Intersection-union Operation to Divide a Network into Communities

Yu-Lan Wang, Chiu-Tang Lin

Abstract—**In this study, we provide a further examination for partitioning a network into communities by intersection-union operation and group action on sets. We point out that the similarity measure proposed by a previously published article contained questionable results and then we present our improvements. Our findings will help realize this research topic to obtain a reasonable division for a network to reveal the inner structure of related communities.**

Index Terms—**Click, Group action on sets, Intersection-union operation, Community in a network**

I. INTRODUCTION

THE motivation for this paper is to study Zhang *et al.* [1] and Lin *et al.* [2]. Zhang *et al.* [1] is the first paper that tried to use group operations to divide networks into communities. Zhang *et al.* [1] used "click" which is a completely connected sub-graph as a generator for group construction. Zhang *et al.* [1] applied elements in their generated group as functions from the domain of all nodes in the network to the range of all nodes in the network. They will separate nodes into two classes: (a) non-isolated nodes, and (b) isolated nodes. For those non-isolated nodes, Zhang *et al.* [1] developed cores as the bases for their communities and then added those isolated nodes to the cores, step by step. Five papers referred to Zhang *et al.* [1] in their references. In the following, we provide a brief review of these five papers. Lin *et al.* [2] examined the development of the finite group generation proposed by Zhang *et al.* [1], and then Lin *et al.* [2] claimed that the group operation will generate too many elements. For example, for the karate club problem, there are thirty-four nodes and seventy-eight links, and then Lin *et al.* [2] predicted that the group proposed by Zhang *et al.* [1] will contain more than four thousand elements. It indicates that applying it to any real-world application problems, they contain thousands of nodes. It reveals that the group operation proposed by Zhang *et al.* [1] is impossible to execute. Consequently, Lin *et al.* [2] invented a new operation, denoted as the "intersect-union operation" to replace the group operation. Li *et al.* [3] applied K-means and principal component analysis to examine a community detection procedure. Ma *et al.* [4] considered traffic congestion problems with networks in community representations. For small-size communities, Li *et al.* [5] derived structural and functional properties. Ma *et al.* [6] studied community structure within networks by hidden metric space algorithm. According to the above brief discussion, we claim that only Lin *et al.* [2] provided a further examination of Zhang *et al.* [1].

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II. REVIEW OF LIN *ET AL.* [2]

Lin *et al.* [2] mentioned that the group action on sets developed by Zhang *et al.* [1] will cause too many cores, and then too many communities will be divided in the network. Lin *et al.* [2] used a well-known example constructed by Girvan and Newman [7] for a network of one hundred and twenty-eight nodes and four given communities to construct a network, and then nine communities were developed by the group action on sets studied by Zhang *et al.* [1]. It indicates that there are too many cores in the first step by the algorithm developed by Zhang *et al.* [1]. Moreover, Lin *et al.* [2] tried to construct a new approach to substitute the difficult group operation mentioned by Zhang *et al.* [1]. Consequently, Lin *et al.* [2] changed the function representation in Zhang *et al.* [1] by the range of the function except for those fixed points that are a set representation. Based on "set representation", Lin *et al.* [2] developed their new operation, and denoted it as an "intersect-union operation".

In this paper, we provide a further examination of the intersect-union operation of Lin *et al.* [2] to find that implicitly Zhang *et al.* [1] and Lin *et al.* [2] both assumed that within a given community, those clicks must intersect such that those clicks can produce only one core within the given community. However, we study the model of Girvan and Newman [7] to find that sometimes within a given community, those clicks do not intersect, and then the intersect-union operation proposed by Lin *et al.* [2] cannot merge those clicks. Hence, too many cores problems are still not improved by the intersect-union operation proposed by Lin *et al.* [2] for the network developed by Girvan and Newman [7].

We recall the results of Lin *et al.* [2] in the following that they tried to test the network one thousand times and to find that there are only one core 281 times, and on the other hand, there are two cores or more in the rest 719 times. Consequently, if Lin *et al.* [2] followed the procedure proposed by Zhang *et al.* [1] with the group action on sets, and then the percentage of there is just one core within each pre-designed community can be evaluated as $(0.281)^4 =$ 0.62% which is a very low possibility to imply the desired four communities.

