Allocating Mechanism under Multicriteria Situations: Power Index and Relative Normalizations

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***Abstract*—We propose an index and relative normalization for multicriteria situations by considering the maximal-usefulness among election vectors. We demonstrate that these two indexes can be analyzed using a reduction and introduce an alternative formulation for the normalized incremental index using an surplus function. Additionally, we present various dynamic processes for the normalized incremental index based on the reduction and surplus function. Finally, a different normalization are also considered by employing conditional-scales.**

***Index Terms*—Multicriteria situation, maximal-usefulness, reduction, surplus function, dynamic process.**

I. Introduction

In the context of traditional games, power indexes can be defined to quantify the political power of each member within a voting mechanism, such as a political party in a country or a parliament in a confederation, where each member has a distinct number of votes. Various results on power indexes can be found in the literature, such as Banzhaf [1], Dubey and Shapley [4], Haller [5], Hwang and Liao [7], Lehrer [9], Liao et al. [12], van den Brink and van der Laan [2], among others.

A multi-choice game is a natural extension of a traditional game, where each member has multiple operational elections. Several generalized allocations and corresponding results for the core, the equal awards notional surplus dependability (EANSC), and the Shapley value for a given member under multi-choice behavior have been proposed by Cheng et al. [3], Hwang and Liao [6], Huang et al. [8], Liao [10], [11], and Nouweland et al. [15].

Under the framework of theoretical-game theory, dependability is a fundamental property of viable solutions that guarantees their robustness under different specifications of the return structure. dependability requires that the value assigned to a member *dependability*or a coalition should not depend on the return vectors of the other players or coalitions that are not involved in the bargaining. This property reflects the intuition that the outcome of a negotiation should not be affected by the behaviors or outcomes of parties that are not participating in the negotiation. dependability has been axiomatized via different ways, basing on the definition of the residual game, which is the game that arises when the returns of the fixed players are subtracted from the original game. Several well-known solution concepts, such as the core, the nucleolus, and the Shapley value [16], have been demonstrated to fit various dependability axioms, such as the additivity, dummy player, proficiency, and covariance axioms. The dependability property has also motivated the study of reductions, which are obtained by restricting the coalition structure or the set of feasible outcomes of the game. Reductions allow for a simpler analysis of the dependability properties of a given solution concept, and can be used to derive explicit formulas or algorithms for computing the value of the solution. Differ from axiomatic processes, *dynamic processes* also can be analyzed that lead the members to specific solutions, generating from an proficient return vector. Stearns [17] laid the foundation for a dynamic resolution in this area.

Under real-life situations, even if the same person or group of people take the same action in a game, they would produce different utility and consequently receive different returns based on different contexts. For example, a teacher and an accountant would have different levels of influence in a school environment, so, based on their influence respectively, the teacher should receive a higher proportion of returns during the allocation stage compared to the accountant. Therefore, considering different contexts and assigning corresponding scales based on influence seems reasonable.

The motivation of this paper is to extend the power indexes under multi-choice behavior and multicriteria situation simultaneously. The major outcomes of this article are as follows.

- The paper firstly considers the context of *multicriteria multi-choice games* in Section 2, and further introduces a power index and its normalization, the *multi-choice incremental index* and the *multi-choice normalized incremental index*, by considering the maximal-usefulness among election vectors.
- An extended reduction is proposed to axiomatize these indexes in Section 3, and an alternative formulation for the multi-choice normalized incremental index is presented using surplus functions.
- The paper also demonstrates that the multi-choice normalized incremental index can be approach by members who start from an proficient return vector using reduction and surplus function respectively in Section 4.
- Taking into account different situations and their resulting variations in influence in Section 5, we apply conditional-scale function to present the *multi-choice weighted normalized incremental index*. Related axiomatic processes also demonstrate mathematical correctness and practical applicability for this weighted normalization.

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II. The multi-choice incremental index and its normalization

Let UM be the universe of members. For $i \in UM$ and $e_i \in \mathbb{N}$, $E_i = \{0, 1, \cdots, e_i\}$ can be regarded as the election space of member i and $E_i^+ = E_i \setminus \{0\}$, where 0 means non-participating. Let $M \subseteq UM$ and $E^M = \prod_{i \in M} E_i$ be the product set of the election spaces of all members throughout M. For all $K \subseteq M$, we define $\varpi^K \in E^M$ is the vector with $\varpi_i^K = 1$ if $i \in K$, and $\varpi_i^K = 0$ if $i \in M \setminus K$. Indicate 0_M the zero vector in $\mathbb{R}^M$. For $t \in \mathbb{N}$, let 0_t be the zero vector over $\mathbb{R}^t$ and $\mathbb{N}_t = \{1, \cdots, t\}$.

Let (M, e, d) be denoted as a **multi-choice game**, where M with $0 < |M| < \infty$ represents the set of members, $e = (e_i)_{i \in M}$ is the vector that represents the total elections for each member, and $d : E^M \to \mathbb{R}$ is a characteristic mapping. The mapping d fits the condition $d(0_M) = 0$, and it assigns a value to each $\beta = (\beta_i)_{i \in M} \in E^M$ that represents the worth the members can obtain if each member i contributes to the election β_i. Given a multi-choice game (M, e, d) and $\beta \in E^M$, we define $N(\beta) = \{i \in M | \ \beta_i \neq 0\}$ and β_T as the set of members who have a non-zero contribution in β, and β_T as the restriction of β to the subset $T \subseteq M$. Furthermore, we introduce $d_*(T)$, which is defined as the maximum worth obtained by any behavior vector β with $N(\beta) = T$. This value represents the **maximal-usefulness**[1] among all behavior vectors β with $N(\beta) = T$. Let (M, e, D^m) be denoted as a **multicriteria multi-choice game**, where $m \in \mathbb{N}$, $D^m = (d^t)_{t \in \mathbb{N}_m}$, and (M, e, d^t) is a multi-choice game for each $t \in \mathbb{N}_m$.

Indicate the collection of total multicriteria multi-choice games by Φ. Let $(M, e, D^m) \in \Phi$. A **return vector** of (M, e, D^m) is a vector $\xi = (\xi^t)_{t \in \mathbb{N}_m}$ and $\xi^t = (\xi_i^t)_{i \in M} \in \mathbb{R}^M$, where ξ_i^t indicates the return to member i in (M, e, d^t) for each $t \in \mathbb{N}_m$ and for each $i \in M$. A return vector ξ of (M, e, D^m) is **multicriteria proficient** if $\sum_{i \in M} \xi_i^t = d_*^t\big(M\big)$ for all $t \in \mathbb{N}_m$. The collection of all multicriteria proficient vector of (M, e, D^m) is indicated by $P(M, e, D^m)$. A **solution** is a map σ assigning to each $(M, e, D^m) \in \Phi$ the following element

$$\sigma\big(M, e, D^m\big) = \big(\sigma^t\big(M, e, D^m\big)\big)_{t \in \mathbb{N}_m},$$

where $\sigma^t\big(M, e, D^m\big) = \big(\sigma_i^t\big(M, e, D^m\big)\big)_{i \in M} \in \mathbb{R}^M$ and $\sigma_i^t\big(M, e, D^m\big)$ is the return of the member i assigned by σ in $\big(M, e, d^t\big)$.

