\max_{t\in[a,b]} g_1(b,s)|, \max_{t\in[a,b]}|g_1(s,s)|\right\}. \quad (18)$$

The calculation results in the following:

$$\begin{aligned}
g_1(s,s) &= (\delta-1)(\psi(b)-\psi(a))^{2-\omega}(\psi(s)-\psi(a))^{\omega-1}\\
&\le (\delta-1)(\psi(b)-\psi(a)) = g_1(b,b), \quad (19)\\
g_1(b,s) &= (\delta-1)(\psi(b)-\psi(a))\\
&\quad-(\omega-1)(\psi(b)-\psi(s)). \quad (20)
\end{aligned}$$

Please note that the function $g_1(b,s)$ is increasing with respect to $s\in[a,b]$. Therefore,

$$g_1(b,a) \le g_1(b,s) \le g_1(b,b),$$

The analysis shows that:

$$\begin{aligned}
g_1(b,a) &= (\delta-\omega)(\psi(b)-\psi(a)) \le 0,\\
g_1(b,b) &= (\delta-1)(\psi(b)-\psi(a)) > 0.
\end{aligned}$$

Now, we get:

$$\begin{aligned}|g_1(b,s)| &\le \max\{g_1(b,b), -g_1(b,a)\}\\ &= (\psi(b)-\psi(a))\max\{\delta-1,\omega-\delta\}.\end{aligned} \tag{21}$$

According to (18)–(21), we get:

$$|g_1(t,s)| \le (\psi(b)-\psi(a))\max\{\delta-1,\omega-\delta\}.$$

Therefore, we conclude that:

$$\begin{aligned}G(t,s) \le &\frac{(\psi(b)-\psi(s))^{\delta-2}(\psi(b)-\psi(a))}{(\omega-1)\Gamma(\delta)}\\ &\times\max\{\omega-\delta,\delta-1\}.\end{aligned}$$

Hence, Lemma 4.4 is proved.

In this section, we present the Lyapunov-type inequalities for problems presented in (8) and (9). We define $(X, ||\cdot||_\infty)$ $X = C[a,b]$ be the Banach space with norm $||x||_\infty = \max\limits_{t\in[a,b]} |x(t)|$.

Theorem 4.1 If the BVP presented in (8) has a nontrivial continuous solution $x(t) \in X$, where $q(t) \in C([a,b],\mathbf{R})$ is a real and continuous function, then:

$$\int_a^b \psi'(s)|q(s)|ds \ge \frac{\Gamma(\delta)(\delta+\omega-2)^{\delta+\omega-2}}{[1+R(b)\sum\limits_{i=1}^{m-2}\sigma_i]A_3}, \tag{22}$$

where, $A_3 = (\delta-1)^{\delta-1}(\omega-1)^{\omega-1}(\psi(b)-\psi(a))^{\delta-1}$.

Proof. It follows from Lemma 4.1 (10) that a nontrivial solution $x(t)$ of BVP presented in (8) satisfies the following integral equation:

$$\begin{aligned}x(t) = &\int_a^b H(t,s)\psi'(s)q(s)x(s)ds + R(t)\sum_{i=1}^{m-2}\sigma_i\\ &\times\int_a^b H(\eta_i,s)\psi'(s)q(s)x(s)ds,\quad t\in[a,b],\end{aligned}$$

therefore,

$$\begin{aligned}|x(t)| \le &\int_a^b |H(t,s)|\psi'(s)|q(s)x(s)|ds + R(t)\sum_{i=1}^{m-2}\sigma_i\\ &\times\int_a^b |H(\eta_i,s)|\psi'(s)|q(s)x(s)|ds,\quad t\in[a,b].\end{aligned}$$

An application of Lemma 4.4 (ii) yields the following:

$$\begin{aligned}||x||_\infty \le &\frac{(\delta-1)^{\delta-1}(\omega-1)^{\omega-1}(\psi(b)-\psi(a))^{\delta-1}}{\Gamma(\delta)(\omega+\delta-2)^{\omega+\delta-2}}\\ &\times\Big[1+R(b)\sum_{i=1}^{m-2}\sigma_i\Big]\int_a^b\psi'(s)|q(s)|ds||x||_\infty,\end{aligned}$$

which implies that (22) holds. This successful proves Theorem 4.1.