The above estimation points out that the procedure developed by Zhang *et al.* [1] is questionable to handle the network proposed by Girvan and Newman [7].

III. FURTHER COMMENTS ON LIN *ET AL.* [2]

We can provide a hypothetical example with the number of cores in the four predesigned communities as 0, 2, 0, and 2 where cores are expressed as

$$\{33,34,35,36\},\ \{37,38,39,40\},$$
$$\{97,98,99,100\},\ \{101,102,103,104\}. \tag{3.1}$$

Following the iterative merging process proposed by Lin *et al.* [2], we assume the final communities as

$$\{1,2,...,36\},\ \{37,38,...,64\},$$
$$\{65,66,...,100\},\ \{101,102,...,128\}. \tag{3.2}$$

We apply the performance formula proposed by Lin *et al.* [2] to find that

$$\max \sum_{j=1}^{4} \left|P_j \cap R_i\right|$$
$$= \left|P_1 \cap R_1\right| + \left|P_2 \cap R_2\right| + \left|P_3 \cap R_3\right| + \left|P_4 \cap R_4\right|$$
$$= 32 + 28 + 32 + 28 = 120, \tag{3.3}$$

with $P_1 = \{1,2,...,32\}$, $R_1 = \{1,2,...,36\}$, $P_2 = \{33,34,...,64\}$, $R_2 = \{37,38,...,64\}$, $P_3 = \{65,66,...,96\}$, $R_3 = \{65,66,...,100\}$, $P_4 = \{97,98,...,128\}$ and $R_4 = \{101,102,...,128\}$.

We find the performance ratio is

$$\frac{120}{128} = 93.75\%, \tag{3.4}$$

that indicates the partition is very close to the predesigned construction proposed by Girvan and Newman [7].

The above hypothetical example with the special character that within each predesigned community, there is no one has just one core. However, the performance ratio can be very good. Hence, Lin *et al.* [2] only use one network to illustrate their prediction is questionable.

The above hypothetical example indicates that the probability of 0.281 proposed by Lin *et al.* [2] for only one core within the predesigned community with 32 nodes is only a partial finding which is not a sound criterion to criticize Zhang *et al.* [1].

IV. EXAMINATION OF THE MEASURE PROPOSED BY LIN *ET AL.* [2]

Girvan and Newman [7] developed a network with four given communities. Researchers applied different approaches to divide the network into communities and then they compared their findings with the original design of Girvan and Newman [7] to show the effectiveness of their approaches.

In the following, based on Song and Singh [8], we cite the Jaccard accuracy measure. We use G_1, G_2,..., G_n, to express those obtained communities and A_1, A_2,..., A_m, to indicate those originally designed communities.

There are three measures in Song and Singh [8]. We cite their first measure that is expressed as J_{G_i,A_j}, then

$$J_{G_i,A_j} = \frac{\left|G_i \cap A_j\right|}{\left|G_i \cup A_j\right|}. \tag{4.1}$$

Based on Equation (4.1), $|\ |$ is the number of how many nodes in the assigned region, to compute the ratio of overlaps between the obtained community G_i and the originally assigned community A_j. The intersection operation is used in the numerator, and the union operation is used in the denominator.

The second measure is defined as obtaining community G_i, that is, expressed as Jac_{G_i}, then

$$Jac_{G_i} = \max_{1\le j\le m} J_{G_i,A_j}. \tag{4.2}$$

Finally, the third measure for the applied approach, which is expressed as the Jaccard accuracy is defined as follows

$$Jac = \frac{\sum_{i=1}^{n} \left|G_i\right| Jac_{G_i}}{\sum_{i=1}^{n} \left|G_i\right|}. \tag{4.3}$$

Based on Equation (4.3), the value is between zero and one, and the higher the better to indicate that your partition is very close to the original design.

We point out that there is a computation error appeared in Lin *et al.* [2] and then we provide our improvement as follows,

$$\max \sum_{j=1}^{4} \left|P_j \cap R_i\right|$$
$$= \left|P_3 \cap R_1\right| + \left|P_4 \cap R_2\right| + \left|P_1 \cap R_3\right| + \left|P_2 \cap R_4\right|$$
$$= 32 + 32 + 15 + 17 = 96. \tag{4.4}$$

We compare the computation in Lin *et al.* [2] with the Jaccard accuracy measure to realize that Lin *et al.* [2] only considered the numerator of Equation (4.1), and overlooked the denominator of Equation (4.1). We begin to show that the similarity measure proposed by Lin *et al.* [2] contained questionable results.