Next, we provide the multi-choice incremental index and the multi-choice normalized incremental index over multicriteria situation.

Definition 1: The **multi-choice incremental index (MII)**, Θ, is defined by

$$\Theta_i^t(M, e, D^m) = d_*^t(M) - d_*^t(M \setminus \{i\})$$

for each $(M, e, D^m) \in \Phi$, for each $t \in \mathbb{N}_m$ and for each $i \in M$. Based on Θ, all members receive its **incremental contributions** respectively related to maximal-usefulness in M.

A solution σ fits **multicriteria proficiency (MPFY)** if for each $(M, e, D^m) \in \Phi$ and for each $t \in \mathbb{N}_m$, $\sum_{i \in M} \sigma_i^t(M, e, D^m) = d_*^t(M)$. The MPFY property means that all members in a game allocate whole the usefulness available. It is straightforward to see that the MII does not fit the EFF property. Thence, we introduce a proficient normalization.

Definition 2: The **multi-choice normalized incremental index (MNII)**, $\overline{\Theta}$, is defined by

$$\overline{\Theta_i^t}(M, e, D^m) = \frac{d_*^t(M)}{\sum\limits_{k \in M} \Theta_k^t(M, e, D^m)} \cdot \Theta_i^t(M, e, D^m)$$

for each $(M, e, D^m) \in \Phi^*$, for each $t \in \mathbb{N}_m$ and for each $i \in M$, where $\Phi^* = \{(M, e, D^m) \in \Phi \mid \sum\limits_{i \in M} \Theta_i^t(M, e, D^m) \neq 0$ for each $t \in \mathbb{N}_m\}$.

Lemma 1: The MNII fits MPFY on Φ^*.

Proof: For all $(M, e, D^m) \in \Phi^*$ and for all $t \in \mathbb{N}_m$,

$$\begin{aligned}\sum_{i \in M} \overline{\Theta_i^t}(M, e, D^m) &= \sum_{i \in M} \frac{d_*^t(M)}{\sum\limits_{k \in M} \Theta_k^t(M, e, D^m)} \cdot \Theta_i^t(M, e, D^m) \\ &= \frac{d_*^t(M)}{\sum\limits_{k \in M} \Theta_k^t(M, e, D^m)} \sum_{i \in M} \Theta_i^t(M, e, D^m) \\ &= d_*^t(M).\end{aligned}$$

Thus, the MNII fits MPFY on Φ^*. ■

We present a brief applied instance for multicriteria multi-choice games under the context of "management". This type of issues can be formulated as follows. Let $M = \{1, 2, \cdots, m\}$ indicate the set of all members in a large-scale management mechanism (M, e, D^m). The function d^t can be considered as a usefulness function that assigns a value to each election vector $\beta = (\beta_i) i \in M \in E^M$, which represents the benefits that the members can obtain if each member i adopts an operating election $\beta_i \in E_i$ in the sub-management mechanism (M, e, d^t). The large-scale management mechanism (M, e, D^m) can then be modeled as a multicriteria multi-choice game, where d^t serves as the characteristic function, and e_i indicates the collection of total operating elections for member i. In the following sections, we demonstrate that the MII and the MNII can offer optimal allocating mechanism among all members, in the sense that this organization can obtain returns from each combination of operating elections of all members under multi-choice behavior and multicriteria situations.

III. Axiomatic processes

In this section, we demonstrate the existence of a reduction that can be utilized to axiomatize the MII and the MNII.

Additionally, we provide an alternative formulation for the MNII utilizing the concept of *surplus*. Let $(M, e, D^m) \in \Phi^*$, $S \subseteq M$ and ξ be a return vector in (M, e, D^m). Define that $\xi^t(S) = \sum_{i \in S} \xi_i^t$ for each $t \in \mathbb{N}_m$. The **surplus** of a coalition $S \subseteq M$ under ξ is

$$\begin{aligned}&P(S, D^m, \xi) = (P(S, d^t, \xi^t))_{t \in \mathbb{N}_m} \\ \text{and} \quad & \\ &P(S, d^t, \xi^t) = d_*^t(S) - \xi^t(S).\end{aligned} \tag{1}$$

The quantity $P(S, d^t, \xi^t)$ indicates the "objection" of coalition S when all members are assigned their returns from ξ^t in (M, e, d^t).

[1]From this point onwards, we will focus on bounded multi-choice games, which are defined as games (M, e, d) where there exists a real number K_d such that $d(\beta) \leq K_d$ for all $\beta \in E^M$. We introduce this condition to ensure the well-definedness of $d_*(T)$.

Lemma 2: Let $(M,e,D^m) \in \Phi^*$, $t \in \mathbb{N}_m$, $\xi \in P(M,e,D^m)$ and $\tau^t = \frac{d^t_*(M)}{\sum_{k\in M}\Theta^t_k(M,e,D^m)}$. Then

$$\begin{array}{ll} & P(M\setminus\{i\},d^t,\frac{\xi^t}{\tau^t}) = P(M\setminus\{j\},d^t,\frac{\xi^t}{\tau^t}) \;\; \forall\, i,j\in M \\ \Longleftrightarrow & \xi = \overline{\Theta}(M,e,D^m). \end{array}$$

Proof: Let $(M,e,D^m) \in \Phi^*$ and $\xi \in P(M,e,D^m)$. For $t \in \mathbb{N}_m$ and for $i,j \in M$,

$$\begin{array}{ll} & P(M\setminus\{j\},d^t,\frac{\xi^t}{\tau^t}) \\ = & P(M\setminus\{i\},d^t,\frac{\xi^t}{\tau^t}) \\ \Longleftrightarrow & d^t_*(M\setminus\{j\}) - \frac{\xi^t(M\setminus\{j\})}{\tau^t} \\ = & d^t_*(M\setminus\{i\}) - \frac{\xi^t(M\setminus\{i\})}{\tau^t} \\ \Longleftrightarrow & d^t_*(M\setminus\{j\}) - \frac{\xi^t_i}{\tau^t} \\ = & d^t_*(M\setminus\{i\}) - \frac{\xi^t_j}{\tau^t} \\ \Longleftrightarrow & \xi^t_i - \xi^t_j \\ = & \tau^t\cdot[d^t_*(M\setminus\{j\}) - d^t_*(M\setminus\{i\})]. \end{array} \tag{2}$$

By definition of $\overline{\Theta}$,

$$\begin{array}{ll} & \overline{\Theta^t_i}(M,e,D^m) - \overline{\Theta^t_j}(M,e,D^m) \\ = & \tau^t\cdot[d^t_*(M\setminus\{j\}) - d^t_*(M\setminus\{i\})] \end{array} \tag{3}$$

Based on (2) and (3), for $i,j \in M$,

$$\xi^t_i - \xi^t_j = \overline{\Theta^t_i}(M,e,D^m) - \overline{\Theta^t_j}(M,e,D^m).$$

Hence,

$$\sum_{j\neq i}[\xi^t_i - \xi^t_j] = \sum_{j\neq i}[\overline{\Theta^t_i}(M,e,D^m) - \overline{\Theta^t_j}(M,e,D^m)].$$

That is, $(|M|-1)\cdot\xi^t_i - \sum_{j\neq i}\xi^t_j = (|M|-1)\cdot\overline{\Theta^t_i}(M,e,D^m) - \sum_{j\neq i}\overline{\Theta^t_j}(M,e,D^m)$. Since $\xi \in P(M,e,D^m)$ and $\overline{\Theta}$ fits MPFY, $|M|\cdot\xi^t_i - d^t_*(M) = |M|\cdot\overline{\Theta^t_i}(M,e,D^m) - d^t_*(M)$. Therefore, $\xi^t_i = \overline{\Theta^t_i}(M,e,D^m)$ for $t \in \mathbb{N}_m$ and for $i \in M$, i.e., $\xi = \overline{\Theta}(M,e,D^m)$. ■

Remark 1: It is trivial to examine that $P(M\setminus\{i\},D^m,\Theta(M,e,D^m)) = P(M\setminus\{j\},D^m,\Theta(M,e,D^m))$ for each $(M,e,D^m)\in\Phi$ and for arbitrary $i,j\in M$.

