

# Fear Effect on a Modified Leslie-Gower Predator-Prey Model with Disease Transmission in Prey Population

Anna Silvia Purnomo, Isnani Darti, Agus Suryanto, and Wuryansari Muharini Kusumawinahyu

**Abstract**—A model that represents the effect of fear on predator-prey interaction with disease transmission in the prey population is considered in this research. The modified Leslie-Gower predator-prey model is adopted here. The prey population is classified into susceptible and infected classes. The fear of prey to the presence of predators reduces the growth rate of susceptible prey as well as the rate of interaction among the prey population. These phenomena are represented by two fear parameters in the model. The first acts as the inhibitor of the growth of susceptible prey, while the second plays as the inhibitor of disease transmission. The existence and local stability of each equilibrium is investigated analytically. It is shown that fear controls the local stability of the two of six equilibrium points, the disease-free and the interior equilibrium point. Furthermore, it is shown that the model undergoes a Hopf bifurcation driven by the fear rate. Some numerical simulations are conducted to illustrate the analytical results.

**Index Terms**—fear, disease transmission, Leslie-Gower, Hopf bifurcation, stability

## I. INTRODUCTION

INTERACTIONS between predators and prey are crucial for determining community structure and preserving ecological variety. For more than a century, predator-prey systems have been extensively studied. For example, Leslie-Gower [1] proposed an additional component to the predator-prey model, referring to the logistic rule-based development of the interrelated population and the relationship between the number of prey and the carrying capacity of the predator. When prey is scarce, the predators can move to find alternative food, but this will limit their ability to develop because their main prey is not available. In [2], Aziz-Aloui and Okiye proposed a modified Leslie-Gower model to address this issue by adding a positive quantity that estimates the environmental defense of the predators.

Recently, eco-epidemiology, which involves the interaction between predator and prey populations with the spreading of disease has been studied. Based on the Leslie-Gower

equations, Zhou, et. al [5] proposed an eco-epidemiological model with disease in the prey population. It is assumed that the spreading of disease follows a bilinear incidence rate and that the predator just consumes infectious prey. Suryanto [6] modified the model of Zhou, et. al [5] by assuming the saturation incidence rate. Additionally, Suryanto, et. al [9] investigated the Leslie-Gower eco-epidemic model, in which a predator only consumes susceptible prey. Purnomo, et. al [7] studied the impact of prey harvesting on the model proposed in [5]. Panigoro, et. al [11] proposed the eco-epidemic model where predator consumes susceptible and infectious prey. In that model, Holling type-I functional response was used. In 2021, Panigoro, et. al [12] developed a model in [11] by using Holling type-II functional response. Meng [13] and Bahlool [14] studied the eco-epidemiological model in which the predator hunts for susceptible and infectious prey with harvesting. Some studies had been conducted on eco-epidemiological models with the spread of disease only on predators [21], [22], [23], as well as on both populations [10], [24], [25], [15], [8].

The majority of studies solely used direct killing. The predators have different ways of affecting prey populations and fear of predation significantly reduces population fitness [3], [4]. Zanette, et. al [26] obtained the result that showed the number of offspring of song sparrows can be reduced by 40% to their fear to the predator. Prey may alter their behavior as a result of this fear. It is possible that prey spends more time keeping an eye out for predators. This behavior causes a reduction in the number of eggs birds and their hatchings [26]. Based on Zanette, et. al [26], Wang, et. al [27] proposed a predator-prey model with fear effect by applying Holling type II functional response. By including age-stage structure, Wang and Zou [28] investigated a prey-predator model with adaptive avoidance of predators. A prey-predator model that included the fear effect and additional food was studied by Mondal, et. al [29]. Mondal and Samantha [34] and Huang, et. al [33] proposed prey-predator systems with refuge. Prey-predator models that incorporate the fear effect with various functional response functions have been the subject of study by some researchers: Beddington-DeAngelis response function [30], Holling type-IV response function [32], Hassel-Varley response function [36], and Holling type-III response function [38]. Further, some researchers have proposed the prey-predator models with fear effect and harvesting [39], [40]. Pal, et. al [31] investigated a Leslie-Gower model incorporating the fear effect and hunting cooperation among predators. It was shown that the fear factor can stabilize the equilibrium. Increasing the strength of hunting cooperation or the cost of fear makes the equilibrium

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a stable focus equilibrium. They noted that by increasing hunting cooperation and fear factors, the sizes of both prey and predator populations will decrease. As well, Pal, et. al [31], Ghosh [48] studied fear effect on Lotka-Volterra prey-predator model with hunting cooperation. Sasmal and Takeuchi [32] studied the Lotka-Volterra model with fear effect and Monod-Haldane type functional response. They discussed the multi-stability, direction, and stability of Hopf-bifurcation. Kumar and Kumari [36] studied the stability and bifurcation analysis of the Lotka-Volterra model with fear effect and Hassel-Varley functional response. In their research, they concluded that the increase in the fear factor can eliminate the limit cycle. Mukharjee [44] and Gökce [45] including intraspecific competition among predator population. In 2022, Mukharjee [43] proposed predator-prey system with fear and Holling type-IV functional response. Khan et al. [41] studied the effects of fear in the predator-prey model, using the model from Ruxton and Lima [42] as the basic model. In [41], the prey population was divided into two subpopulations which are the breeder and suppressor prey. Their results showed the cost of fear to the predator by which suppressors turn into breeders and vice versa affected the dynamics of the solution model. Some research involved the time delay in the predator-prey model with fear [37], [46], [47].

One of the intriguing recent areas in mathematical modeling research is the investigation of the eco-epidemiological model adding the fear effect. Sha, et. al [20] investigated a Lotka-Volterra eco-epidemiological model incorporating fear factors and disease in the prey population. In the model, it is presumable that fear has two effects: it slows both the rate of prey population growth and the rate of disease transmission. Their results showed that fear can exhibit bistability and backward bifurcation. In order to study the impact of fear in the model, Sha, et. al [20] considered two situations, when the eco-epidemiological system is in stable dynamics and when the system in limit cycle dynamics. When the endemic equilibrium is stable, increasing the fear strength causes the disease to go out of the system and the system becomes an ecological model. The same results became evident with limit cycle dynamics. The fear effect also has several impacts on backward bifurcation and bistability. A Lotka-Volterra eco-epidemiological model with disease transmission in the prey population and the fear effect was taken into consideration by Liu, et. al [17]. They used Holling type-I functional response. Similar to the results of Sha, et. al [20], in this model fear effect can affect the stability of the system. In [21], [22], [23], the disease spread only in the predator population. Barman et al. [21] obtained the results that a high rate of fear induced by infected predators can destabilize the system. Pal [22] found that a high level of the strength of fear can stabilize the equilibrium by excluding the oscillations solution. In [16], [18], and [19], the eco-epidemiological model with fear, for which it was assumed that predators consume susceptible and infectious prey was investigated. Baishya, et. al [24] and [25] studied the dynamics of the eco-epidemiological model in which the disease spread in both populations.

In a modified Leslie-Gower eco-epidemiological model by [5], which is presented in the following section, we investigate the effects of the fear effect. Section III presents studies on the positivity and boundedness of the model.

Sections IV and V, respectively, present the conditions for the existence of each equilibrium as well as its local stability. The Hopf bifurcation caused by the fear parameter is examined in the following section. In Section VII, we run some numerical simulations to validate our analytical findings. We discuss the biological significance of our findings in the final section.

## II. THE MODEL

Zhou, et. al in 2009 considered a modified Leslie-Gower predator-prey model incorporating disease transmission in the prey population. There are two categories of prey: susceptible and infected prey. The disease is thought to spread horizontally or through close physical contact between susceptible and infected individuals. Infected individuals can not recover. The incidence rate is assumed following bilinear incidence with  $\beta$  being the rate of transmission. It is also assumed that predator only consumes infected prey with Holling-type II schemes. The following eco-epidemiological model has been considered in [5]:

$$\begin{aligned} \frac{du_1(t)}{dt} &= \left( r_1 \left( 1 - \frac{u_1(t) + u_2(t)}{K} \right) - \beta u_2(t) \right) u_1(t), \\ \frac{du_2(t)}{dt} &= \left( \beta u_1(t) - e - \frac{bv(t)}{u_2(t) + K_1} \right) u_2(t), \\ \frac{dv(t)}{dt} &= \left( r_2 - \frac{a_2 v(t)}{u_2(t) + K_2} \right) v(t), \end{aligned} \quad (1)$$

where  $u_1(t)$ ,  $u_2(t)$ , and  $v(t)$  are the size of the susceptible prey, infectious prey, and predator at time  $t$ , respectively. Here,  $r_1$  is the intrinsic growth rate of the susceptible prey and  $K$  is the carrying capacity of the prey population. Parameter  $e$  represents the death rate of the infectious prey due to the disease. The predation rate and competition rate are denoted by  $b$  and  $a_2$ , respectively.  $K_i$  ( $i=1,2$ ) are the parameters representing environment protection for infectious prey and predator, respectively. The growth rate of predators is  $r_2$ .