We hypothetically construct a network related to Girvan and Newman [7] with the following four communities listed in the following Table 1.

Based on Table 1, we know that

$$Jac_{R_1} = J_{R_1,P_4} = \frac{1}{32}, \tag{4.5}$$

$$Jac_{R_2} = J_{R_2,P_4} = \frac{31}{32}, \tag{4.6}$$

$$Jac_{R_3} = J_{R_3,P_1} = J_{R_3,P_2} = J_{R_3,P_3} = \frac{15}{62}, \tag{4.7}$$

and

$$Jac_{R_4} = J_{R_4,P_1} = J_{R_4,P_2} = J_{R_4,P_3} = \frac{17}{66}. \tag{4.8}$$

Based on our findings of Equations (4.5-4.8), we obtain the Jaccard accuracy for the partition as follows,

Table 1. A hypothetical result for the network of Girvan and Newman [7].

	P_1	P_2	P_3	P_4
R_1				97
R_2				98,…,128
R_3	1 2 3 4 8 10 12 13 18 19 20 22 27 29 30	33 34 36 38 39 40 45 46 48 51 54 55 61 63 64	65 66 67 68 69 70 71 72 73 74 75 76 77 78 79	
R_4	5 6 7 9 11 14 15 16 17 21 23 24 25 26 28 31 32	35 37 41 42 43 44 47 49 50 52 53 56 57 58 59 60 62	80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96	

$$\text{Jac} = \frac{1(1/32) + 31(31/32) + 45(15/62) + 51(17/66)}{128} = 0.422547. \quad (4.9)$$

On the other hand, we apply the similarity measure proposed by Lin *et al.* [2] to find that

$$|P_1 \cap R_3| = |P_2 \cap R_3| = |P_3 \cap R_3| = 15, \quad (4.10)$$

and

$$|P_1 \cap R_4| = |P_2 \cap R_4| = |P_3 \cap R_4| = 17. \quad (4.11)$$

To evaluate $\max \sum_{j=1}^{4} |P_j \cap R_i|$, we know that $|P_4 \cap R_3| = 31$ is the best result. For the rest P_1, P_2 and P_3, there are six different possible outcomes:

$$|P_1 \cap R_4| + |P_2 \cap R_3| = 17 + 15, \quad (4.12)$$

$$|P_1 \cap R_4| + |P_3 \cap R_3| = 17 + 15, \quad (4.13)$$

$$|P_2 \cap R_4| + |P_1 \cap R_3| = 17 + 15, \quad (4.14)$$

$$|P_2 \cap R_4| + |P_3 \cap R_3| = 17 + 15, \quad (4.15)$$

$$|P_3 \cap R_4| + |P_1 \cap R_3| = 17 + 15, \quad (4.16)$$

and

$$|P_3 \cap R_4| + |P_2 \cap R_3| = 17 + 15. \quad (4.17)$$

Under the finding of Equation (16), researchers cannot further apply $|P_3 \cap R_4| = 17$, or $|P_3 \cap R_3| = 15$ such that researchers must use $|P_3 \cap R_1| = |P_3 \cap R_2| = 0$ to imply that

$$\max \sum_{j=1}^{4} |P_j \cap R_i| = 17 + 15 + 0 + 31 = 63 \quad (4.18)$$

and then we find the similarity measure proposed by Lin *et al.* [2] as

$$\frac{63}{128} = 0.492. \quad (4.19)$$

Now, we compare the findings of Equations (4.9) and (4.19) to derive that Lin *et al.* [2] also overestimated the similarity values for this hypothetical network.

In the following, we construct our second example to demonstrate the similarity measure proposed by Lin *et al.* [2] contained questionable results.

We observe Table 2 to find that

$$|P_1 \cap R_1| = |P_2 \cap R_1| = |P_3 \cap R_1| = |P_4 \cap R_1| = 15, \quad (4.20)$$

and

$$|P_1 \cap R_2| = |P_2 \cap R_2| = |P_3 \cap R_2| = |P_4 \cap R_2| = 17, \quad (4.21)$$

Table 2. Our second example for the network of Girvan and Newman [7].