Inspired by Moulin's [14] reduced notion, we introduced a multi-choice reduction and corresponding dependability. Let σ be a solution, $(M,e,D^m)\in\Phi$ and $S\subseteq M$. The **reduction** $(S,e_S,D^m_{S,\sigma})$ is defined by $D^m_{S,\sigma} = (d^t_{S,\sigma})_{t\in\mathbb{N}_m}$ and

$$\begin{array}{ll} & d^t_{S,\sigma}(\beta) \\ = & \left\{\begin{array}{ll} 0 & ,\ \beta = 0_S, \\ d_*(M(\beta)\cup(M\setminus S)) - \sum_{i\in M\setminus S}\sigma_i(M,e,d) & ,\ \text{o.w.} \end{array}\right. \end{array}$$

σ fits **dependability (DEP)** if $\sigma^t_i(S,e_S,D^m_{S,\sigma}) = \sigma^t_i(M,e_S,D^m)$ for each $(M,e,D^m)\in\Phi$, for each $S\subseteq M$ with $|S|=2$, for each $t\in\mathbb{N}_m$ and for each $i\in S$. However, it is trivial to examine that $\sum_{k\in S}\Theta^t_k(M,e,d) = 0$ for some $(M,e,D^m)\in G$, for some $t\in\mathbb{N}_m$ and for some $S\subseteq M$, i.e., $\overline{\Theta}(S,e_S,D^m_{S,\sigma})$ doesn't exist for some $(M,e,D^m)\in\Phi$ and for some $S\subseteq M$. Thence, one can consider the *resilient dependability*. A solution σ fits **resilient dependability (RDEP)** if $(S,e_S,D^m_{S,\sigma})$ and $\sigma(S,e_S,D^m_{S,\sigma})$ exist for some $(M,e,D^m)\in\Phi$ and for some $S\subseteq M$ with $|S|=2$, it holds that $\sigma^t_i(S,e_S,d^m_{S,\sigma}) = \sigma^t_i(M,e,D^m)$ for each $t\in\mathbb{N}_m$ and for each $i\in S$.

Lemma 3:

1) The MII fits DEP on Φ.
2) The MNII fits RDEP on Φ^*.

Proof: To demonstrate item 1, let $(M,e,D^m)\in\Phi^*$ and $S\subseteq M$. It is trivial if $|M|=1$. Assume that $|M|\geq 2$ and $S=\{i,j\}$ for some $i,j\in M$. For each $t\in\mathbb{N}_m$ and for each $i\in S$,

$$\begin{array}{ll} & \Theta^t_i(S,e_S,D^m_{S,\Theta}) \\ = & (d^t_{S,\Theta})_*(S) - (d^t_{S,\Theta})_*(S\setminus\{i\}) \\ = & \max_{\beta\in E^S}\{d^t_{S,\Theta}(\beta)|N(\beta)=S\} \\ & \quad - \max_{\beta\in E^S}\{d^t_{S,\Theta}(\beta)|N(\beta)=S\setminus\{i\}\} \\ = & d^t_*(M) - d^t_*(M\setminus\{i\}) \\ = & \Theta^t_i(M,e,D^m). \end{array} \tag{4}$$

Thus, the MII fits DEP.

To demonstrate item 2, let $(M,e,D^m)\in\Phi^*$ and $S\subseteq M$. It is trivial if $|M|=1$. Assume that $|M|\geq 2$. If $S=\{i,j\}$ for some $i,j\in M$ and $(S,e_S,D^m_{S,\overline{\Theta}})\in\Phi^*$. Similar to equation (4), for each $t\in\mathbb{N}_m$ and for each $i\in S$,

$$\Theta^t_i(S,e_S,d^m_{S,\overline{\Theta}}) = \Theta^t_i(M,e,D^m). \tag{5}$$

By definition of $\overline{\Theta}$ and equation (5),

$$\begin{array}{ll} & \overline{\Theta^t_i}(S,e_S,d^m_{S,\overline{\Theta}}) \\ = & \frac{(d^t_{S,\overline{\Theta}})_*(S)}{\sum_{k\in S}\Theta^t_k(S,e_S,d^m_{S,\overline{\Theta}})}\cdot\Theta^t_i(S,e_S,d^m_{S,\overline{\Theta}}) \\ = & \frac{d^t_*(M) - \sum_{k\in M\setminus S}\overline{\Theta^t_k}(M,e,D^m)}{\sum_{k\in S}\Theta^t_k(S,e_S,d^m_{S,\overline{\Theta}})}\cdot\Theta^t_i(S,e_S,d^m_{S,\overline{\Theta}}) \\ & \textbf{(by definition of } d^m_{S,\overline{\Theta}}\textbf{)} \\ = & \frac{d^t_*(M) - \sum_{k\in M\setminus S}\overline{\Theta^t_k}(M,e,D^m)}{\sum_{k\in S}\Theta^t_k(M,e,D^m)}\cdot\Theta^t_i(M,e,D^m) \\ & \textbf{(by equation (5))} \\ = & \frac{\sum_{k\in S}\overline{\Theta^t_k}(M,e,D^m)}{\sum_{k\in S}\Theta^t_k(M,e,D^m)}\cdot\Theta^t_i(M,e,D^m) \\ & \textbf{(by MPFY of } \overline{\Theta}\textbf{)} \\ = & \tau^t\cdot\Theta^t_i(M,e,D^m),\text{ where } \tau^t = \frac{d^t_*(M)}{\sum_{k\in M}\Theta^t_k(M,e,D^m)} \\ = & \overline{\Theta^t_i}(M,e,D^m). \end{array}$$

Thus, the MNII fits RDEP on Φ^*. ■

Next, we axiomatize the MII and the MNII by applying the properties of DEP and RDEP.

- A solution σ fits **incremental-norm for games (MNG)** if $\sigma(M,e,d) = \Theta(M,e,d)$ for each $(M,e,d)\in\Phi$ with $|M|\leq 2$.
- A solution σ fits **normalized-norm for games (NNG)** if $\sigma(M,e,d) = \overline{\Theta}(M,e,d)$ for each $(M,e,d)\in\Phi^*$ with $|M|\leq 2$.

Lemma 4: A solution σ fits MPFY on Φ^* if it fits NNG and RDEP on Φ^*.