In this article, we investigate the role that fear plays in the eco-epidemiological model (1). It is assumed that fear reduces the growth rate of susceptible prey and reduces interaction between susceptible and infectious prey. To capture these phenomenons, we multiply  $\frac{1}{1 + m_1 v}$  with the growth term of the susceptible prey and multiply  $\frac{1}{1 + m_2 v}$  with the term of interaction among susceptible and infectious prey. The modified the model (1) is represented as the following system:

$$\begin{aligned} \frac{u_1(t)}{dt} &= \left( \frac{r_1}{1 + m_1 v(t)} \left( 1 - \frac{u(t)}{K} \right) - \frac{\beta u_2(t)}{1 + m_2 v(t)} \right) u_1(t), \\ \frac{u_2(t)}{dt} &= \left( \frac{\beta u_1(t)}{1 + m_2 v(t)} - e - \frac{bv(t)}{u_2(t) + K_1} \right) u_2(t), \\ \frac{v(t)}{dt} &= \left( r_2 - \frac{a_2 v(t)}{u_2(t) + K_2} \right) v(t), \end{aligned} \quad (2)$$

where  $u(t) = u_1(t) + u_2(t)$ ,  $m_1$  and  $m_2$  represent the level of fear reducing susceptible's growth and the level of fear reduces interaction between susceptible and infectious individuals.

III. POSITIVITY AND BOUNDEDNESS

**Theorem 1.** The system (2) with initial condition  $u_1(0) > 0, u_2(0) > 0, v(0) > 0$  is positively invariant for all  $t > 0$ .

**Proof.** The solution of (2) is as follows.

$$\begin{aligned}
 u_1(t) &= u_1(0) \exp \int_0^t \left( \frac{r_1}{1+m_1v(q)} \left( 1 - \frac{u(q)}{K} \right) - \frac{\beta u_2(q)}{1+m_2v(q)} \right) dq > 0, \\
 u_2(t) &= u_2(0) \exp \int_0^t \left( \frac{\beta u_1(q)}{1+m_2v(q)} - e - \frac{bv(q)}{u_2(q)+K_1} \right) dq > 0, \\
 u_3(t) &= u_3(0) \exp \int_0^t \left( r_2 - \frac{a_2v(q)}{u_2(q)+K_2} \right) dq > 0.
 \end{aligned} \tag{3}$$

From (3), it is concluded that any solution of system (2) with initial condition  $u_1(0) > 0, u_2(0) > 0, v(0) > 0$  is element of  $\mathbb{R}_3^+$ .

**Theorem 2.** With assumption  $u(0) \geq K$ , either

- 1)  $u(t) \geq K$  for all non-negative  $t$  and hence, as  $t \rightarrow \infty$ ,  $(u_1(t), u_2(t), v(t)) \rightarrow \left( K, 0, \frac{r_2K_2}{a_2} \right)$ , or
- 2) there is a positive value  $t_1$  such that  $u(t) < K$  for all  $t > t_1$ .

If  $u(0) < K$ , then  $u(t) < K$  for all positive  $t$ .

**Proof.** It is considered that  $u(t) \geq K$  for all  $t \geq 0$ . From system (2), we get

$$\frac{du(t)}{dt} = \frac{r_1u_1(t)}{1+m_1v(t)} \left( 1 - \frac{u(t)}{K} \right) - eu_2(t) - \frac{bu_2(t)v(t)}{u_2(t)+K_1}. \tag{4}$$

Therefore, for all  $t \geq 0$ , we obtain  $u'(t) \leq 0$ . Let

$$\lim_{t \rightarrow +\infty} u(t) = \zeta. \tag{5}$$

For  $\zeta > K$ , by using the Barbălat lemma, we have

$$\begin{aligned}
 0 &= \lim_{t \rightarrow +\infty} \frac{d}{dt} u(t) \\
 &= \lim_{t \rightarrow +\infty} \left[ \frac{r_1u_1(t)}{1+m_1v(t)} \left( 1 - \frac{u(t)}{K} \right) - eu_2(t) - \frac{bu_2(t)v(t)}{u_2(t)+K_1} \right] \\
 &= \lim_{t \rightarrow +\infty} \left[ \frac{r_1u_1(t)}{1+m_1v(t)} \left( 1 - \frac{\zeta}{K} \right) - eu_2(t) - \frac{bu_2(t)v(t)}{u_2(t)+K_1} \right] \\
 &\leq \lim_{t \rightarrow +\infty} \left[ r_1 \left( 1 - \frac{\zeta}{K} \right) - eu_2(t) - \frac{bu_2(t)v(t)}{u_2(t)+K_1} \right] \\
 &= - \lim_{t \rightarrow +\infty} \left[ r_1 \left( \frac{\zeta}{K} - 1 \right) + eu_2(t) + \frac{bu_2(t)v(t)}{u_2(t)+K_1} \right] \\
 &\leq - \min \left\{ r_1 \left( \frac{\zeta}{K} - 1 \right), e \right\} \lim_{t \rightarrow +\infty} u(t) \\
 &= -\zeta \min \left\{ r_1 \left( \frac{\zeta}{K} - 1 \right), e \right\} < 0.
 \end{aligned}$$

There is a contradiction. Furthermore, it is shown that  $\zeta = K$ . Furthermore,

$$\lim_{t \rightarrow +\infty} u(t) = K. \tag{6}$$

Due to  $u(t)$  is differentiable and the derivative of  $u(t)$  is uniformly continuous for positive  $t$  and (6), we have

$$\lim_{t \rightarrow +\infty} u'(t) = 0. \tag{7}$$

We know that (6) and (7) are applied if and only if  $\lim_{t \rightarrow +\infty} u_2(t) = 0$  and  $\lim_{t \rightarrow +\infty} u_1(t) = K$ . It implies  $v(t) \rightarrow \frac{r_2K_2}{a_2}$  as  $t \rightarrow +\infty$ .

We presume that Theorem 2's point 1 does not apply. There exists the positive  $t_1$  which is at  $t = t_1$  the value of  $u(t) = K$  for the first time. Based on (3), we have

$$\frac{d}{dt} u(t) |_{t=t_1} = -eu_2(t_1) - \frac{bu_2(t_1)v(t_1)}{u_2(t_1)+K_1} < 0. \tag{8}$$

This result implies that once the value  $u(t) = K$  then for all  $t > t_1$  it is shown that

$$u(t) < K. \tag{9}$$

The last one, if  $u(0) < K$ , by applying the previous result, we get  $u(t) < K$  for all  $t > t_1$ . The proof is complete.

**Theorem 3.** There exists value  $M$  such that for any positive solution  $u_1(t), u_2(t), v(t)$  of (2),  $v(t) < M$  for all large  $t$ .

**Proof.** For  $v(0) > 0$  we have  $v(t) > 0$ . From the third equation of (2) and the previous results, we obtain

$$\frac{dv(t)}{dt} \leq \left( r_2 - \frac{a_2v(t)}{K + \epsilon + K_2} \right) v(t). \tag{10}$$

By taking  $\epsilon \rightarrow 0$ , we have

$$\frac{dv(t)}{dt} \leq \left( r_2 - \frac{a_2v(t)}{K + K_2} \right) v(t). \tag{11}$$

Furthermore, by applying the standard comparison theorem, we get

$$v(t) \leq M, \tag{12}$$

with  $M = \frac{r_2(K + K_2)}{a_2}$ .