Theorem 4.2 If the BVP presented in (9) has a nontrivial continuous solution $x(t) \in X$, where $q(t) \in C([a,b],\mathbf{R})$ is a real and continuous function, then:

$$\int_a^b (\psi(b)-\psi(s))^{\delta-2}\psi'(s)|q(s)|ds \ge \frac{(\omega-1)\Gamma(\delta)}{A_4}, \tag{23}$$

where,

$$\begin{aligned}A_4 = &(\psi(b)-\psi(a))\max\{\omega-\delta,\delta-1\}\\ &\times[1+Q(b)\sum_{i=1}^{m-2}\lambda_i].\end{aligned}$$

Proof. From Lemma 4.2, it follows that a solution of BVP presented in (9) satisfies the following integral equation:

$$\begin{aligned}x(t) = &\int_a^b G(t,s)\psi'(s)q(s)x(s)ds + Q(t)\sum_{i=1}^{m-2}\lambda_i\\ &\times\int_a^b G(\xi_i,s)\psi'(s)q(s)x(s)ds,\quad t\in[a,b],\end{aligned}$$

therefore,

$$\begin{aligned}|x(t)| \le &\int_a^b |G(t,s)|\psi'(s)|q(s)||x(s)|ds + |Q(t)\sum_{i=1}^{m-2}\lambda_i\\ &\times|\int_a^b |G(\xi_i,s)|\psi'(s)|q(s)|x(s)|ds,\quad t\in[a,b].\end{aligned}$$

Using the maximum value of $G(t,s)$ obtained in Lemma 4.4 (iii) yields the desired inequality.

$$\begin{aligned}||x||_\infty \le &\frac{(\psi(b)-\psi(a))}{(\omega-1)\Gamma(\delta)}\max\{\omega-\delta,\delta-1\}\times\Big[1+\\ &Q(b)\sum_{i=1}^{m-2}\lambda_i\Big]\int_a^b(\psi(b)-\psi(s))^{\delta-2}|q(s)|ds||x||_\infty.\end{aligned}$$

This prove Theorem 4.2.

According to Theorem 4.1, we have the following result:

Corollary 4.1 If a nontrivial solution of the fractional ψ-Hilfer BVP

$$\begin{cases}({}^H D_{a+}^{\alpha,\beta,\psi}x)(t) + q(t)x(t) = 0,\quad t\in(a,b),\\ x(a) = 0,\quad x(b) = \sum_{i=1}^{m-2}\sigma_i x(\eta_i),\end{cases} \tag{24}$$

exists, where, $q(t) \in C([a,b],\mathbf{R})$, and ${}^H D_{a+}^{\alpha,\beta,\psi}$ represents the ψ-Hilfer fractional derivative of order α and type β, $\alpha\in(1,2]$, $\beta\in[0,1]$, $a<\eta_1<\cdots<\eta_{m-2}<b$, $\sigma_i\ge 0$ $(i=1,2,\cdots,m-2)$, $(\psi(b)-\psi(a))^{1-(2-\alpha)(1-\beta)} > \sum\limits_{i=1}^{m-2}\sigma_i(\psi(\eta_i)-\psi(a))^{1-(2-\alpha)(1-\beta)}$, then

$$\int_a^b \psi'(s)|q(s)|ds \ge \frac{O_1}{\Delta_1[1+M_1(b)\sum_{i=1}^{m-2}\sigma_i]}, \tag{25}$$