	P_1	P_2	P_3	P_4
R_1	1 2 3 4 8 10 12 13 18 19 20 22 27 29 30	33 34 36 38 39 40 45 46 48 51 54 55 61 63 64	65 66 67 68 69 70 71 72 73 74 75 76 77 78 79	97 98 99 100 101 102 103 104 105 106 107 108 109 110 111
R_2	5 6 7 9 11 14 15 16 17 21 23 24 25 26 28 31 32	35 37 41 42 43 44 47 49 50 52 53 56 57 58 59 60 62	80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96	112 113 114 115 116 117 118 119 120 121 122 123 124 125 126 127 128

such that when we compute $\max \sum_{j=1}^{4} \left| P_j \cap R_i \right|$ if we decide $\left| P_1 \cap R_2 \right| = 17$ and $\left| P_2 \cap R_1 \right| = 15$, then we cannot further evaluate $\max \sum_{j=3}^{4} \left| P_j \cap R_i \right|$ because we run out all possible R_i for i=1, 2. Therefore, we construct a partition such that the similarity measure proposed by Lin *et al.* [2] failed to find its similarity measure. Hence, our second hypothetical example demonstrates the similarity measure proposed by Lin *et al.* [2] containing severe problems.

V. Direction for future research

Newman [9] studied modularity and community structure in networks and Radicchi *et al.* [10] defined and identified communities in networks that are worthy to mention to provide possible directions for future research.

VI. A Related Open Problem

In the second part of this paper, a solution to the open problem studied in the paper by Chang *et al.* [11] in which the algebraic method is applied to find the optimal solution for inventory models, published in the International Journal of Production Economics, is provided. The findings presented in this part will help practitioners accept the replenishment policy for inventory models without a background in calculus. Grubbström [12] applied the algebraic method to solve the EOQ model without referring to calculus so that more practitioners can realize inventory models without the knowledge of differential equations. Grubbström and Erdem [13] followed this trend to find the optimal solution for EOQ models with shortages. Cárdenas-Barrón [14] generalized to find the optimal solution for EPQ models. Ronald *et al.* [15] showed that the approaches of Grubbström and Erdem [13] and Cárdenas-Barrón [14] are too sophistic thus ordinary readers may not be able to follow their solution method and apply it to their research. Ronald *et al.* [15] provided a new approach, first to locate the minimum on each ray to construct a new minimum and then solve their new minimum problem. Chang *et al.* [11] pointed out that the two-stage algebraic procedure of Ronald *et al.* [15] is very difficult and presented a simplified method. We provided an outline of their method.

Based on Cárdenas-Barrón [14], Chang *et al.* [11] tried to solve the following inventory model,

$$C(Q,B) = -hB + B^2 \frac{h+b}{2Q\rho}$$
$$cD + \frac{KD}{Q} + Q\frac{\rho h}{2}, \quad (6.1)$$

where Q is the economic production quantity; B is the maximum shortage level; D is demand rate per unit time; P is production rate per unit time, with $P > D$; ρ is an abbreviation with $\rho = 1 - \frac{D}{P}$; b is the backorder cost; h is the holding cost per unit, per unit of time; K is the setup cost; c is the production cost per unit.

Chang *et al.* [11] rewrote Equation (6.1) as

$$C(Q,B) = \frac{h+b}{2Q\rho}\left(\frac{Q\rho h}{h+b} - B\right)^2$$
$$+ cD + \frac{KD}{Q} + \frac{Q\rho hb}{2(h+b)}, \quad (6.2)$$

to derive that

$$B = B(Q) = Q\frac{\rho h}{h+b}, \quad (6.3)$$

and then rewrote Equation (6.2) in one variable Q as

$$C(Q,B(Q)) = cD + \frac{DK}{Q} + \frac{Q\rho hb}{2(h+b)}. \quad (6.4)$$

Chang *et al.* [11] derived that

$$C(Q,B(Q)) = \left(\sqrt{\frac{KD}{Q}} - \sqrt{Q\frac{\rho hb}{2(h+b)}}\right)^2$$
$$+ cD + \sqrt{\frac{2KD\rho hb}{h+b}} \quad (6.5)$$