IV. EQUILIBRIA

Model (2) has six equilibria as follows:

- 1) the trivial equilibrium  $E_1(0, 0, 0)$ ,
- 2) the disease-free and predator-free equilibrium  $E_2(K, 0, 0)$ ,
- 3) the prey-free equilibrium  $E_3\left(0, 0, \frac{r_2K_2}{a_2}\right)$ ,
- 4) the predator-free equilibrium  $E_4\left(\frac{e}{\beta}, \frac{r_1(\beta K - e)}{\beta(r_1 + \beta K)}, 0\right)$  which is exists when  $\beta K > e$ ,
- 5) the disease-free equilibrium  $E_5\left(K, 0, \frac{r_2K_2}{a_2}\right)$ ,
- 6) the interior equilibrium  $E^*(u_1^*, u_2^*, v^*)$ , where

$$\begin{aligned}
 u_1^* &= \frac{a_2 + m_2r_2(u_2^* + K_2)}{\beta a_2} \left( e + \frac{br_2(u_2^* + K_2)}{a_2(u_2^* + K_1)} \right) \\
 v^* &= \frac{r_2u_2^*}{a_2} + \frac{r_2K_2}{a_2}.
 \end{aligned}$$

$u_2^*$  is the positive root(s) of the following polynomial

$$Q_3(u_2^*)^3 + Q_2(u_2^*)^2 + Q_1(u_2^*) + Q_0 = 0, \tag{13}$$

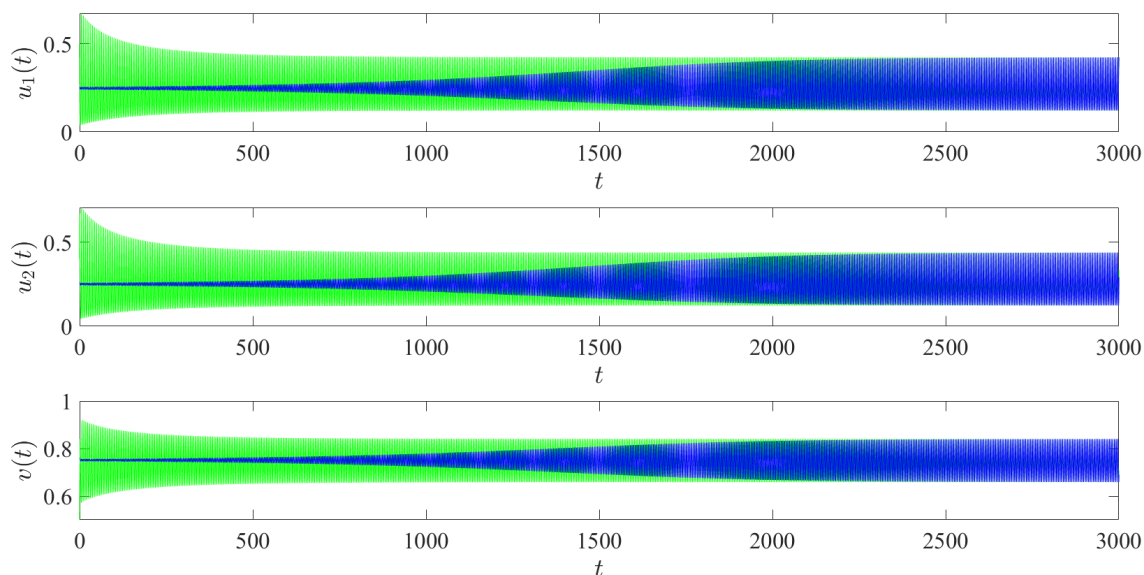


Fig. 1. Time series when  $m_2 = 0.62 < 0.6432 = m_2^*$  and  $m_1 = 0.5$ . The solution is chaotic. The initial point of blue is  $(0.254, 0.258, 0.76)$  while the green is  $(0.5, 0.4, 0.5)$ .

with

$$\begin{aligned}
 Q_3 &= -r_2\beta(K\beta m_1 + m_2 r_1)a_2^2 - r_2^2 m_2^2 r_1 e a_2 \\
 &\quad - r_2^3 b m_2^2 r_1 \\
 Q_2 &= -\beta(K\beta + r_1)a_2^3 - 3r_2^3 b m_2^2 r_1 K_2 \\
 &\quad - a_2(2r_2^2 m_2^2 r_1 K_2 e + r_2^2 m_2 r_1(K_1 e m_2 + 2b)) \\
 &\quad - a_2^2(r_2 K_2 \beta(K\beta m_1 + m_2 r_1)) \\
 &\quad - a_2^2(r_2(-(K\beta - K_1\beta - 2e)m_2 r_1)) \\
 &\quad - a_2^2 r_2 K \beta^2 K_1 m_1 \\
 Q_1 &= -a_2^3(\beta K_1(\beta K - e) - r_1(\beta K - e)) \\
 &\quad - a_2^2 r_2 K_2(-(K\beta - K_1\beta - 2e)m_2 r_1) \\
 &\quad - a_2^2 r_2 K_2(K\beta^2 K_1 m_1) \\
 &\quad + a_2^2 r_1 r_2(K_1(\beta K - 2e)m_2 - b) \\
 &\quad - a_2(2r_2^2 m_2 r_1 K_2(K_1 e m_2 + 2b)) \\
 &\quad - a_2 r_2^2 m_2^2 r_1 K_2^2 e - 3r_2^3 b m_2^2 r_1 K_2^2 \\
 Q_0 &= K_1 r_1(\beta K - e)a_2^3 \\
 &\quad + r_2 K_2(K_1(\beta K - 2e)m_2 - b)r_1 a_2^2 \\
 &\quad - r_2^2 m_2 r_1 K_2^2(K_1 e m_2 + 2b)a_2 - r_2^3 b m_2^2 r_1 K_2^3.
 \end{aligned}$$

First, we express (13) as following equation.

$$(u_2^*)^3 + W_2(u_2^*)^2 + W_1(u_2^*) + W_0 = 0, \quad (14)$$

where

$$\begin{aligned}
 W_2 &= \frac{Q_2}{3Q_3}, \\
 W_1 &= \frac{Q_1}{3Q_3}, \\
 W_0 &= \frac{Q_0}{Q_3}.
 \end{aligned}$$

By using the transformation  $z^* = u_2^* + W_2$ , we have

$$(z^*)^3 + 3z^*\Phi_1 + \Phi_2 = 0, \quad (15)$$

where  $\Phi_1 = W_1 - W_2^2$  dan  $\Phi_2 = 2W_2^3 - 3W_1W_2 + W_0$ ,  $\Phi_1 = W_1 - W_2^2$ , and  $\Phi_2 = 2W_2^3 - 3W_1W_2 + W_0$ . The existence of positive root(s) of equation (15) can be determined by Cardan's method [49] as follows.

- 1) If  $\Phi_2 < 0$  then (15) has a unique positive root.
- 2) If  $\Phi_2 > 0, \Phi_1 < 0$ :
  - a) if  $\Phi_2^2 + 4\Phi_1^3 = 0$  then (15) has two same positive roots,
  - b) if  $\Phi_2^2 + 4\Phi_1^3 < 0$  then (15) has two distinct positive roots.

If  $\Phi_2 = 0$  and  $\Phi_1 < 0$  then (15) has a unique positive root.

The positive root(s) of (15) is (are)

$$z_1^* = \frac{\sqrt[3]{(-\Phi_2 + 4\sqrt{4\Phi_1^3 + \Phi_2^2})^2 - 4\Phi_1}}{2\sqrt[3]{(-\Phi_2 + 4\sqrt{4\Phi_1^3 + \Phi_2^2})^2}}$$

$$\left(\text{and } z_2^* = -\frac{z_1^*}{2} + \frac{\sqrt{z_1^{*3} + 4\Phi_2}}{2\sqrt{z_1^*}}\right).$$

## V. LOCAL STABILITY

Let  $\hat{E}(u_1, u_2, v)$  is the equilibrium. The Jacobian matrix at  $\hat{E}$  is given by

$$J(\hat{E}) = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix},$$

where

$$\begin{aligned}
 j_{11} &= \frac{r_1}{1+m_1\hat{v}} \left( 1 - \frac{2\hat{u}_1}{K} - \frac{\hat{u}_2}{K} \right) - \frac{\beta\hat{u}_2}{1+m_2\hat{v}}, \\
 j_{12} &= - \left( \frac{r_1\hat{u}_1}{K(1+m_1\hat{v})} + \frac{\beta\hat{u}_1}{1+m_2\hat{v}} \right), \\
 j_{13} &= \frac{r_1m_1\hat{u}_1}{(1+m_1\hat{v})^2} \left( \frac{\hat{u}_1+\hat{u}_2}{K} - 1 \right) + \frac{\beta\hat{u}_1\hat{u}_2m_2}{(1+m_1\hat{v})^2}, \\
 j_{21} &= \frac{\beta\hat{u}_2}{1+m_2\hat{v}}, \\
 j_{22} &= \frac{\beta\hat{u}_1}{1+m_2\hat{v}} - e - \frac{b\hat{v}}{\hat{u}_2+K_1} + \frac{\beta\hat{u}_2\hat{v}}{(\hat{u}_2+K_1)^2}, \\
 j_{23} &= - \left( \frac{\beta m_2\hat{u}_1\hat{u}_2}{(1+m_2\hat{v})^2} + \frac{b\hat{u}_2}{\hat{u}_2+K_1} \right), \\
 j_{31} &= 0, \\
 j_{32} &= \frac{a_2\hat{v}^2}{(\hat{u}_2+K_2)^2}, \\
 j_{33} &= r_2 - \frac{2a_2\hat{v}}{\hat{u}_2+K_2}.
 \end{aligned}$$

**Theorem 4.** The trivial equilibrium  $E_1(0, 0, 0)$ , the disease-free and predator-free equilibrium  $E_2(K, 0, 0)$ , and prey-free equilibrium  $E_3\left(0, 0, \frac{r_2K_2}{a_2}\right)$  of system (2) are the saddle point.