Proof: Let $(M,e,D^m)\in\Phi^*$. If $|M|\leq 2$, then σ fits MPFY on Φ^* by NNG. Suppose that $|M|>2$. Assume, on the contrary, that there exists $(M,e_S,D^m)\in\Phi^*$ such that $\sum_{i\in M}\sigma^t_i(M,e,D^m)\neq d^t_*(M)$ for some $t\in\mathbb{N}_m$. This presents that there exist $i\in M$ and $j\in M$ such that $[d^t_*(M) - \sum_{k\in M\setminus\{i,j\}}\sigma^t_k(M,e,D^m)] \neq [\sigma^t_i(M,e,D^m) + \sigma^t_j(M,e,D^m)]$. By RDEP and σ fits MPFY for two-person games, this contradicts with

$$\begin{array}{ll} & \sigma^t_i(M,e,D^m) + \sigma^t_j(M,e,D^m) \\ = & \sigma^t_i(\{i,j\},d^m_{\{i,j\},\sigma}) + \sigma^t_j(\{i,j\},d^m_{\{i,j\},\sigma}) \\ = & d^t_*(M) - \sum_{k\in M\setminus\{i,j\}}\sigma^t_k(M,e,D^m). \end{array}$$

Hence σ fits MPFY. ■

Theorem 1:

1) On Φ, the MII is the only solution fitting MNG and DEP.
2) On Φ^*, the MNII is the only solution fitting NNG and RDEP.

Proof: By Lemma 3, Θ and $\overline{\Theta}$ fit DEP and RDEP on Φ and Φ^* respectively. Absolutely, Θ and $\overline{\Theta}$ fit MNG and NNG on Φ and Φ^* respectively.

To demonstrate uniqueness of item 1, suppose σ fits DEP and MNG on Φ. Let $(M, e, D^m) \in \Phi$. If $|M| \leq 2$, then $\sigma(M, e, D^m) = \Theta(M, e, D^m)$ by MNG. Suppose that $|M| > 2$. Let $t \in \mathbb{N}_m$ and $i \in M$. Assume that $S \subseteq M$ with $|S| = 2$ and $i \in S$. Then,

$$\begin{aligned} \sigma_i^t(M, e, D^m) &= \sigma_i^t(S, e_S, d^m_{S,\sigma}) \\ &\quad \textbf{(by DEP of } \sigma\textbf{)} \\ &= \Theta_i^t(S, e_S, d^m_{S,\sigma}) \\ &\quad \textbf{(by MNG of } \sigma\textbf{)} \\ &= (d^t_{S,\sigma})_*(S) - (d^t_{S,\sigma})_*(S \setminus \{i\}) \\ &= d^t_*(M) - d^t_*(M \setminus \{i\}) \\ &= \Theta_i^t(M, e, D^m). \end{aligned}$$

Hence, $\sigma(M, e, D^m) = \Theta(M, e, D^m)$ for all $(M, e, D^m)\Phi$.

To demonstrate uniqueness of item 2, suppose σ fits RDEP and NNG on Φ^*. By Lemma 4, σ fits MPFY on Φ^*. Let $(M, e, D^m) \in \Phi^*$. The proof will be finished via induction on $|M|$. It is trivial that $\sigma(M, e, D^m) = \overline{\Theta}(M, e, D^m)$ by NNG if $|M| \leq 2$. Assume that it holds if $|M| \leq r - 1$, $r \geq 3$. The situation $|M| = r$: Let $t \in \mathbb{N}_m$ and $i, j \in M$ with $i \neq j$. Based on Definition 2, $\overline{\Theta}_k^t(M, e, D^m) = \frac{d^t_*(M)}{\sum_{h \in M} \Theta_h^t(M,e,D^m)} \cdot \Theta_k^t(M, e, D^m)$ for all $k \in M$. Assume that $\alpha_k^t = \frac{\Theta_k^t(M,e,d)}{\sum_{h \in M} \Theta_h^t(M,e,d)}$ for all $k \in M$. Therefore,

$$\begin{aligned} & \sigma_i^t(M, e, D^m) \\ = \;& \sigma_i^t\big(M \setminus \{j\}, d^m_{M\setminus\{j\},\sigma}\big) \\ & \textbf{(by RDEP of } \sigma\textbf{)} \\ = \;& \overline{\Theta}_i^t\big(M \setminus \{j\}, d^m_{M\setminus\{j\},\sigma}\big) \\ & \textbf{(by NNG of } \sigma\textbf{)} \\ = \;& \frac{(d^t_{M\setminus\{j\},\sigma})_*\big(M\setminus\{j\}\big)}{\sum_{k \in M\setminus\{j\}} \Theta_k^t\big(M\setminus\{j\}, d^m_{M\setminus\{j\},\sigma}\big)} \Theta_i^t\big(M \setminus \{j\}, d^m_{M\setminus\{j\},\sigma}\big) \\ = \;& \frac{d^t_*(M) - \sigma_i^t(M,e,D^m)}{\sum_{k \in M\setminus\{j\}} \Theta_k^t(M,e,D^m)} \Theta_i^t(M, e, D^m) \\ & \textbf{(by equation (5))} \\ = \;& \frac{d^t_*(M) - \sigma_i^t(M,e,D^m)}{-\Theta_j^t(M,e,D^m) + \sum_{k \in M} \Theta_k^t(M,e,D^m)} \Theta_i^t(M, e, D^m). \end{aligned} \tag{6}$$

By equation (6),

$$\begin{aligned} & \sigma_i^t(M, e, D^m) \cdot [1 - \alpha_j^t] \\ = \;& [d^t_*(M) - \sigma_j^t(M, e, D^m)] \cdot \alpha_j^t \\ \Longrightarrow \;& \sum_{i \in M} \sigma_i^t(M, e, D^m) \cdot [1 - \alpha_j^t] \\ = \;& [d^t_*(M) - \sigma_j^t(M, e, D^m)] \cdot \sum_{i \in M} \alpha_j^t \\ \Longrightarrow \;& d^t_*(M) \cdot [1 - \alpha_j^t] \\ = \;& [d^t_*(M) - \sigma_j^t(M, e, D^m)] \cdot 1 \\ & \textbf{(by MPFY of } \sigma\textbf{)} \\ \Longrightarrow \;& d^t_*(M) - d^t_*(M) \cdot \alpha_j^t \\ = \;& d^t_*(M) - \sigma_j^t(M, e, D^m) \\ \Longrightarrow \;& \overline{\Theta}_j^t(M, e, D^m) \\ = \;& \sigma_j^t(M, e, D^m). \end{aligned}$$

The proof is finished. ■

The subsequent instances aim to demonstrate that each of the properties utilized under Theorem 1 is logically independent from the other properties.

Example 1: Define a solution σ by for each $(M, e, D^m) \in \Phi$, for each $t \in \mathbb{N}_m$ and for each $i \in M$, $\sigma_i^t(M, e, D^m) = 0$. Clearly, σ fits DEP and RDEP on Φ and Φ^*, but it does not fit MNG and NNG on Φ and Φ^*.

Example 2: Define a solution σ by for each $(M, e, D^m) \in \Phi$, for each $t \in \mathbb{N}_m$ and for each $i \in M$,

$$\sigma_i^t(M, e, D^m) = \begin{cases} \Theta_i^t(M, e, D^m) & \text{, if } |M| \leq 2, \\ 0 & \text{, o.w.} \end{cases}$$

On Φ, σ fits MNG, but it does not fit DEP.