to locate the optimal production quantity,

$$Q^* = \sqrt{\frac{2KD(h+b)}{\rho hb}}, \quad (6.6)$$

and the optimal total cost,

$$C(Q,B(Q^*)) = cD + \sqrt{\frac{2KD\rho hb}{h+b}}, \quad (6.7)$$

and then the optimal maximum shortage level,

$$B^* = \sqrt{\frac{2KD\rho h}{(h+b)b}}. \quad (6.8)$$

Chang *et al.* [11] mentioned an alternate solution approach which was to rewrite Equation (6.1) as

$$C(Q,B) = \left(\sqrt{Q\frac{\rho h}{2}} - \sqrt{\left(B^2\frac{h+b}{2\rho} + KD\right)\frac{1}{Q}}\right)^2$$
$$- hB + cD + \sqrt{B^2(h+b)h + 2KD\rho h}, \quad (6.9)$$

to derive that

$$Q = Q(B) = \sqrt{\frac{2}{h\rho}\left(KD + B^2\frac{h+b}{2\rho}\right)}, \quad (6.10)$$

and then rewrote Equation (6.9) as

$$C(Q(B),B) = -Bh$$
$$cD + \sqrt{B^2(h+b)h + 2KDh\rho}. \quad (6.11)$$

Chang *et al.* [11] raised an open question of how to handle an optimal solution of Equation (6.11) in the variable B by algebraic methods. In the second part of this paper, a solution is provided for this open question proposed by Chang *et al.* [11].

This research problem is a hot topic for many papers because many practitioners are not familiar with the analytical skills to realize a system of partial differential equations, Riemann integration in one variable, and transformation from double integration to iterative integrations. We list some related studies in the following.

Lin [16] showed the questionable results of Chung and Cárdenas-Barrón [17], Cárdenas-Barrón [18], and Omar *et al.* [19], and for their incomplete algebraic methods. Lin [20] pointed out that the dispute proposed by Leung [21] to Teng [22] is not valid. The amendment developed by Leung [21] to separate the domain into three different regions are unnecessary. To separate the domain into three different regions is the calculus procedure since calculus cannot deal with critical solutions occurring on boundaries.

Feng [23] showed that Sphicas [24] did not offer any reasonings to explain the motivations proposed mentioned by Sphicas [25]. Lin [26] showed that Cárdenas-Barrón [27] violated the criteria of Cárdenas-Barrón [28] to produce a simple solution process. Lin [29] showed that Teng [22] implicitly used the findings of Wee et al. [30] to produce a simplified solution method without clearly informing the readers of the source of Wee et al. [30].

Lin *et al.* [31] claimed that Cárdenas-Barrón [18] derived an excellent algebraic method through the famous Cauchy-Bunyakovsky-Schwarz inequality for EOQ models. However, Cárdenas-Barrón [18] repeatedly applied the same excellent algebraic method for EPQ models that were redundant.

Lin and Deng [32] constructed a compact process to avoid the tedious algebraic algorithm developed by Wee *et al.* [33].

Tuan and Himalaya [34] studied a supply chain inventory model proposed by Xiao *et al.* [35]. Lan *et al.* [36] derived a compact version to replace the tedious solution process used by Cárdenas-Barrón [37].

Yen [38] constructed a simple method that offered an intuitive approach to reduce those difficult computations presented by Ronald *et al.* [15], Chang *et al.* [11], Luo and Chou [39], and Grubbström and Erdem [13]. Luo and Chou [39] dealt with an open problem mentioned by Chiu *et al.* [40] and Lau *et al.* [41]. Chiu *et al.* [40] developed revisions for several questionable findings in Lau *et al.* [41].

Lau *et al.* [41] generalized the open problem mentioned in Chang *et al.* [11] to find the conditions to guarantee the existence of the optimal minimum solution.

VII. Our Solution Approach

Based on Equation (6.11), we minimize $C(B)$, where

$$C(B) = -hB + cD + \sqrt{B^2(h+b)h + 2KD\rho h}\,. \quad (7.1)$$

We rearrange Equation (7.1) as

$$\sqrt{2KD\rho h + B^2(h+b)h} = Bh - cD + C(B) \quad (7.2)$$

and then square both sides and simplify the expression to yield

$$B^2hb + 2B(cD - C(B))h + 2h\rho KD = (cD - C(B))^2\,. \quad (7.3)$$

The square for the variable B is then completed, to treat $C(B)$ as a constant, that is, overlooking that $C(B)$ contains the variable B for the moment.