**Proof.** The Jacobian matrix of system (2) at  $E_1(0, 0, 0)$  is

$$J(E_1) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & -e & 0 \\ 0 & 0 & r_2 \end{bmatrix}.$$

The eigenvalues of  $J(E_1)$  are  $\lambda_1 = r_1, \lambda_2 = -e$ , and  $\lambda_3 = r_2$ . Due to  $\lambda_1$  and  $\lambda_3$  are positive, the extinction equilibrium  $E_1(0, 0, 0)$  is a saddle point. On the other hand, the following matrix is the Jacobian matrix of system (2) at  $E_2(K, 0, 0)$ .

$$J(E_2) = \begin{bmatrix} -r_1 & -r_1 - \beta K & 0 \\ 0 & \beta K - e & 0 \\ 0 & 0 & r_2 \end{bmatrix}.$$

There are three eigenvalues of  $J(E_2)$ , i.e.  $\lambda_1 = -r_1, \lambda_2 = \beta K - e$ , and  $\lambda_3 = r_2$ . There is an eigenvalue that is always positive, that is  $\lambda_3$ . It is proven that  $E_2(K, 0, 0)$  is a saddle point. The last one, the Jacobian matrix of system (2) at  $E_3\left(0, 0, \frac{r_2K_2}{a_2}\right)$  is given as follows.

$$J(E_3) = \begin{bmatrix} \frac{r_1a_2}{a_2+m_1r_2K_2} & 0 & 0 \\ 0 & -e - \frac{br_2k_2}{K_1a_2} & 0 \\ 0 & \frac{r_2}{a_2} & -r_2 \end{bmatrix}.$$

The eigenvalues of  $J(E_3)$  is  $\lambda_1 = \frac{r_1a_2}{a_2+m_1r_2K_2}$ ,  $\lambda_2 = -e - \frac{br_2K_2}{K_1a_2}$ , and  $\lambda_3 = -r_2$ . Clearly, we know that there is a definite positive eigenvalue, i.e.  $\lambda_1$ . Furthermore, we can conclude that the prey-free equilibrium is a saddle point. The proof has been completed.

**Theorem 5.** The predator-free equilibrium  $E_4\left(\frac{e}{\beta}, \frac{r_1(\beta K - e)}{\beta(r_1 + \beta K)}, 0\right)$  is unstable.

**Proof.** The characteristic of the Jacobian matrix of system (2) at predator-free equilibrium  $E_4\left(\frac{e}{\beta}, \frac{r_1(\beta K - e)}{\beta(r_1 + \beta K)}, 0\right)$  is

$$\begin{vmatrix} -\frac{r_1e}{\beta K} - \lambda & -e - \frac{r_1}{\beta K} & 0 \\ \frac{r_1(\beta K - e)}{r_1 + \beta K} & -\lambda & A \\ 0 & 0 & r_2 - \lambda \end{vmatrix} = 0,$$

where  $A = \frac{r_1X[m_2e(r_1X + K_1\beta Y) - \beta bY]}{\beta Y(\beta^2K_1K + r_1(K + K_1)\beta - r_1e)}$ , and  $X = \beta K - e$  and  $Y = r_1 + \beta K$ . Obviously,  $\lambda = r_2$  is always positive. It implies  $E_4$  is unstable.

**Theorem 6.** The disease-free equilibrium  $E_5\left(K, 0, \frac{r_2K_2}{a_2}\right)$  is a saddle point if  $R_p > 1$ . If  $R_p < 1$  then  $E_5$  is locally asymptotically stable. The value of  $R_p$  is  $\frac{K_1a_2^2(\beta K - e)}{r_2K_2a_2(em_2K_1 + b) + br_2^2K_2^2m_2}$ .

**Proof.** The Jacobian matrix of model (2) at  $E_5$  is given by following matrix.

$$J(E_5) = \begin{bmatrix} -\frac{r_1a_2}{a_2+m_1r_2K_2} & B & 0 \\ 0 & C & 0 \\ 0 & \frac{r_2^2}{a_2} & -r_2 \end{bmatrix},$$

where  $B = -\frac{a_2(r_1a_2 + r_1m_2r_2K_2 + \beta Ka_2 + \beta Km_1r_2K_2)}{(a_2 + m_1r_2K_2)(a_2 + m_2r_2K_2)}$  and  $C = \frac{K_1a_2^2(\beta K - e) - r_2K_2a_2(em_2K_1 + b) - br_2^2K_2^2m_2}{a_2K_1(a_2 + m_2r_2K_2)}$ .

We know that the eigenvalues of  $J(E_5)$  is  $\lambda_1 = -\frac{r_1a_2}{a_2+m_1r_2K_2}, \lambda_2 = C$ , and  $\lambda_3 = -r_2$ . If  $R_p > 1$  then  $E_5$  is a saddle point, but if  $R_p < 1$  then the equilibrium  $E_5$  is locally asymptotically stable, where

$$R_p = \frac{K_1a_2^2(\beta K - e)}{r_2K_2a_2(em_2K_1 + b) + br_2^2K_2^2m_2}.$$

Next, we study the local stability of the interior equilibrium  $E^*(u_1^*, u_2^*, v^*)$ . For the interior equilibrium  $E^*(u_1^*, u_2^*, v^*)$ , we get this following Jacobian matrix.

$$J(E^*) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix},$$

where  $a_{11} = -\frac{r_1u_1^*}{K(1+m_1v^*)} < 0, a_{12} = -\frac{r_1u_1^*}{K(1+m_1v^*)} - \frac{\beta u_1^*}{1+m_2v^*} < 0, a_{13} = \frac{r_1m_1u_1^*}{(1+m_1v^*)^2} \left(-1 + \frac{u_1^*}{K} + \frac{u_2^*}{K}\right) + \frac{\beta u_1^*u_2^*m_2}{(1+m_2v^*)^2} > 0,$   
 $a_{21} = \frac{\beta u_2^*}{1+m_2v^*} > 0, a_{22} = -\frac{bu_2^*v^*}{(u_2^*+K_1)^2} < 0, a_{23} = -\frac{\beta u_1^*u_2^*m_2}{(1+m_2v^*)^2} - \frac{bu_2^*}{u_2^*+K_1} < 0, a_{32} = \frac{a_2(v^*)^2}{(u_2^*+K_2)^2} > 0,$   
 and  $a_{33} = -\frac{a_2v^*}{u_2^*+K_2} < 0$ . The characteristic equation of  $J(E^*)$  is

$$\lambda^3 + R_1\lambda^2 + R_2\lambda + R_3 = 0, \tag{16}$$

where  $R_1 = a_{11} + a_{22} - a_{33}, R_2 = (a_{11} + a_{22})a_{33} + a_{11}a_{22} - a_{21}a_{12} - a_{23}a_{32}$ , and  $R_3 = a_{11}a_{23}a_{32} - a_{11}a_{22}a_{33} -$

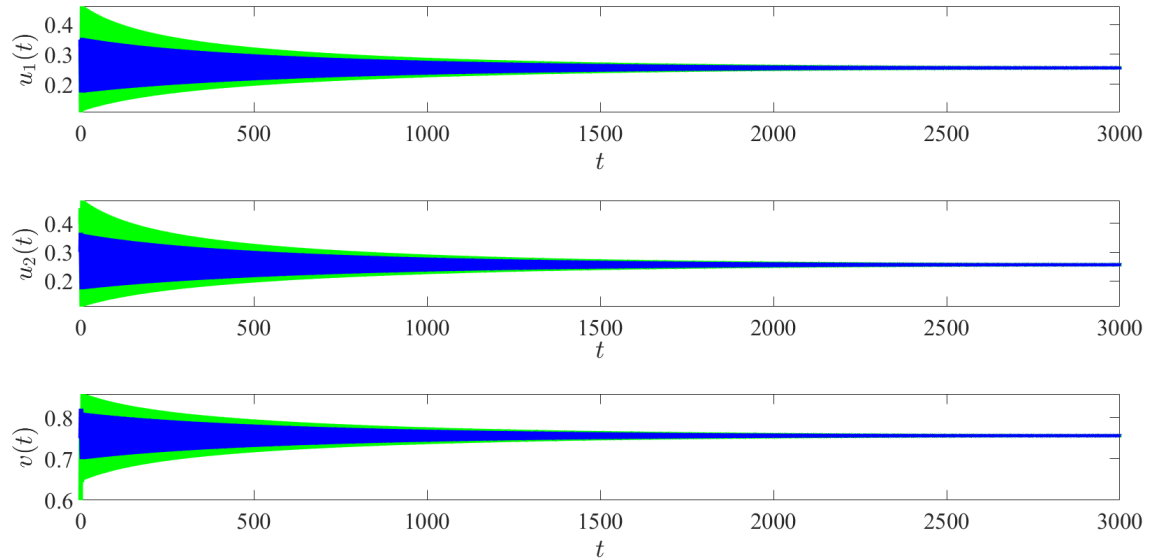


Fig. 2. Time series when  $m_2 = 0.67 > 0.6432 = m_2^*$  and  $m_1 = 0.5$ . The interior equilibrium is asymptotically stable. The initial point of blue is  $(0.3, 0.4, 0.6)$  while the green is  $(0.35, 0.3, 0.75)$ .