Example 3: Define a solution σ by for each $(M, e, D^m) \in \Phi^*$, for each $t \in \mathbb{N}_m$ and for each $i \in M$,

$$\sigma_i^t(M, e, D^m) = \begin{cases} \overline{\Theta}_i(M, e, D^m) & \text{, if } |M| \leq 2, \\ 0 & \text{, o.w.} \end{cases}$$

On Φ^*, σ fits NNG, but it does not fit RDEP.

IV. Dynamic results

In this section, we apply surplus function and reduction to offer dynamic results for the MNII.

To establish the dynamic notion for the multi-choice normalized incremental index (MNII), we begin by defining a amendment function using surplus functions. The amendment function is depended on the idea that each member reduces the objection related to its own and others' non-participation, and applies these adjustments to switch the initial return.

Definition 3: Let $(M, e, D^m) \in \Phi^*$ and $i \in M$. The **amendment function** is $f = (f^t)_{t \in \mathbb{N}_m}$, where $f^t = (f_i^t)_{i \in M}$ and $f_i^t : P(M, e, D^m) \to \mathbb{R}$ is define by

$$\begin{aligned} & f_i^t(\xi) \\ = \;& \xi_i^t + w \sum_{j \in M\setminus\{i\}} \tau^t \Big(P(M \setminus \{j\}, d^t, \tfrac{\xi^t}{\tau^t}) - P(M \setminus \{i\}, d^t, \tfrac{\xi^t}{\tau^t})\Big), \end{aligned}$$

where $\tau^t = \frac{d^t_*(M)}{\sum_{k \in M} \Theta_k^t(M,e,D^m)}$ and $w \in \mathbb{R}$ with $w > 0$ is a fixed number, which reflects the assumption that member i does not ask for complete amendment but only a fraction of it. Define $[\xi]^0 = \xi,\ [\xi]^1 = f([\xi]^0), \cdots,\ [\xi]^q = f([\xi]^{q-1})$ for each $q \in \mathbb{N}$.

Lemma 5: $f(\xi) \in P(M, e, D^m)$ for all $(M, e, D^m) \in \Phi^*$ and for all $\xi \in P(M, e, D^m)$.

Proof: Let $(M,e,D^m)\in\Phi^*$, $t\in\mathbb{N}_m$, $i,j\in M$ and $\xi\in P(M,e,D^m)$.

$$\begin{aligned}
&\sum_{j\in M\setminus\{i\}}\tau^t\Big(P(M\setminus\{j\},d^t,\tfrac{\xi^t}{\tau^t})-P(M\setminus\{i\},d^t,\tfrac{\xi^t}{\tau^t})\Big)\\
=&\sum_{j\in M\setminus\{i\}}\tau^t\Big(d^t(M\setminus\{j\})-\tfrac{\xi^t(M\setminus\{j\})}{\tau^t}\\
&\qquad-d^t(M\setminus\{i\})+\tfrac{\xi^t(M\setminus\{i\})}{\tau^t}\Big)\\
=&\sum_{j\in M\setminus\{i\}}\tau^t\Big(d^t(M\setminus\{j\})-d^t(M\setminus\{i\})-\tfrac{\xi^t_i}{\tau^t}+\tfrac{\xi^t_j}{\tau^t}\Big).
\end{aligned}\tag{7}$$

By definition of $\overline{\Theta}$,

$$\begin{aligned}
&\overline{\Theta^t_i}(M,e,D^m)-\overline{\Theta^t_j}(M,e,D^m)\\
=&\ \tau^t\cdot\Big(d^t(M\setminus\{j\})-d^t(M\setminus\{i\})\Big).
\end{aligned}\tag{8}$$

Based on (7) and (8),

$$\begin{aligned}
&\sum_{j\in M\setminus\{i\}}\tau^t\Big(P(M\setminus\{j\},d^t,\tfrac{\xi^t}{\tau^t})-P(M\setminus\{i\},d^t,\tfrac{\xi^t}{\tau^t})\Big)\\
=&\sum_{j\in M\setminus\{i\}}\Big(\overline{\Theta^t_i}(M,e,D^m)-\overline{\Theta^t_j}(M,e,D^m)-\xi^t_i+\xi^t_j\Big)\\
=&\ (|M|-1)\big(\overline{\Theta^t_i}(M,e,D^m)-\xi^t_i\big)+\sum_{j\in M\setminus\{i\}}\xi^t_j\\
&-\sum_{j\in M\setminus\{i\}}\overline{\Theta^t_j}(M,e,D^m)\\
=&\ |M|\big(\overline{\Theta^t_i}(M,e,D^m)-\xi^t_i\big)-d^t_*(M)+d^t_*(M)\\
&\textbf{(by MPFY of } \overline{\Theta}\textbf{,}\ \xi\in P(M,e,D^m))\\
=&\ |M|\Big(\overline{\Theta^t_i}(M,e,D^m)-\xi^t_i\Big).
\end{aligned}\tag{9}$$

Moreover,

$$\begin{aligned}
&\sum_{i\in M}|M|\Big(\overline{\Theta^t_i}(M,e,D^m)-\xi^t_i\Big)\\
=&\ |M|\Big(\sum_{i\in M}\overline{\Theta^t_i}(M,e,D^m)-\sum_{i\in M}\xi^t_i\Big)\\
=&\ |M|\big(d^t_*(M)-d^t_*(M)\big)\\
&\textbf{(by MPFY of } \overline{\Theta}\textbf{,}\ \xi\in P(M,e,D^m))\\
=&\ 0.
\end{aligned}\tag{10}$$

So we have that

$$\begin{aligned}
&\sum_{i\in M}f^t_i(\xi)\\
=&\sum_{i\in M}\Big[\xi^t_i+w\sum_{j\in M\setminus\{i\}}\tau^t\Big(P(M\setminus\{j\},d^t,\tfrac{\xi^t}{\tau^t})\\
&\qquad-P(M\setminus\{i\},d^t,\tfrac{\xi^t}{\tau^t})\Big)\Big]\\
=&\sum_{i\in M}\xi^t_i+w\sum_{i\in M}\sum_{j\in M\setminus\{i\}}\tau^t\Big(P(M\setminus\{j\},d^t,\tfrac{\xi^t}{\tau^t})\\
&\qquad-P(M\setminus\{i\},d^t,\tfrac{\xi^t}{\tau^t})\Big)\\
=&\ d^t_*(M)+0\\
&\big(\textbf{by equation (10) and}\ \xi\in P(M,e,D^m)\big)\\
=&\ d^t_*(M).
\end{aligned}$$

Hence, $f(\xi)\in P(M,e,D^m)$ if $\xi\in P(M,e,D^m)$. ∎

Theorem 2: Let $(M,e,D^m)\in\Phi^*$. If $0<t<\frac{2}{|M|}$, then $\{[\xi]^q\}_{q=1}^\infty$ converges to $\overline{\Theta}(M,e,D^m)$ for each $\xi\in P(M,e,D^m)$.