We derive that

$$hb\left(\left(\frac{C(B)-cD}{b}\right)^2 - 2\frac{B}{b}(C(B)-cD) + B^2\right) + 2KDh\rho = \frac{h+b}{b}(C(B)-cD)^2\,. \quad (7.4)$$

From Equation (7.1) with $\sqrt{B^2h(h+b)} \geq Bh$, it is known that $C(B) > cD$.

Based on Equation (7.4), if we take

$$B = \frac{C(B)-cD}{b} \quad (7.5)$$

then the left-hand side of Equation (7.4) will attain its minimum. Under the condition of Equation (7.5), Equation (7.4) is rewritten as

$$(C(B)-cD)^2 = 2KD\frac{\rho hb}{h+b}\,. \quad (7.6)$$

Owing to the condition $C(B) > cD$, we obtain that

$$C(B) = cD + \sqrt{KD\frac{2\rho hb}{h+b}}\,, \quad (7.7)$$

and then we rewrite Equation (7.5) as

$$B = \sqrt{KD\frac{2\rho h}{(h+b)b}}\,. \quad (7.8)$$

By comparing the results of Equation (7.7) and Equation (6.6) it is found that the same optimal cost is derived from the solution proposed in this paper. Moreover, comparing the findings of Equation (7.8) and Equation (6.8), then we also derive the optimal maximum shortage level.

The above comparisons demonstrate that the algebraic method proposed in this paper has handled an open problem mentioned by Chang *et al.* [11].

VIII. Comments for our method

Our approach provides an answer to an open problem raised by Chang *et al.* [11] concerning quadratic polynomials containing the square root. Our derivation has a special character such that we obtain the optimal cost first and then we find the optimal production quantity and optimal maximum shortage level.

This distinct feature should be useful for some traffic systems. Under that environment, researchers may overlook the optimal solutions for headway, optimal length, and optimal solution for width of the rectangular service area, but concentrate on the social benefit of the bus transit system.

Several related papers deserve to be mentioned: Kang *et al.* [42], Al-Sharu *et al.* [43], Yuan *et al.* [44], Canot *et al.* [45], Yen [46], Bu and Xu [47], Gaketem *et al.* [48], and Du and Sun [49] to help practitioners realize the current research trend.

IX. Construction of Linear Independent Family

There is a series of papers by Yen et al. [50], Chu et al. [51], Hung et al. [52], and Chou [53] that tried to develop algorithms to solve pattern recognition problems under discrete environments. During their development, they needed to construct a family of linearly independent vectors.

In the following, we will try to construct a new family in which they are linearly independent of each other.

X. Recall of Previous Results

First, we recall a theorem by Chu et al. [51] that mentioned that if $x + y + z = 1$, and $w \neq 0$, according to

$$\det\begin{bmatrix} x & y & z \\ x+2w & y+w & z-3w \\ x+w & y+2w & z-3w \end{bmatrix} = 3w^2(x+y+z) = 3w^2 \neq 0. \quad (10.1)$$

Based on the findings of Equation (10.1), we know that three vectors (x, y, z), $(x+2w,\ y+w,\ z-3w)$, and $(x+w,\ y+2w,\ z-3w)$ are three independent vectors.

Following the above discussion of Chu et al. [51], we compute the following

$$\det\begin{bmatrix} w_1 & w_2 & w_3 \\ w_1+\alpha & w_2-\alpha & w_3 \\ w_1+\alpha & w_2 & w_3-\alpha \end{bmatrix} = \det\begin{bmatrix} w_1 & w_2 & w_3 \\ \alpha & -\alpha & 0 \\ \alpha & 0 & -\alpha \end{bmatrix} = w_1\alpha^2 - w_2(-\alpha^2) + w_3(0-(-\alpha^2)) = (w_1+w_2+w_3)\alpha^2 = \alpha^2. \quad (10.2)$$

Based on Equation (10.2), we know that if $w_1 + w_2 + w_3 = 1$ and $\alpha \neq 0$, then three vectors (w_1, w_2, w_3), $(w_1+\alpha, w_2-\alpha, w_3)$, and $(w_1+\alpha, w_2, w_3-\alpha)$ are independent.