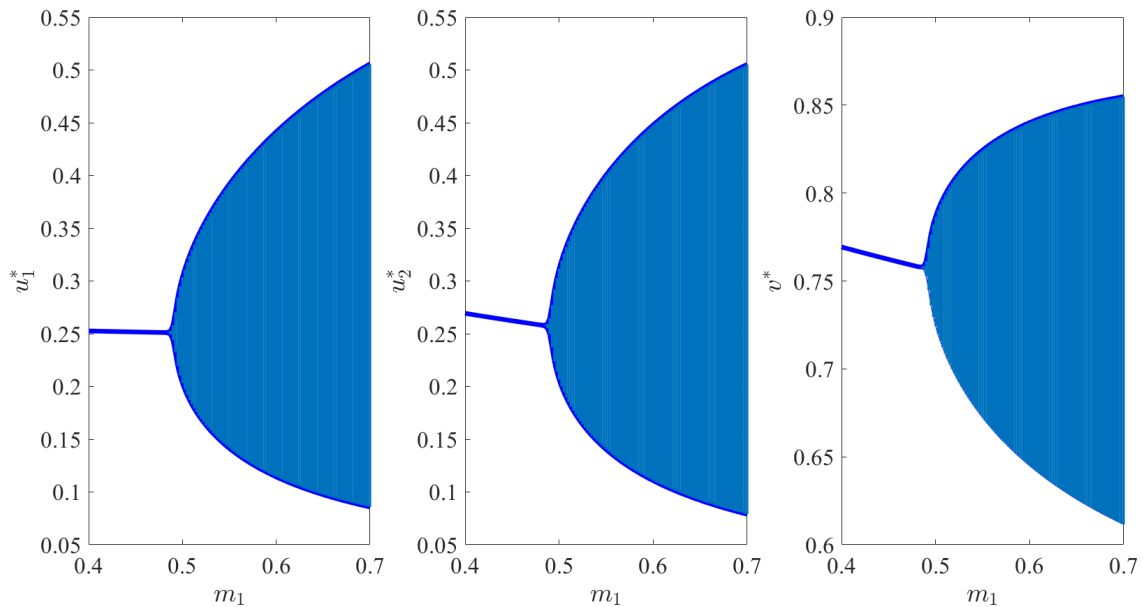


Fig. 3. Bifurcation diagram of the system corresponding to the bifurcation parameter  $m_1$  for the populations  $u_1, u_2, v$ .

$a_{13}a_{21}a_{32}$ . By using the Routh-Hurwitz criterion, all eigenvalues of matrix  $J(E^*)$  have the negative real parts when

$$R_1 > 0, R_3 > 0, R_1 R_2 - R_3 > 0. \quad (17)$$

Based on the previous result we get the following theorem.  
**Theorem 7.** When the interior equilibrium point  $E^*(u_1^*, u_2^*, v^*)$  exists, it would locally asymptotically stable if (17) is satisfied.

## VI. HOPF BIFURCATION

The possibility of the Hopf bifurcation with fear as the bifurcating parameter has been studied in this section.

**Theorem 8.** When the fear parameter  $m_1$  crosses a critical

value, the system (2) undergoes a Hopf bifurcation around the endemic equilibrium point. The necessary and sufficient conditions for the existence of Hopf-bifurcation at  $m_1 = m_1^*$  are

- 1)  $\sigma_1(m_1^*) \sigma_2(m_1^*) - \sigma_3(m_1^*) = 0$ ,
- 2)  $\frac{d}{dm_1} (\text{Re}(r(m_1)))|_{m_1=m_1^*} \neq 0$ ,

where  $r$  is the root of the characteristic equation corresponding to the coexisting equilibrium point.

**Proof:** For  $m_1 = m_1^*$ , we can write the characteristic equation  $r^3 + \sigma_1 r^2 + \sigma_2 r + \sigma_3 = 0$  as  $(r^2 + \sigma_2)(r + \sigma_1) = 0$ . This equation has three roots  $r_1 = i\sqrt{\sigma_2}$ ,  $r_2 = -i\sqrt{\sigma_2}$ , and  $r_3 = -\sigma_1$ .

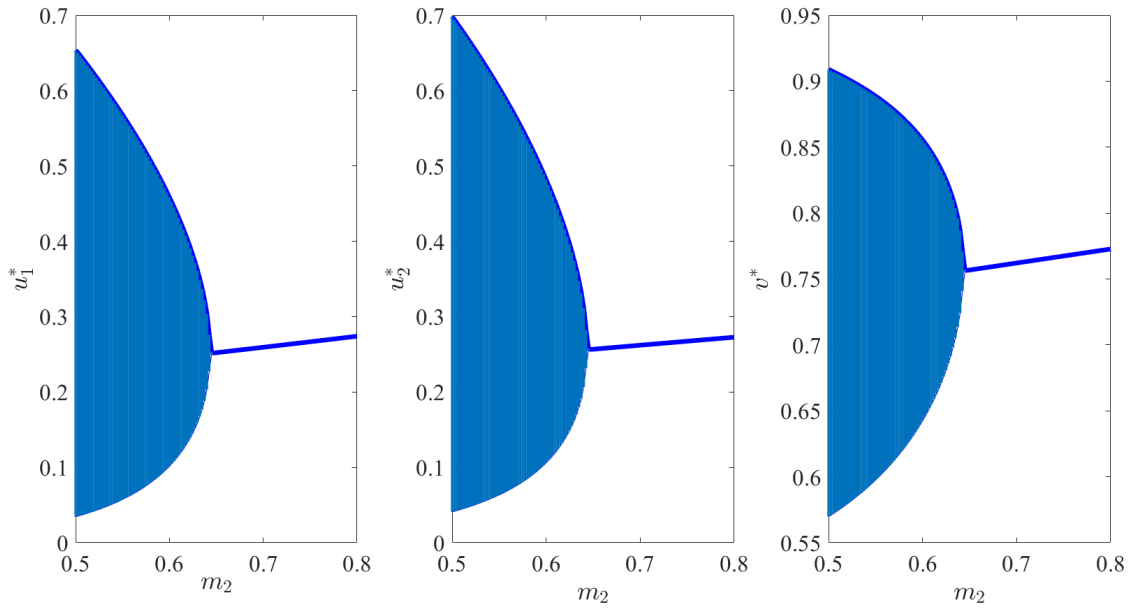


Fig. 4. Bifurcation diagram of the system corresponding to the bifurcation parameter  $m_2$  for the populations  $u_1, u_2, v$ .

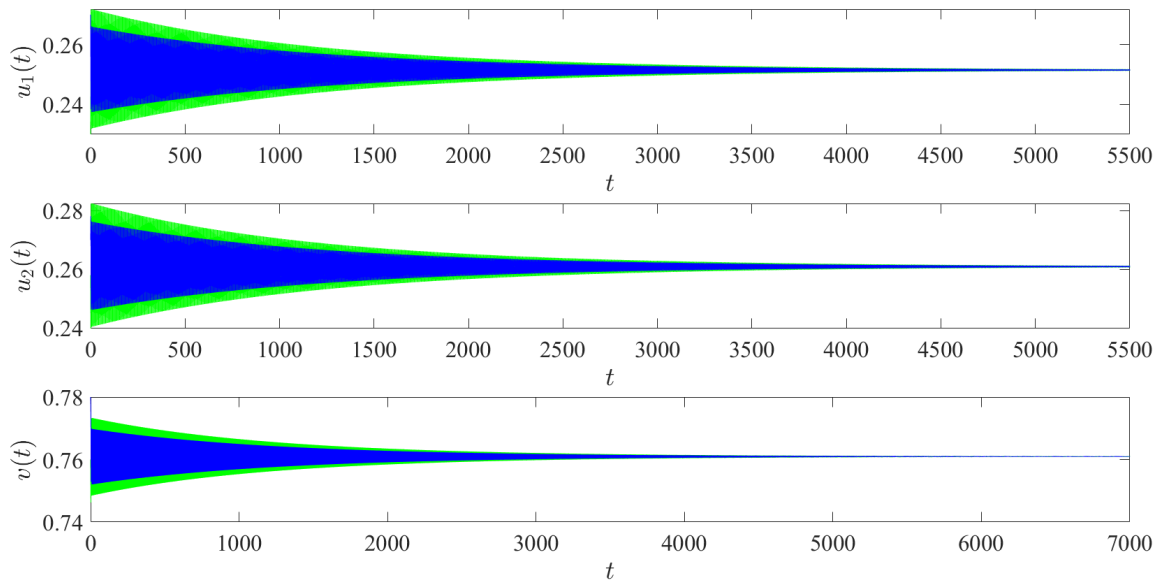


Fig. 5. Time series when  $m_1 = 0.46 < 0.49 = m_1^*$  and  $m_2 = 0.64$ . The interior equilibrium is asymptotically stable. The initial point of blue is  $(0.23, 0.258, 0.76)$  while the green is  $(0.27, 0.27, 0.78)$ .