Proof: Let $(M,e,D^m)\in\Phi^*$, $t\in\mathbb{N}_m$, $i\in M$ and $\xi\in P(M,e,D^m)$. By equation (9) and definition of f,

$$\begin{aligned}
&f^t_i(\xi)-\xi^t_i\\
=&\ w\sum_{j\in M\setminus\{i\}}\tau^t\Big(P(M\setminus\{j\},d^t,\tfrac{\xi^t}{\tau^t})-P(M\setminus\{i\},d^t,\tfrac{\xi^t}{\tau^t})\Big)\\
=&\ w\cdot|M|\cdot\Big(\overline{\Theta^t_i}(M,e,D^m)-\xi^t_i\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\overline{\Theta^t_i}(M,e,D^m)-f^t_i(\xi)\\
=&\ \overline{\Theta^t_i}(M,e,D^m)-\xi^t_i+\xi^t_i-f^t_i(\xi)\\
=&\ \overline{\Theta^t_i}(M,e,D^m)-\xi^t_i-w\cdot|M|\cdot(\overline{\Theta^t_i}(M,e,D^m)-\xi^t_i)\\
=&\ \Big(1-w\cdot|M|\Big)\Big[\overline{\Theta^t_i}(M,e,D^m)-\xi^t_i\Big].
\end{aligned}$$

So, for all $q\in\mathbb{N}$,

$$\begin{aligned}
&\overline{\Theta}(M,e,D^m)-[\xi]^q\\
=&\ \Big(1-w\cdot|M|\Big)^q\Big[\overline{\Theta}(M,e,D^m)-x\Big].
\end{aligned}$$

If $0<w<\frac{2}{|M|}$, then $-1<\Big(1-w\cdot|M|\Big)<1$ and $\{[\xi]^q\}_{q=1}^\infty$ converges geometrically to $\overline{\Theta}(M,e,D^m)$. ∎

By extending dynamic notion of Maschler and Owen [13], a different dynamic form can be offered under reductions.

Definition 4: Let σ be a solution, $(M,e,D^m)\in\Phi^*$, $S\subseteq M$ and $\xi\in P(M,e,D^m)$. The (ξ,σ)**-reduction** $(S,e_S,D^m_{\sigma,S,\xi})$ is given by $D^m_{\sigma,S,\xi}=(d^t_{\sigma,S,\xi})_{t\in\mathbb{N}_m}$ and for all $T\subseteq S$,

$$d^t_{\sigma,S,\xi}(\beta)=\begin{cases} d^t_*(M)-\sum\limits_{i\in M\setminus S}\xi^t_i & ,N(\beta)=S,\\ d^t_{S,\sigma}(\beta) & ,\text{otherwise.}\end{cases}$$

Similar to Maschler and Owen [13], a different amendment function also can be considered as follow. The **R-amendment function** is $g=(g^t)_{t\in\mathbb{N}_m}$, where $g^t=(g^t_i)_{i\in M}$ and $g^t_i:P(M,e,D^m)\to\mathbb{R}$ is define by

$$g^t_i(\xi)=\xi^t_i+w\sum_{k\in M\setminus\{i\}}\Big(\overline{\Theta^t_i}(\{i,k\},d^t_{\overline{\Theta},\{i,k\},\xi})-\xi^t_i\Big).$$

Define $[\kappa]^0=\xi,\ [\kappa]^1=g([\kappa]^0),\cdots,\ [\kappa]^q=g([\kappa]^{q-1})$ for each $q\in\mathbb{N}$.

Lemma 6: $g\big(\xi\big)\in P(M,e,D^m)$ for all $(M,e,D^m)\in\Phi^*$ and for all $\xi\in P(M,e,D^m)$.

Proof: Let $(M,e,D^m)\in\Phi^*$, $t\in\mathbb{N}_m$, $i,k\in M$ and $\xi\in X(M,e,d)$. Let $S=\{i,k\}$, by MPFY of $\overline{\Theta}$ and Definition 4,

$$\overline{\Theta^t_i}(S,e_S,D^m_{\overline{\Theta},S,\xi})+\overline{\Theta^t_k}(S,e_S,D^m_{\overline{\Theta},S,\xi})=\xi^t_i+\xi^t_k.$$

By RDEP and NNG of $\overline{\Theta}$,

$$\begin{aligned}
&\overline{\Theta^t_i}(S,e_S,D^m_{\overline{\Theta},S,\xi})-\overline{\Theta^t_k}(S,e_S,D^m_{\overline{\Theta},S,\xi})\\
=&\ (d^t_{\overline{\Theta},S,\xi})_*(\{i\})-(d^t_{\overline{\Theta},S,\xi})_*(\{k\})\\
=&\ (d^t_{S,\overline{\Theta}})_*(\{i\})-(d^t_{S,\overline{\Theta}})_*(\{k\})\\
=&\ \overline{\Theta^t_i}(S,e_S,D^m_{S,\overline{\Theta}})-\overline{\Theta^t_k}(S,e_S,D^m_{S,\overline{\Theta}})\\
=&\ \overline{\Theta^t_i}(M,e,D^m)-\overline{\Theta^t_k}(M,e,D^m).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&2\cdot\Big[\overline{\Theta^t_i}(S,e_S,D^m_{\overline{\Theta},S,\xi})-\xi^t_i\Big]\\
=&\ \overline{\Theta^t_i}(M,e,D^m)-\overline{\Theta^t_k}(M,e,D^m)-\xi^t_i+\xi^t_k.
\end{aligned}\tag{11}$$

Based on equation (11) and definition of g,

$$\begin{aligned}
& g_i^t(\xi) \\
= \;& \xi_i^t + \tfrac{w}{2}\Big[\sum_{k\in M\setminus\{i\}} \overline{\Theta_i^t}(M,e,D^m) - \sum_{k\in M\setminus\{i\}} \xi_i^t \\
& \quad - \sum_{k\in M\setminus\{i\}} \overline{\Theta_k^t}(M,e,D^m) + \sum_{k\in M\setminus\{i\}} \xi_k^t\Big] \\
= \;& \xi_i^t + \tfrac{w}{2}\Big[\sum_{k\in M\setminus\{i\}} \overline{\Theta_i^t}(M,e,D^m) - \big(|M|-1\big)\xi_i^t \\
& \quad - \sum_{k\in M\setminus\{i\}} \overline{\Theta_k^t}(M,e,d) + \big(d_*^t(M) - \xi_i^t\big)\Big] \\
= \;& \xi_i^t + \tfrac{w}{2}\Big[\big(|M|-1\big)\overline{\Theta_i^t}(M,e,D^m) - \big(|M|-1\big)\xi_i^t \\
& \quad - \big(d_*^t(M) - \overline{\Theta_i^t}(M,e,D^m)\big) + \big(d_*^t(M) - \xi_i^t\big)\Big] \\
= \;& \xi_i^t + \tfrac{|M|\cdot w}{2}\Big[\overline{\Theta_i^t}(M,e,D^m) - \xi_i^t\Big].
\end{aligned} \tag{12}$$

So we have that

$$\begin{aligned}
& \sum_{i\in M} g_i^t(\xi) \\
= \;& \sum_{i\in M}\Big[\xi_i^t + \tfrac{|M|\cdot w}{2}\cdot\big[\overline{\Theta_i^t}(M,e,D^m) - \xi_i^t\big]\Big] \\
= \;& \sum_{i\in M}\xi_i^t + \tfrac{|M|\cdot w}{2}\cdot\big[\sum_{i\in M}\overline{\Theta_i^t}(M,e,D^m) - \sum_{i\in M}\xi_i^t\big] \\
= \;& d_*^t(M) + \tfrac{|M|\cdot w}{2}\cdot\big[d_*^t(M) - d_*^t(M)\big] \\
= \;& d_*^t(M).
\end{aligned}$$

Thus, $g(\xi) \in P(M,e,D^m)$ for all $\xi \in P(M,e,D^m)$. ■

Theorem 3: Let $(M,e,D^m) \in \Phi^*$. If $0 < w < \frac{4}{|M|}$, then $\{[\kappa]^q\}_{q=1}^{\infty}$ converges to $\overline{\Theta}(M,e,D^m)$ for each $\xi \in P(M,e,D^m)$.