The pending problem is to generalize the above results to any finite-dimensional vector space.

XI. Our new results

We will extend our results from three independent vectors to generalize to four independent vectors.

The determinant of four vectors, (w_1, w_2, w_3, w_4), $(w_1+\alpha, w_2-\alpha, w_3, w_4)$, $(w_1+\alpha, w_2, w_3-\alpha, w_4)$ and $(w_1+\alpha, w_2, w_3, w_4-\alpha)$ is evaluated as follows,

$$\det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ w_1+\alpha & w_2-\alpha & w_3 & w_4 \\ w_1+\alpha & w_2 & w_3-\alpha & w_4 \\ w_1+\alpha & w_2 & w_3 & w_4-\alpha \end{bmatrix}. \quad (11.1)$$

We apply the row operation to simplify Equation (11.1) to yield that

$$\text{determinant} = \det\begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ \alpha & -\alpha & 0 & 0 \\ \alpha & 0 & -\alpha & 0 \\ \alpha & 0 & 0 & -\alpha \end{bmatrix}. \quad (11.2)$$

We apply the first-row expansion of Equation (11.2) to derive that the determinant,

$$w_1(-\alpha^3) - w_2\alpha^3 + w_3(-\alpha^3) - w_4\alpha^3 = (w_1+w_2+w_3+w_4)(-\alpha^3) = -\alpha^3, \quad (11.3)$$

under the condition of $w_1 + w_2 + w_3 + w_4 = 1$.

Based on our above derivations, we will solve the following open question for the future researcher:

To show that the determinant of $(w_1, w_2, \ldots, w_n)$, $(w_1+\alpha, w_2-\alpha, \ldots, w_4)$, $(w_1+\alpha, w_2, w_3-\alpha, \ldots, w_4)$ and $(w_1+\alpha, w_2, w_3, \ldots, w_{n-1}, w_4-\alpha)$ is $(-1)^{n-1}\alpha^{n-1}(w_1+w_2+\ldots+w_n)$.

We assume that a matrix, $(a_{ij})_{n\times n}$ with $a_{1j} = w_j$, $w_j \geq 0$ and $\sum_{i=1}^{n} w_i = 1$ for $j = 1,\ldots,n$ and for $i = 2,\ldots,n$, $a_{ij} = w_j + \alpha\Delta_{ij}$ with $\alpha > 0$, $\Delta_{i1} = 1$, $\Delta_{ii} = -1$, $\Delta_{ij} = 0$ otherwise, and then we will prove $\det(a_{ij})_{n\times n} = (-\alpha)^{n-1}$.

The third kind of elementary row operation will not change the value of a determinant so if we use $(-1)R_1 + R_i$ to work on the i th row for the matrix $(a_{ij})_{n\times n}$, for $i = 2,\ldots,n$, then $\det(a_{ij})_{n\times n} = \det(b_{ij})_{n\times n}$ where $b_{1j} = w_j$ for $j = 1,\ldots,n$ and for $i = 2,\ldots,n$, $b_{ij} = \alpha\Delta_{ij}$.

The second kind of elementary row operation will change the value of a determinant. If we multiply one row by x that will result in x multiple the original determinant to be the new determinant. Hence, $\det(a_{ij})_{n\times n} = \alpha^{n-1}\det(c_{ij})_{n\times n}$ where $c_{ij} = b_{ij}/\alpha$ for $i = 2,\ldots,n$. We derive that $c_{1j} = w_j$ for $j = 1,\ldots,n$, $c_{i1} = 1$ and $c_{ii} = -1$, for $i = 2,\ldots,n$.

For the determinant of the matrix $(c_{ij})_{n\times n}$, we will use the first column expansion, such that we assume $(d_{ij}^{k})_{(n-1)\times(n-1)}$ for $k = 1,\ldots n$ such that $(d_{ij}^{k})_{(n-1)\times(n-1)}$ is the remaining matrix of $(c_{ij})_{n\times n}$ to delete the first column and the k th row, and then we know that

$$\det(c_{ij})_{n\times n} = (-1)^{1+1} w_1 \det(d_{ij}^{1})_{(n-1)\times(n-1)} + \sum_{k=2}^{n}(-1)^{1+k}\det(d_{ij}^{k})_{(n-1)\times(n-1)}. \quad (11.4)$$