For all  $m_1$ , the roots are in general of the form

$$\begin{aligned} r_1(m_1) &= \mu_1(m_1) + i\mu_2(m_1), \\ r_2(m_1) &= \mu_1(m_1) - i\mu_2(m_1), \\ r_3(m_1) &= \sigma_1. \end{aligned}$$

Now, we shall verify the transversality condition

$$\frac{d}{dm_1} (\text{Re}(r_j(m_1))) \Big|_{m_1=m_1^*} \neq 0, \text{ for } j = 1, 2$$

Substituting  $r_1(m_1) = \mu_1(m_1) + i\mu_2(m_1)$  and  $r_2(m_1) = \mu_1(m_1) - i\mu_2(m_1)$ , into the characteristic equation and calculating the derivative,

we have

$$\begin{aligned} A_1(m_1)\mu_1'(m_1) - A_2(m_1)\mu_2'(m_1) + B_1(m_1) &= 0, \\ A_2(m_1)\mu_1'(m_1) + A_1(m_1)\mu_2'(m_1) + B_2(m_1) &= 0, \end{aligned}$$

where

$$\begin{aligned} A_1(m_1) &= 3\mu_1^2(m_1) + 2\sigma_1(m_1)\mu_1(m_1) + \sigma_2(m_1) \\ &\quad - 3\mu_2^2(m_1), \\ A_2(m_1) &= 6\mu_1(m_1)\mu_2(m_1) + 2\sigma_1(m_1)\mu_2, \\ B_1(m_1) &= \mu_1(m_1)\sigma_1'(m_1) + \sigma_2'(m_1)\mu_1(m_1) + \sigma_3'(m_1) \\ &\quad - \sigma_1'(m_1)\mu_2^2(m_1), \\ B_2(m_1) &= 2\mu_1(m_1)\mu_2(m_1) + \sigma_2'(m_1)\mu_2(m_1). \end{aligned}$$

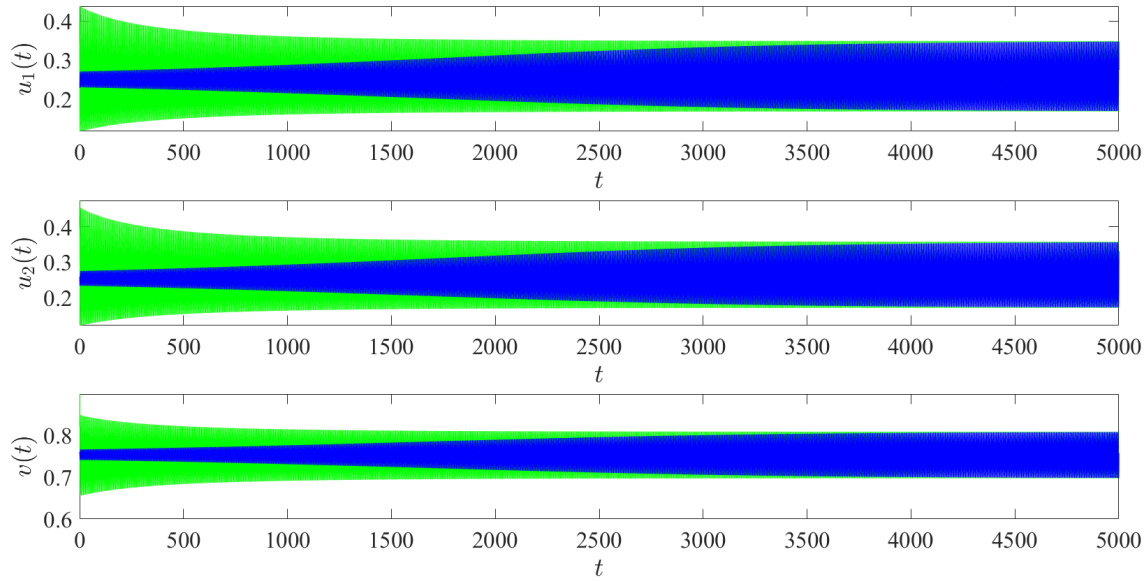


Fig. 6. Time series when  $m_1 = 0.52 > 0.49 = m_1^*$  and  $m_2 = 0.64$ . The solution is chaotic. The initial point of blue is  $(0.23, 0.258, 0.76)$  while the green is  $(0.4, 0.4, 0.82)$ .

Noticing that  $\mu_1(m_1^*) = 0, \mu_2(m_1^*) = \sqrt{\sigma_2(m_1^*)}$ , we have  $A_1(m_1^*) = -2\sigma_2(m_1^*), A_2(m_1^*) = 2\sigma_1(m_1^*)\sqrt{\sigma_2(m_1^*)}, B_1(m_1^*) = \sigma_3'(m_1) - \sigma_1'(m_1)\sigma_2(m_1^*)$  and  $B_2(m_1^*) = \sigma_2'(m_1^*)\sqrt{\sigma_2(m_1^*)}$ . Now,

$$\begin{aligned} & \frac{d}{dm_1} (\text{Re}(r(m_1))) \Big|_{m_1=m_1^*} \\ &= \frac{A_2(m_1^*)B_2(m_1^*) + A_1(m_1^*)A_2(m_1^*)}{A_1(m_1^*)^2 + A_2(m_1^*)^2} \\ &= \frac{\sigma_1(m_1^*)\sigma_2'(m_1^*) - \sigma_3'(m_1^*) + \sigma_1'(m_1^*)\sigma_2(m_1^*)}{2(\sigma_2(m_1^*) + (\sigma_1(m_1^*))^2)} \\ &\neq 0, \text{ if } \frac{d}{dm_1} (\text{Re}(r(m_1))) \Big|_{m_1=m_1^*} \neq 0, \end{aligned}$$

and  $r_3(m_1^*) = -\sigma_1(m_1^*) \neq 0$ .

**Theorem 9.** When the fear parameter  $m_2$  crosses a critical value, the system (2) undergoes a Hopf bifurcation around the endemic equilibrium point. The necessary and sufficient conditions for the existence of Hopf-bifurcation at  $m_2 = m_2^*$  are

- 1)  $\sigma_1(m_2^*)\sigma_2(m_2^*) - \sigma_3(m_2^*) = 0$ ,
- 2)  $\frac{d}{dm_2} (\text{Re}(r(m_2))) \Big|_{m_2=m_2^*} \neq 0$

where  $r$  is the root of the characteristic equation corresponding to the coexisting equilibrium point.

**Proof:** For  $m_2 = m_2^*$ , we can write the characteristic equation  $r^3 + \sigma_1 r^2 + \sigma_2 r + \sigma_3 = 0$  as  $(r^2 + \sigma_2)(r + \sigma_1) = 0$ . This equation has three roots  $r_1 = i\sqrt{\sigma_2}, r_2 = -i\sqrt{\sigma_2}$ , and  $r_3 = -\sigma_1$ .