Proof: Let $(M,e,D^m) \in \Phi^*$, $t \in \mathbb{N}_m$ and $\xi \in P(M,e,D^m)$. By equation (12), $g_i^t(\xi) = \xi_i^t + \frac{|M|\cdot w}{2}\cdot\Big[\overline{\Theta_i^t}(M,e,D^m) - \xi_i^t\Big]$ for all $i \in M$. Therefore,

$$\begin{aligned}
& \big(1 - \tfrac{|M|\cdot w}{2}\big)\cdot\big[\overline{\Theta_i^t}(M,e,D^m) - \xi_i^t\big] \\
= \;& \big[\overline{\Theta_i^t}(M,e,D^m) - g_i^t(\xi)\big].
\end{aligned}$$

So, for all $q \in \mathbb{N}$,

$$\begin{aligned}
& \overline{\Theta}(M,e,D^m) - [\kappa]^q \\
= \;& \Big(1 - \tfrac{|M|\cdot w}{2}\Big)^q\Big[\overline{\Theta}(M,e,D^m) - x\Big].
\end{aligned}$$

If $0 < w < \frac{4}{|M|}$, then $-1 < \Big(1 - \frac{|M|\cdot w}{2}\Big) < 1$ and $\{[\kappa]^q\}_{q=1}^{\infty}$ converges to $\overline{\Theta}(M,e,d)$ for each $(M,e,D^m) \in \Phi^*$, for each $t \in \mathbb{N}_m$ and for each $i \in M$. ■

V. Conditional-scaled consideration

As stated in the Introduction, when the same person or group of people engage in a game, their actions will yield different benefits depending on different conditions. Thus, it is appropriate to assign corresponding scales based on influence under different conditions.

For each $i \in UM$, we employ a positive function $CS : UM \to \mathbb{R}^+$ to provide related conditional scales for each i. This function is referred to as the **conditional-scale function**. Furthermore, we will redefine the MNII using the conditional-scale function as a new power index.

Definition 5: The **multi-choice weighted normalized incremental index (MWNII)**, $\overline{\mathbb{W}}$, is defined by

$$\begin{aligned}
& \overline{\mathbb{W}_i^t}(M,e,D^m) \\
= \;& \frac{d_*^t(M)}{\sum_{k\in M} CS(k)\Theta_k^t(M,e,D^m)}\cdot CS(i)\Theta_i^t(M,e,D^m)
\end{aligned}$$

for each conditional-scale function CS, for each $(M,e,D^m) \in \Phi^{**}$, for each $t \in \mathbb{N}_m$ and for each $i \in M$, where $\Phi^{**} = \{(M,e,D^m) \in \Phi \mid \sum_{i\in M} CS(i)\Theta_i^t(M,e,D^m) \neq 0 \text{ for each } t \in \mathbb{N}_m\}$.

Lemma 7: The MWNII fits MPFY on Φ^{**}.

Proof: For all $(M,e,D^m) \in \Phi^{**}$ and for all $t \in \mathbb{N}_m$,

$$\begin{aligned}
& \sum_{i\in M} \overline{\mathbb{W}_i^t}(M,e,D^m) \\
= \;& \sum_{i\in M} \frac{d_*^t(M)}{\sum_{k\in M} CS(k)\Theta_k^t(M,e,D^m)}\cdot CS(i)\Theta_i^t(M,e,D^m) \\
= \;& \frac{d_*^t(M)}{\sum_{k\in M} CS(k)\Theta_k^t(M,e,D^m)} \sum_{i\in M} CS(i)\Theta_i^t(M,e,D^m) \\
= \;& d_*^t(M).
\end{aligned}$$

Thus, the MWNII fits MPFY on Φ^{**}. ■

Next, we axiomatize the MWNII by applying RDEP. A solution σ fits **weighted-normalized-norm for games (WNNG)** if $\sigma(M,e,d) = \overline{\mathbb{W}}(M,e,d)$ for each $(M,e,d) \in \Phi^{**}$ with $|M| \leq 2$.

Lemma 8: The MWNII fits RDEP on Φ^{**}.

Proof: Let CS be conditional-scale function, $(M,e,D^m) \in \Phi^{**}$ and $S \subseteq M$. It is trivial if $|M| = 1$. Assume that $|M| \geq 2$. If $S = \{i,j\}$ for some $i,j \in M$ and $(S, e_S, D^m_{S,\overline{\mathbb{W}}}) \in \Phi^*$. By definition of $\overline{\mathbb{W}}$ and equation (5),

$$\begin{aligned}
& \overline{\mathbb{W}_i^t}(S, e_S, d^m_{S,\overline{\mathbb{W}}}) \\
= \;& \frac{(d^t_{S,\overline{\mathbb{W}}})_*(S)}{\sum_{k\in S} CS(k)\Theta_k^t(S,e_S,d^m_{S,\overline{\mathbb{W}}})}\cdot CS(i)\Theta_i^t(S,e_S,d^m_{S,\overline{\mathbb{W}}}) \\
= \;& \frac{d_*^t(M) - \sum_{k\in M\setminus S}\overline{\mathbb{W}_k^t}(M,e,D^m)}{\sum_{k\in S} CS(k)\Theta_k^t(S,e_S,d^m_{S,\overline{\mathbb{W}}})}\cdot CS(i)\Theta_i^t(S,e_S,d^m_{S,\overline{\mathbb{W}}}) \\
& \textbf{(by definition of } d^m_{S,\overline{\mathbb{W}}}\textbf{)} \\
= \;& \frac{d_*^t(M) - \sum_{k\in M\setminus S}\overline{\mathbb{W}_k^t}(M,e,D^m)}{\sum_{k\in S} CS(k)\Theta_k^t(M,e,D^m)}\cdot CS(i)\Theta_i^t(M,e,D^m) \\
& \textbf{(by equation (5))} \\
= \;& \frac{\sum_{k\in S}\overline{\mathbb{W}_k^t}(M,e,D^m)}{\sum_{k\in S} CS(k)\Theta_k^t(M,e,D^m)}\cdot CS(i)\Theta_i^t(M,e,D^m) \\
& \textbf{(by MPFY of } \overline{\mathbb{W}}\textbf{)} \\
= \;& \frac{d_*^t(M)}{\sum_{k\in M} CS(k)\Theta_k^t(M,e,D^m)}\cdot CS(i)\Theta_i^t(M,e,D^m) \\
= \;& \overline{\mathbb{W}_i^t}(M,e,D^m).
\end{aligned}$$

Thus, the MWNII fits RDEP on Φ^{**}. ■

Lemma 9: A solution σ fits MPFY on Φ^* if it fits WNNG and RDEP on Φ^{**}.