where,

$$M_1(b) = \frac{(\psi(b)-\psi(a))^{1-(2-\alpha)(1-\beta)}}{N_1},$$
$$O_1 = \Gamma(\alpha)[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)},$$

$$\begin{aligned}N_1 = &(\psi(b)-\psi(a))^{1-(2-\alpha)(1-\beta)}\\ &-\sum_{i=1}^{m-2}\sigma_i(\psi(\eta_i)-\psi(a))^{1-(2-\alpha)(1-\beta)},\end{aligned}$$

$$\begin{aligned}\Delta_1 = &(\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}\\ &\times(\psi(b)-\psi(a))^{\alpha-1}.\end{aligned}$$

Proof. Let $\alpha = \beta$ in Theorem 4.1, then we obtain the following expression:

$$\int_a^b \psi'(s)|q(s)|ds \geq \frac{P}{\nabla_1[1+R_1(b)\sum_{i=1}^{m-2}\sigma_i]},$$

where, $M_1(b) = R_1(b)$,

$$P = \Gamma(\alpha)[2(\alpha-1)+\mu(2-\alpha)]^{2(\alpha-1)+\mu(2-\alpha)},$$

$$\nabla_1 = (\alpha-1)^{\alpha-1}[\alpha-1+\mu(2-\alpha)]^{\alpha-1+\mu(2-\alpha)} \times(\psi(b)-\psi(a))^{\alpha-1}.$$

Here, the proof of Corollary 4.1 is completed.

Corollary 4.2 If a nontrivial solution of the fractional Hilfer BVP

$$\begin{cases} (D_{a+}^{\alpha,\beta}x)(t)+q(t)x(t)=0, \quad t\in(a,b), \\ x(a)=0, \quad x(b)=\sum_{i=1}^{m-2}\sigma_i x(\eta_i), \end{cases} \tag{26}$$

exists, where, $q(t) \in C([a,b],\mathbf{R})$, and $D_{a+}^{\alpha,\beta}$ represents the Hilfer fractional derivative of order α and type β, $\alpha \in (1,2]$, $\beta \in [0,1]$, $a < \eta_1 < \cdots < \eta_{m-2} < b$, $\sigma_i \geq 0$ $(i = 1,2,\cdots,m-2)$, $(b-a)^{1-(2-\alpha)(1-\beta)} > \sum_{i=1}^{m-2}\sigma_i(\eta_i-a)^{1-(2-\alpha)(1-\beta)}$, then

$$\int_a^b |q(s)|ds \geq \frac{O_1}{\Delta_2[1+M_2(b)\sum_{i=1}^{m-2}\sigma_i]}, \tag{27}$$

where,

$$M_2(b) = \frac{(b-a)^{1-(2-\alpha)(1-\beta)}}{N_2},$$
$$O_1 = \Gamma(\alpha)[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)},$$
$$N_2 = (b-a)^{1-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2}\sigma_i(\eta_i-a)^{1-(2-\alpha)(1-\beta)},$$
$$\Delta_2 = (\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}(b-a)^{\alpha-1}.$$

Proof. Using $\alpha = \beta$ and $\psi(x) = x$ in Theorem 4.1, we obtain the following:

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)[2(\alpha-1)+\mu(2-\alpha)]^{2(\alpha-1)+\mu(2-\alpha)}}{\nabla_2[1+R_2(b)\sum_{i=1}^{m-2}\sigma_i]},$$

where,

$$M_2(b) = R_2(b),$$
$$\nabla_2 = (\alpha-1)^{\alpha-1}[\alpha-1+\mu(2-\alpha)]^{\alpha-1+\mu(2-\alpha)}(b-a)^{\alpha-1}.$$

Here, the proof of Corollary 4.2 is completed.