For all  $m_2$ , the roots are in general of the form

$$\begin{aligned} r_1(m_2) &= \mu_1(m_2) + i\mu_2(m_2), \\ r_2(m_2) &= \mu_1(m_2) - i\mu_2(m_2), \\ r_3(m_2) &= \sigma_1. \end{aligned}$$

Now, we shall verify the transversality condition

$$\frac{d}{dm_2} (\text{Re}(r_j(m_2))) \Big|_{m_2=m_2^*} \neq 0, \text{ for } j = 1, 2$$

Substituting  $r_1(m_2) = \mu_1(m_2) + i\mu_2(m_2)$  and  $r_2(m_2) = \mu_1(m_2) - i\mu_2(m_2)$ , into the characteristic equation and calculating the derivative, we have

$$\begin{aligned} A_1(m_2)\mu_1'(m_2) - A_2(m_2)\mu_2'(m_2) + B_1(m_2) &= 0, \\ A_2(m_2)\mu_1'(m_2) + A_1(m_2)\mu_2'(m_2) + B_2(m_2) &= 0, \end{aligned}$$

where

$$\begin{aligned} A_1(m_2) &= 3\mu_1^2(m_2) + 2\sigma_1(m_2)\mu_1(m_2) + \sigma_2(m_2) \\ &\quad - 3\mu_2^2(m_2), \\ A_2(m_2) &= 6\mu_1(m_2)\mu_2(m_2) + 2\sigma_1(m_2)\mu_2, \\ B_1(m_2) &= \mu_1(m_2)\sigma_1'(m_2) + \sigma_2'(m_2)\mu_1(m_2) + \sigma_3'(m_2) \\ &\quad - \sigma_1'(m_2)\mu_2^2(m_2), \\ B_2(m_2) &= 2\mu_1(m_2)\mu_2(m_2) + \sigma_2'(m_2)\mu_2(m_2). \end{aligned}$$

Noticing that  $\mu_1(m_2^*) = 0, \mu_2(m_2^*) = \sqrt{\sigma_2(m_2^*)}$ , we have  $A_1(m_2^*) = -2\sigma_2(m_2^*), A_2(m_2^*) = 2\sigma_1(m_2^*)\sqrt{\sigma_2(m_2^*)}, B_1(m_2^*) = \sigma_3'(m_2) - \sigma_1'(m_2)\sigma_2(m_2^*)$  and  $B_2(m_2^*) = \sigma_2'(m_2^*)\sqrt{\sigma_2(m_2^*)}$ . Now,

$$\begin{aligned} & \frac{d}{dm_2} (\text{Re}(r(m_2))) \Big|_{m_2=m_2^*} \\ &= \frac{A_2(m_2^*)B_2(m_2^*) + A_1(m_2^*)A_2(m_2^*)}{A_1(m_2^*)^2 + A_2(m_2^*)^2} \\ &= \frac{\sigma_1(m_2^*)\sigma_2'(m_2^*) - \sigma_3'(m_2^*) + \sigma_1'(m_2^*)\sigma_2(m_2^*)}{2(\sigma_2(m_2^*) + (\sigma_1(m_2^*))^2)} \\ &\neq 0, \text{ if } \frac{d}{dm_2} (\text{Re}(r(m_2))) \Big|_{m_2=m_2^*} \neq 0, \end{aligned}$$



## VII. NUMERICAL SIMULATION

We demonstrate some numerical simulations in this section to support our analytical findings. The numerical simulations were carried out using MATLAB. We take the hypothetical parameters values as  $r_1 = 2, K = 3, \beta_1 = 7, e = 0.3, b = 1, K_1 = 0.6, r_2 = 1, a_2 = 1$ , dan  $K_2 = 0.5$ . By using these parameters values, the model (2) has five equilibria, i.e. the trivial equilibrium  $E_1(0, 0, 0)$ , the disease-free and predator-free equilibrium  $E_2(3, 0, 0)$ , the prey equilibrium  $E_3(0, 0, 0.5)$ , the predator-free equilibrium  $E_4(0.0429, 0, 0.0484)$ , and the disease-free equilibrium  $(3, 0, 0.5)$ .

First, we take  $m_2 = 0.62 < 0.6432 = m^*$  and  $m_1 = 0.5$ . Now, it is important to study the dynamical behavior around the coexistence equilibrium point  $E^*$ . The system shows unstable behavior near the coexistence equilibrium by producing limit cycle oscillation (see Fig. 1). If we keep changing the value of parameter  $m_2$ , the system shows asymptotically stable behavior by washing out the limit cycle oscillation after passing the critical value  $m_2^* = 0.6432$  (see Fig. 2). Fear parameter  $m_1$  plays a crucial role in driving the dynamics of the system (2). Figure 4 verifies that the fear parameter  $m_2$  enhances the stability of the system dynamics. The system enters the stable zone when  $m_2$  passes the critical value  $m_2^*$ . From an ecological, when  $m_2$  is less than the critical value, all the populations co-exist in a periodic manner, but when  $m_2$  is greater than the critical value, all populations co-exist in an asymptotically stable way.

Now, we are interested to know the system dynamics near the co-existence equilibrium with the variety of fear parameter value  $m_1$ . We have known that the system shows stable behavior around  $E^*$  for  $m_1 = 0.46 < 0.49 = m_1^*$  as shown in Fig. 3. From Fig. 3, it is clear that the system enters the unstable area when  $m_1$  crosses the critical value  $m_1^* = 0.49$  and the further increase of value  $m_1$  make the system unstable. Fig. 6 shows that the system produces a limit cycle when  $m_1 > m_1^*$ .

## VIII. CONCLUSION

In this article, we have suggested an eco-epidemiological model with disease among the prey. There are two subpopulations of prey: susceptible prey and infectious prey. There are two different parameters for fear. The first one is fear as the inhibitor of the growth of the prey population ( $m_1$ ) and the second one is the inhibitor of interaction among the prey population ( $m_2$ ). Both  $m_1$  and  $m_2$  affect the existence and stability of the interior equilibrium. Increasing interaction among prey population can lead the solution to a disease-free situation. We have shown that the model undergoes a Hopf bifurcation that is driven by fear rate. The critical value of bifurcation  $m_1^*$  and  $m_2^*$  have been calculated. Numerical simulations show that if  $m_1 < m_1^*$ , the interior equilibrium is asymptotically stable, but if  $m_1 > m_1^*$  it stable in periodic way. On the other hand, when  $m_2 < m_2^*$  all populations co-exist in a limit cycle, and after  $m_2$  passes  $m_2^*$  the interior equilibrium is asymptotically stable.