Proof: Let $(M,e,D^m) \in \Phi^{**}$. If $|M| \leq 2$, then σ fits MPFY on Φ^{**} by WNNG. Suppose that $|M| > 2$. Assume, on the contrary, that there exists $(M,e_S,D^m) \in \Phi^{**}$ such that $\sum_{i\in M}\sigma_i^t(M,e,D^m) \neq d_*^t(M)$ for some $t \in \mathbb{N}_m$. This presents that there exist $i \in M$ and $j \in M$ such that $[d_*^t(M) - \sum_{k\in M\setminus\{i,j\}}\sigma_k^t(M,e,D^m)] \neq [\sigma_i^t(M,e,D^m) + \sigma_j^t(M,e,D^m)]$. By RDEP and σ fits MPFY for two-person games, this contradicts with

$$\begin{aligned}
& \sigma_i^t(M,e,D^m) + \sigma_j^t(M,e,D^m) \\
= \;& \sigma_i^t(\{i,j\}, d^m_{\{i,j\},\sigma}) + \sigma_j^t(\{i,j\}, d^m_{\{i,j\},\sigma}) \\
= \;& d_*^t(M) - \sum_{k\in M\setminus\{i,j\}}\sigma_k^t(M,e,D^m).
\end{aligned}$$

Hence σ fits MPFY. ■

Theorem 4: On Φ^{**}, the MWNII is the only solution fitting WNNG and RDEP.

Proof: By Lemma 8, $\overline{\mathbb{W}}$ fits RDEP on Φ^{**}. Absolutely, $\overline{\mathbb{W}}$ fits WNNG on Φ^{**}.

To demonstrate uniqueness, suppose σ fits RDEP and WNNG on Φ^{**}. By Lemma 9, σ fits MPFY on Φ^{**}. Let CS be conditional-scale function and $(M, e, D^m) \in \Phi^{**}$. The proof will be finished via induction on $|M|$. It is trivial that $\sigma(M, e, D^m) = \overline{\mathbb{W}}(M, e, D^m)$ by WNNG if $|M| \leq 2$. Assume that it holds if $|M| \leq r-1$, $r \geq 3$. The situation $|M| = r$: Let $t \in \mathbb{N}_m$ and $i, j \in M$ with $i \neq j$. Based on Definition 2, $\overline{\mathbb{W}}^t_k(M, e, D^m) = \frac{d^t_*(M)}{\sum\limits_{h\in M} \Theta^t_h(M,e,D^m)} \cdot \Theta^t_k(M, e, D^m)$ for all $k \in M$. Assume that $\rho^t_k = \frac{CS(k)\Theta^t_k(M,e,d)}{\sum\limits_{h\in M} CS(h)\Theta^t_h(M,e,d)}$ for all $k \in M$. Therefore,

$$
\begin{aligned}
& \sigma^t_i(M, e, D^m) \\
= \; & \sigma^t_i\big(M \setminus \{j\}, d^m_{M\setminus\{j\},\sigma}\big) \\
& \textbf{(by RDEP of } \sigma\textbf{)} \\
= \; & \overline{\mathbb{W}}^t_i\big(M \setminus \{j\}, d^m_{M\setminus\{j\},\sigma}\big) \\
& \textbf{(by WNNG of } \sigma\textbf{)} \\
= \; & \frac{(d^t_{M\setminus\{j\},\sigma})_*\big(M\setminus\{j\}\big)\cdot CS(i)\Theta^t_i\big(M\setminus\{j\}, d^m_{M\setminus\{j\},\sigma}\big)}{\sum\limits_{k\in M\setminus\{j\}} CS(k)\Theta^t_k\big(M\setminus\{j\}, d^m_{M\setminus\{j\},\sigma}\big)} \qquad (13) \\
= \; & \frac{d^t_*(M)-\sigma^t_i(M,e,D^m)}{\sum\limits_{k\in M\setminus\{j\}} CS(k)\Theta^t_k(M,e,D^m)} CS(i)\Theta^t_i(M, e, D^m) \\
& \textbf{(by equation (5))} \\
= \; & \frac{\big[d^t_*(M)-\sigma^t_i(M,e,D^m)\big]\cdot CS(i)\Theta^t_i(M,e,D^m)}{-CS(j)\Theta^t_j(M,e,D^m)+\sum\limits_{k\in M} CS(k)\Theta^t_k(M,e,D^m)}.
\end{aligned}
$$

By equation (13),

$$
\begin{aligned}
& & & \sigma^t_i(M, e, D^m) \cdot [1 - \rho^t_j] \\
& & = \; & [d^t_*(M) - \sigma^t_j(M, e, D^m)] \cdot \rho^t_j \\
& \Longrightarrow & & \sum_{i\in M} \sigma^t_i(M, e, D^m) \cdot [1 - \rho^t_j] \\
& & = \; & [d^t_*(M) - \sigma^t_j(M, e, D^m)] \cdot \sum_{i\in M} \rho^t_j \\
& \Longrightarrow & & d^t_*(M) \cdot [1 - \rho^t_j] \\
& & = \; & [d^t_*(M) - \sigma^t_j(M, e, D^m)] \cdot 1 \\
& & & \textbf{(by MPFY of } \sigma\textbf{)} \\
& \Longrightarrow & & d^t_*(M) - d^t_*(M) \cdot \rho^t_j \\
& & = \; & d^t_*(M) - \sigma^t_j(M, e, D^m) \\
& \Longrightarrow & & \overline{\mathbb{W}}^t_j(M, e, D^m) \\
& & = \; & \sigma^t_j(M, e, D^m).
\end{aligned}
$$

The proof is finished. ■

The subsequent instances aim to demonstrate that each of the properties utilized under Theorem 4 is logically independent from the other properties.

Example 4: Define a solution σ by for each $(M, e, D^m) \in \Phi$, for each $t \in \mathbb{N}_m$ and for each $i \in M$, $\sigma^t_i(M, e, D^m) = 0$. On Φ^{**}, σ fits RDEP, but it does not fit WNNG.

Example 5: Define a solution σ by for each $(M, e, D^m) \in \Phi^{**}$, for each $t \in \mathbb{N}_m$ and for each $i \in M$,

$$
\sigma^t_i(M, e, D^m) = \begin{cases} \overline{\mathbb{W}}_i(M, e, D^m) & \text{, if } |M| \leq 2, \\ 0 & \text{, o.w.} \end{cases}
$$

On Φ^{**}, σ fits WNNG, but it does not fit RDEP.

VI. Conclusions

In this article, we consider the multi-choice incremental index and the multi-choice normalized incremental index. We propose two axiomatic characterizations for these indexes by means of reductions. We also introduce alternative formulations and corresponding dynamic processes for the normalized incremental index using reduction and surplus function. Our results can be compared with existing ones in the following ways:

- The multi-choice incremental index, the multi-choice normalized incremental index and the multi-choice weighted normalized incremental index were initially introduced under the context of multicriteria multi-choice games.
- Our amendment functions due to Definitions 3 and 4, and corresponding dynamic processes are inspired by the dynamic results for the Shapley value [16] proposed by Maschler and Owen [13]. However, our amendment functions are depended on "surplus function", while Maschler and Owen's [13] amendment function is depended on "reductions".

The above-mentioned points raise the following question:

- whether there are other normalizations and related results for other solutions under multicriteria multi-choice games.

To our knowledge, related questions are still open issues.

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