Corollary 4.3 If a nontrivial solution of the fractional ψ-Caputo BVP

$$\begin{cases} ({}^C D_{a+}^{\alpha,\psi}x)(t)+q(t)x(t)=0, \quad t\in(a,b), \\ x(a)=0, \quad x(b)=\sum_{i=1}^{m-2}\sigma_i x(\eta_i), \end{cases} \tag{28}$$

exists, where, $q(t) \in C([a,b],\mathbf{R})$, and ${}^C D_{a+}^{\alpha,\psi}$ denotes the ψ-Caputo fractional derivative of order α, $\alpha \in (1,2]$, $a < \eta_1 < \cdots < \eta_{m-2} < b$, $\sigma_i \geq 0$ $(i = 1,2,\cdots,m-2)$, $(\psi(b)-\psi(a)) > \sum_{i=1}^{m-2}\sigma_i(\psi(\eta_i)-\psi(a))$, then

$$\int_a^b \psi'(s)|q(s)|ds \geq \frac{\Gamma(\alpha)\alpha^\alpha}{\Delta_3[1+M_3(b)\sum_{i=1}^{m-2}\sigma_i]}, \tag{29}$$

where,

$$\Delta_3 = (\alpha-1)^{\alpha-1}(\psi(b)-\psi(a))^{\alpha-1},$$
$$M_3(b) = \frac{(\psi(b)-\psi(a))}{(\psi(b)-\psi(a)) - \sum_{i=1}^{m-2}\sigma_i(\psi(\eta_i)-\psi(a))}.$$

Proof. If we use $\mu = 1$ in Theorem 4.1, then

$$\int_a^b \psi'(s)|q(s)|ds \geq \frac{\Gamma(\alpha)\alpha^\alpha}{N_3[1+R_3(b)\sum_{i=1}^{m-2}\sigma_i]},$$

where,

$$M_3(b) = R_3(b),\ N_3 = (\alpha-1)^{\alpha-1}(\psi(b)-\psi(a))^{\alpha-1}.$$

Here, the proof of Corollary 4.3 is completed.

Theorem 4.2 gives the following corollaries:

Corollary 4.4 If a nontrivial solution of the fractional ψ-Hilfer BVP

$$\begin{cases} ({}^H D_{a+}^{\alpha,\beta,\psi}x)(t)+q(t)x(t)=0, \quad a<t<b, \\ x(a)=0, \quad \frac{1}{\psi'(t)}\frac{d}{dt}x(t)|_{t=b}=\sum_{i=1}^{m-2}\lambda_i x(\xi_i), \end{cases} \tag{30}$$

exists, where, $q(t) \in C([a,b],\mathbf{R})$, and ${}^H D_{a+}^{\alpha,\beta,\psi}$ denotes the ψ-Hilfer fractional derivative of order α and type β, $\alpha \in (1,2]$, $\beta \in [0,1]$, $a < \xi_1 < \cdots < \xi_{m-2} < b$, $\lambda_i \geq 0$ $(i = 1,2,\cdots,m-2)$, $(\psi(b)-\psi(a))^{1-(2-\alpha)(1-\beta)} > \sum_{i=1}^{m-2}\lambda_i(\psi(\xi_i)-\psi(a))^{1-(2-\alpha)(1-\beta)}$, then

$$\int_a^b (\psi(b)-\psi(s))^{\alpha-2}\psi'(s)|q(s)|ds \geq \frac{O_2}{P_2}, \tag{31}$$

where,

$$O_2 = \Gamma(\alpha)[1-(2-\alpha)(1-\beta)],$$
$$P_2 = \Omega_1[1+T_1(b)\sum_{i=1}^{m-2}\lambda_i],$$
$$T_1(b) = \frac{(\psi(t)-\psi(a))^{1-(2-\alpha)(1-\beta)}}{N_4},$$
$$\Omega_1 = (\psi(b)-\psi(a))\max\{\beta(2-\alpha),\alpha-1\},$$
$$N_4 = [1-(2-\alpha)(1-\beta)])(\psi(b)-\psi(a))^{-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2}\lambda_i(\psi(\xi_i)-\psi(a))^{1-(2-\alpha)(1-\beta)}.$$