We know that $(d_{ij}^{1})_{(n-1)\times(n-1)} = (-1)I_{(n-1)\times(n-1)}$ where $I_{(n-1)\times(n-1)}$ is the identity matrix then

$$\det(d_{ij}^{1})_{(n-1)\times(n-1)} = (-1)^{n-1}. \quad (11.5)$$

For $k = 2,\ldots,n$,

$$d_{ij}^{k} = \begin{cases} c_{i(j+1)} : i < k \\ c_{(i+1)(j+1)} : i \geq k \end{cases}, \quad (11.6)$$

and then it results in $d_{1j}^{k} = w_{j+1}$ for $j = 1,\ldots,n-1$, and

$$d_{ij}^{k}=\begin{cases}-1: i=j+1<k\\ -1: i=j\geq k\\ 0: \text{otherwise}\end{cases}, \qquad (11.7)$$

for $i=2,...,n-1$.
We know that

$$d_{i(k-1)}^{k}=\begin{cases}w_k: i=1\\ 0: \text{otherwise}\end{cases}, \qquad (11.8)$$

owing to $i=(k-1)+1<k$ and $i=k-1\geq k$ both will not happen.

With $k=2,...,n$, for the determinant of the matrix $\left(d_{ij}^{k}\right)_{(n-1)\times(n-1)}$, we use the $(k-1)$th column expansion. Hence, we assume that $\left(e_{ij}^{k}\right)_{(n-2)\times(n-2)}$ as $\left(d_{ij}^{k}\right)_{(n-1)\times(n-1)}$ deleting the first row and the $(k-1)$th column implies that

$$e_{ij}^{k}=\begin{cases}d_{(i+1)(j+1)}^{k}: j\geq k-1\\ d_{(i+1)j}^{k}: j\leq k-2\end{cases}. \qquad (11.9)$$

We derive the following four cases:
(a) when $j\geq k-1$ then $e_{ij}^{k}=d_{(i+1)(j+1)}^{k}=-1$ and $i+1=j+1+1<k$ will not happen;
(b) when $j\geq k-1$ then $e_{ij}^{k}=d_{(i+1)(j+1)}^{k}=-1$ and $i+1=j+1\geq k$ that is valid for $i=j=k-1,...,n$;
(c) when $j\leq k-2$ then $e_{ij}^{k}=d_{(i+1)j}^{k}=-1$ and $i+1=j+1<k$ that is valid for $i=j=1,...,k-2$;
(d) when $j\leq k-2$ then $e_{ij}^{k}=d_{(i+1)j}^{k}=-1$ and $i+1=j\geq k$ will not happen. Therefore, we obtain that $\left(e_{ij}^{k}\right)_{(n-2)\times(n-2)}=(-1)I_{(n-2)\times(n-2)}$.

Consequently, we find that

$$\det\left(d_{ij}^{k}\right)_{(n-1)\times(n-1)}=(-1)^{1+k-1}w_k\det\left(e_{ij}^{k}\right)_{(n-2)\times(n-2)}$$
$$=(-1)^{k}w_k(-1)^{n-2}, \qquad (11.10)$$

and then

$$\det\left(c_{ij}\right)_{n\times n}=w_1(-1)^{n-1}+\sum_{k=2}^{n}(-1)^{k+1}(-1)^{k}w_k(-1)^{n-2}$$
$$=(-1)^{n-1}\sum_{k=1}^{n}w_k=(-1)^{n-1}. \qquad (11.11)$$

At last, we obtain that

$$\det\left(a_{ij}\right)_{n\times n}=(-\alpha)^{n-1}. \qquad (11.12)$$

Owing to $\det\left(a_{ij}\right)_{n\times n}\neq 0$, we know that $\{R_1,...,R_n\}$ are linear independent where R_i is the ith row of the matrix $\left(a_{ij}\right)_{n\times n}$.

XII. Conclusion

In this paper, we point out several questionable results in Lin *et al.* [2] that will help researchers to realize (a) the group action on sets proposed by Zhang *et al.* [1], and (b) the intersection-union operation proposed by Lin *et al.* [2] to provide an improved partition of networks into communities. Moreover, we provide a positive answer to an open problem developed by Chang *et al.* [11]. Last, but not least, we construct a new family of independent vectors.

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