## REFERENCES

- [1] P.H. Leslie, "Some Further Notes on the Use of Matrices in Population Mathematics," *Biometrika*, vol. 35:3-4, pp. 213-245, 1948.
- [2] M.A. Aziz-Aloui, M.D. Okiye, "Boundedness and Global Stability for a Predator-Prey Model with Modified Leslie-Gower and Holling-Type II Schemes," *Applied Mathematics Letters*, vol. 16, no. 7, pp. 1069-1075, 2022.
- [3] P.A. Abrams, "Foraging Time Optimization and Interactions in Food Webs," *The American Naturalist*, vol. 124, no. 1, 80-96, 1983.
- [4] P.A. Abrams, "Strengths of Indirect Effects Generated by Optimal Foraging," *Oikos*, vol. 62, no. 2, 161-176, 1991.
- [5] X. Zhou, J. Cui, X. Shi, X. Song, "A Modified Leslie-Gower Predator-Prey Model with Prey Infection," *Journal of Applied Mathematics and Computing*, vol. 33, pp. 471-487, 2009.
- [6] A. Suryanto, "Dynamics of an Eco-Epidemiological Model with Saturated Incidence Rate," *AIP Conference Proceedings*, vol. 1825, no. 1, 020021, 2017.
- [7] A.S. Purnomo, I. Darti, A. Suryanto, "Dynamics of Eco-Epidemiological Model with Harvesting," *AIP Conference Proceedings*, vol. 1913, no. 1, pp. 020018, 2017.
- [8] K. Zhuang, "The Effect of Time Delay on the Stability of a Diffusive Eco-Epidemiological System," *IAENG International Journal of Applied Mathematics*, vol. 4, no. 4, 394-400, 2018.
- [9] A. Suryanto, I. Darti, "Dynamics of Leslie-Gower Pest-Predator with Disease in Pest including Pest-Harvesting and Optimal Implementation of Pesticide," *International Journal of Mathematics and Mathematical Sciences*, vol. 2019, 5079171, 2019.
- [10] L.F. de Jesus, C.M. Silva, H. Vilarinho, "Dynamics of a Discrete Eco-Epidemiological Model with Disease in the Prey," *Journal of Difference Equations and Applications*, vol. 27, pp. 132-155, 2021.
- [11] H.S. Panigoro, A. Suryanto, W.M. Kusumawinahyu, I. Darti, "Dynamics of a Fractional-Order Predator-Prey model with Infectious Diseases in Prey," *Communication in Biomathematical Sciences*, vol. 2, 105-117, 2019.
- [12] H.S. Panigoro, A. Suryanto, W.M. Kusumawinahyu, I. Darti, "Dynamics of an Eco-Epidemic Predator-Prey Model Involving Fractional Derivatives with Power-Law and Mittag-Leffler Kernel," *Symmetry*, vol. 13, 785, 2021.
- [13] X.Y. Meng, N.N. Qin, H.F. Huo "Dynamics Analysis of a Predator-Prey System with Harvesting Prey and Disease in Prey Species," *Journal of Biological Dynamics*, vol. 12, no. 1, pp. 342-374, 2018.
- [14] D.K. Bahloul, H.A. Satar, H.A. Ibrahim, "Order and Chaos in a Predator Model Incorporating Refuge, Disease, and Harvesting," *Journal of Applied Mathematics*, vol. 2020, 5373817, 2020.
- [15] M.G. Cojocaru, T. Migot, A. Jaber, "Controlling Infection in Predator-Prey Systems with Transmission Dynamics," *Infectious Disease Modelling*, vol. 2020, pp. 1-11, 2019.
- [16] M. Hossain, N. Pal, S. Sumanta, "Impact of Fear on an Eco-Epidemiological Model," *Chaos, Solitons and Fractional*, vol. 134, 109718, 2020.
- [17] J. Liu, B. Liu, Lv. Pan, T. Zhang, "An Eco-Epidemiological Model with Fear Effect and Hunting Cooperation," *Chaos, Solitons and Fractals*, vol. 142, 110494, 2021.
- [18] S.X. Wu, X.Y. Meng, "Dynamics of a Delayed Predator-Prey System with Fear Effect, Herd Behavior and Disease in the Susceptible Prey," *Mathematics*, vol. 6, no. 4, pp. 3654-3685, 2021.
- [19] S. Nazmul, S. Pal, "Dynamics of an Infected Prey-Generalist Predator System with the Effects of Fear, Refuge, and Harvesting: Deterministic and Stochastic Approach," *The European Physical Journal Plus*, vol. 137, 138, 2022.
- [20] A. Sha, S. Samanta, M. Martcheva, J. Chattopadhyay, "Backward Bifurcation, Oscillations and Chaos in an Eco-Epidemiological Model with Fear Effect," *Journal of Biological Dynamics*, vol. 13, no. 1, 301-327, 2019.
- [21] D. Barman, J. Roy, S. Alam, "Dynamical Behaviour of an Infected Predator-Prey Model with Fear Effect," *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 45, 309-325, 2021.
- [22] A.K. Pal, "Effect of Fear on a Modified Leslie-Gower Predator-Prey Eco-Epidemiological Model with Disease in Predator," *Journal of Applied Mathematics and Informatics*, vol. 38, pp. 375-406, 2020.
- [23] S. Rana, S. Samanta, S. Bhattacharya, "The Interplay of Allee Effect in an Eco-Epidemiological System with Disease in Predator Population," *Bulletin Calcutta Mathematical Society*, vol. 108, pp. 103-122, 2016.
- [24] C. Baishya, S.J. Achar, P. Veerasha, D.G. Prakasha, "Dynamics of a Fractional Epidemiological Model with Disease Infection in Both Populations," *Chaos*, vol. 31, 043130, 2021.
- [25] L. Han, Z. Ma, "Four Predator Prey Models with Infectious Diseases," *Mathematical and Computer Modelling*, vol. 34, 849-858, 2000.
- [26] L. Zanette, A.F. White, M.C. Allen, M. Clinchy, "Perceived Predation Risk Reduces the Number of Offspring Songbirds Produce per Year," *Science*, vol. 334, 1398-1401, 2011.
- [27] X. Wang, L. Zanette, X. Zou, "Modeling the Fear Effect in Predator-Prey Interactions," *Journal of Mathematical Biology*, vol. 73, 1179-1204, 2016.

- [28] X. Wang, X. Zou, "Modeling the Fear Effect in Predator-Prey Interactions with Adaptive Avoidance of Predators," *Bulletin of Mathematical Biology*, vol. 79, 1325-1359, 2017.
- [29] S. Mondal, A. Maiti, G.P. Samanta, "Effects of Fear and Additional Food in a Delayed Predator-Prey Model," *Biophysical Reviews and Letters*, vol. 13, 157-177, 2018.
- [30] S. Pal, S. Majhi, S. Mandal, N. Pal, "Role of Fear in a Predator-Prey Model with Beddington-De Angelis Functional Response," *Zeitschrift für Naturforschung A*, vol. 74, no. 7, pp. 581-595, 2019.
- [31] S. Pal, N. Pal, S. Sumanta, J. Chattopadhyay, "Fear Effect in Prey and Hunting Cooperation among Predators in a Leslie-Gower Model," *Mathematical Biosciences and Engineering*, vol. 16, 5146-5179, 2019.
- [32] S.K. Sasmal, Y. Takeuchi, "Dynamics of a Predator-Prey Systems with Fear and Group Defense," *Journal of Mathematical Analysis and Applications*, vol. 481, 123471, 2019.
- [33] Y. Huang, Z. Zhu, Z. Li, "Modeling the Allee Effect and Fear Effect in Predator-Prey System incorporating a Prey Refuge," *Advances in Difference Equations*, vol. 2020, 321, 2020.
- [34] S. Mondal, G.P. Samantha, "Dynamics of a Delayed Predator-Prey Interaction incorporating Nonlinear Prey Refuge under the Influence of Fear Effect and Additional Food," *Journal of Physics A: Mathematical and Theoretical*, vol. 59, 295601, 2020.
- [35] K. Sarkar, S. Khanjanchi, "Impact of Fear Effect on the Growth of Prey in a Predator-Prey Interaction Model," *Ecological Complexity*, vol. 42, 100826, 2020.
- [36] V. Kumar, N. Kumari, "Stability and Bifurcation Analysis of Hassel-Varley Prey-Predator System with Fear Effect," *International Journal of Applied and Computational Mathematics*, vol. 6, 150, 2020.
- [37] J. Liu, P. Lv, B. Liu, T. Zhang, "Dynamics of a Predator-Prey with Fear Effect and Time Delay," *Complexity*, vol. 2021, 9184193, 2021.
- [38] B. Xie, N. Zhang, "Influence of Fear Effect on a Holling type III Prey-Predator System with the Prey Refuge," *AIMS Mathematics*, vol. 7, 1811-1830, 2021.
- [39] Y. Tian, H.M. Li, "The Study of a Predator-Prey Model with Fear Effect Based on State-Dependent Harvesting Strategy," *Complexity*, vol. 2022, 9496599, 2022.
- [40] B. Belew, D. Melese, "Modelling and Analysis of Predator-Prey Model with Fear Effect in Prey and Hunting Cooperation among Predators and Harvesting," *Journal of Applied Mathematics*, vol. 2022, no. 2, 1-14, 2022.
- [41] Q.J.A. Khan, F.A. Al-Kharousi, H. Al Shamsi, "Breeding Suppression in Prey due to Fear of Predator," *IAENG International Journal of Applied Mathematics*, vol. 51:2, 386-393, 2021.
- [42] G.D. Ruxton, S.L. Lima, "Predator-induced Breeding Suppression and Its Consequences for Predator-Prey Population Dynamics," *Proceedings of the Royal Society B*, vol. 264:1380, 409-415, 1997.
- [43] D. Mukharjee, "Fear Induced Dynamics on Leslie-Gower Predator-Prey System with Holling-type IV Functional Response," *Jambura Journal of Biomathematics*, vol. 3, no. 2, 49-57, 2022.
- [44] D. Mukharjee, "Role of Fear in Predator-Prey System with Intraspecific Competition," *Mathematics and Computers in Simulation*, vol. 177, 263-275, 2020.
- [45] A. Gökce, "Dynamical Behaviour of a Predator-Prey System encapsulating the Fear Affecting Death Rate of Prey and Intraspecific Competition: Revisited in a Fluctuating Environment," *Journal of Computational and Applied Mathematics*, vol. 421, 114849, 2022.
- [46] Y. Wang, Y. Shao, C. Chai, "Dynamics of a Predator-Prey Model with Fear Effects and Gestation Delays," *AIMS Mathematics*, vol. 8, no. 3, 7535-7559, 2023.
- [47] P. Majumdar, B. Mondal, S. Debnath, S. Sarkar, U. Ghosh, "Effect of Fear and Delay on a Prey-Predator Model with Predator Harvesting," *Computational and Applied Mathematics*, vol. 41, no. 357, 2022.
- [48] U. Ghosh, A.A. Thirtar, B. Mondal, P. Majumdar, "Effect of Fear, Treatment, and Hunting Cooperation on an Eco-Epidemiological Model: Memory Effect in Terms of Fractional Derivative," *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 46, no. 357, 1541-1554, 2022.
- [49] S. Ganti, S. Gopinathan, "A Note on the Solutions of Cubic Equations of State in Low-Temperature Region," *Journal of Molecular Liquids*, vol. 315, 113808, 2020.