Proof. Using $\alpha = \beta$ in Theorem 4.2, we obtain the following:

$$\int_a^b (\psi(b)-\psi(s))^{\alpha-2}\psi'(s)|q(s)|ds \geq \frac{U}{\Lambda_1[1+Q_1(b)\sum_{i=1}^{m-2}\lambda_i]},$$

where,

$$U = [1-(2-\alpha)(1-\mu)]\Gamma(\alpha),\ T_1(b) = Q_1(b),$$
$$\Lambda_1 = (\psi(b)-\psi(a))\max\{\mu(2-\alpha),\alpha-1\}.$$

The proof of Corollary 4.4 is completed.

Corollary 4.5 If a nontrivial solution of the fractional Hilfer BVP

$$\begin{cases} (D_{a+}^{\alpha,\beta}x)(t)+q(t)x(t)=0, \quad a<t<b, \\ x(a)=0, \quad x'(b)=\sum_{i=1}^{m-2}\lambda_i x(\xi_i), \end{cases} \tag{32}$$

exists, where, $q(t) \in C([a,b],\mathbf{R})$, and $D_{a+}^{\alpha,\beta}$ denotes the Hilfer fractional derivative of order α and type β, $\alpha \in (1,2]$, $\beta \in [0,1]$, $a < \xi_1 < \cdots < \xi_{m-2} < b$, $\lambda_i \geq 0$ $(i = 1,2,\cdots,m-2)$, $(b-a)^{1-(2-\alpha)(1-\beta)} > \sum\limits_{i=1}^{m-2} \lambda_i(\xi_i - a)^{1-(2-\alpha)(1-\beta)}$, then

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geq \frac{\Gamma(\alpha)[1-(2-\alpha)(1-\beta)]}{\Omega_2[1+T_2(b)\sum_{i=1}^{m-2}\lambda_i]}, \quad (33)$$

where,

$$\Omega_2 = (b-a)\max\{\beta(2-\alpha),\alpha-1)\},$$
$$T_2(b) = \frac{(t-a)^{1-(2-\alpha)(1-\beta)}}{N_5 - \sum\limits_{i=1}^{m-2}\lambda_i(\xi_i-a)^{1-(2-\alpha)(1-\beta)}},$$
$$N_5 = [1-(2-\alpha)(1-\beta)](b-a)^{-(2-\alpha)(1-\beta)}.$$

Proof. Using $\alpha = \beta$ and $\psi(x) = x$ in Theorem 4.2, we get:

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geq \frac{[1-(2-\alpha)(1-\mu)]\Gamma(\alpha)}{\Lambda_2[1+Q_1(b)\sum\limits_{i=1}^{m-2}\lambda_i]},$$

where,

$$T_2(b) = Q_2(b),\ \Lambda_2 = (b-a)\max\{\mu(2-\alpha),\alpha-1\}.$$

Here, the proof of Corollary 4.5 is completed.

V. Conclusion

In this work, we present a new fractional derivative, namely bi-ordinal ψ-Hilfer fractional derivative. Based on this proposed fractional derivative, we consider two kinds of fractional BVPs, and obtain related Lyapunov-type inequalities. In this process, we first convert m-point BVPs of bi-ordinal ψ-Hilfer fractional differential equations into equivalent integral equations based on the corresponding Green's functions. Afterwards, we derive the properties of the Green's functions. Finally, we obtain the desired results and provide several corollaries to show that the results of the proposed method extend and enrich the previous literature. In the future, a lot of research is required. For instance, we will discuss the Lyapunov-type inequalities for a nonlinear fractional anti-periodic BVPs of bi-ordinal ψ-Hilfer fractional derivative, and will consider the Lyapunov-type inequalities for sequential fractional BVP in the frame of bi-ordinal ψ-Hilfer fractional derivative, and so on.

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